

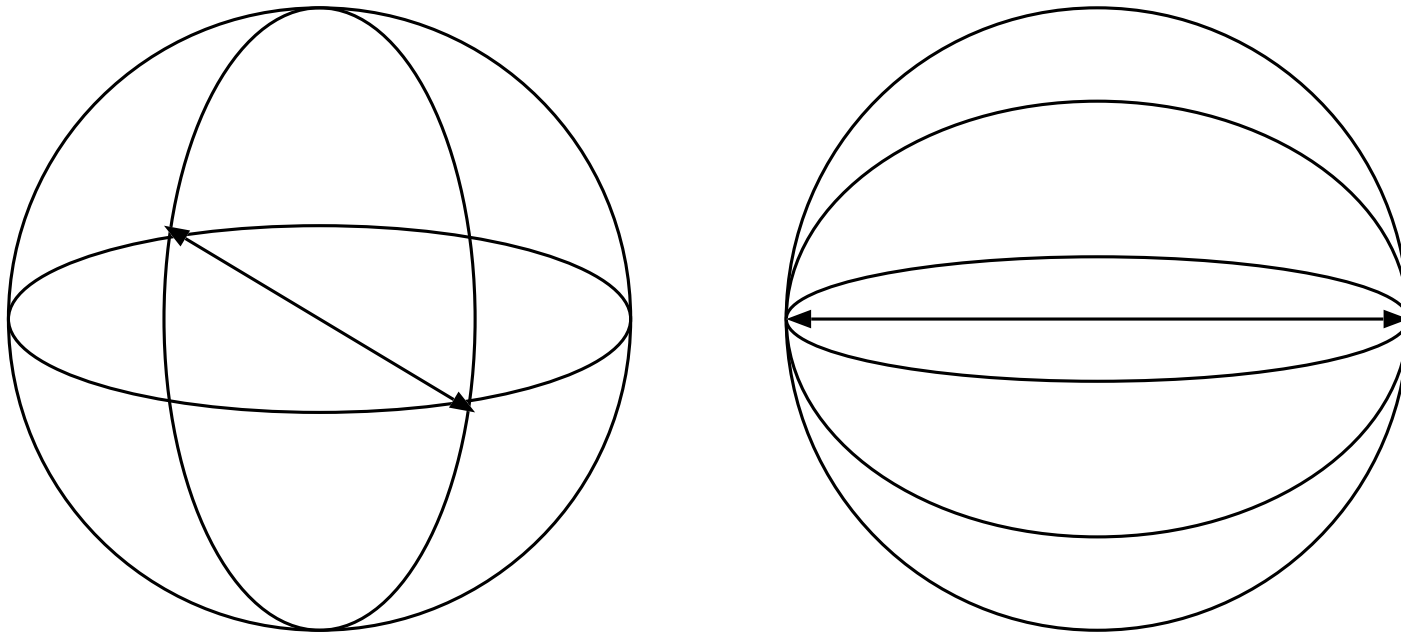
The intersection of two real forms in Hermitian symmetric spaces of compact type

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16 April 2010

Great circles



Pair of **antipodal points**

Complex hyperquadric : T.

General case : joint work with Tanaka

Fundamental items :

real form : Leung, Takeuchi

antipodal set : Chen-Nagano

2-number and topology : Takeuchi

1. Main results and related topics

$\bar{M}$: Hermitian symmetric space

M : **real form** of $\bar{M}$

$\exists$ involutive anti-holomorphic isometry

$$\sigma : \bar{M} \rightarrow \bar{M}$$

$$M = \{x \in \bar{M} \mid \sigma(x) = x\}$$

real form

: totally geodesic Lagrangian submanifold

Examples of real forms

$$\mathbb{R}^n \subset \mathbb{C}^n,$$

Lagrangian subspace,

$$\mathbb{R}P^n \subset \mathbb{C}P^n,$$

$$\mathbb{R}P^1 \subset \mathbb{C}P^1 : \text{the first example}$$

Hermitian symmetric spaces
of compact type
classification of real forms

: Leung, Takeuchi

$$\mathbb{C}P^n : \mathbb{R}P^n$$

$$G_r^{\mathbb{C}}(\mathbb{C}^{n+r}) : G_r^{\mathbb{R}}(\mathbb{R}^{n+r}),$$

$$G_q^{\mathbb{H}}(\mathbb{H}^{m+q}) \quad (n = 2m, r = 2q),$$

$$U(n) \quad (n = r)$$

M : Riemannian symmetric space

$S \subset M$: antipodal set

$$\Leftrightarrow \forall x, y \in S \quad s_x y = y$$

$\#_2 M$: 2-number of M

$$= \sup\{\#S \mid S \text{ is antipodal}\}$$

S : great antipodal set

$$\Leftrightarrow \#_2 M = \#S$$

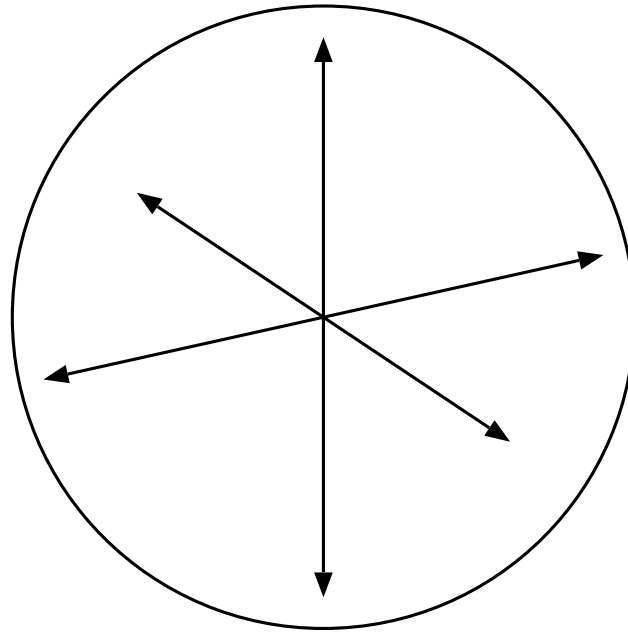
(These are introduced by Chen-Nagano)

Examples of antipodal sets

pair of antipodal points in S^n

$$\#_2 S^n = 2$$

$$\#_2 \mathbb{R}P^2 = 3$$



Chen-Nagano : $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$$\#_2 G_r^{\mathbb{K}}(\mathbb{K}^{n+r}) = \binom{n+r}{r}$$

Takeuchi :

M is a **symmetric R -space**

$$\Rightarrow \#_2 M = \dim H_*(M, \mathbb{Z}_2)$$

real form of Hermitian symmetric

space of compact type

: symmetric R -space

Thorem 1.1

M : Hermitian symmetric space
of compact type

L_1, L_2 : real forms of M
transversally intersect

$\Rightarrow$

$L_1 \cap L_2$

: antipodal set of L_1 and L_2

Thorem 1.2

M : Hermitian symmetric space
of compact type

L_1, L_2 : **congruent** real forms of M
transversally intersect

$\Rightarrow$

$L_1 \cap L_2$

: **great** antipodal set of L_1 and L_2

Theorem 1.3

M : irreducible Hermitian symmetric
space of compact type

L_1, L_2 : real forms of M

transversally intersect, $\#_2 L_1 \leq \#_2 L_2$

(1) $(M, L_1, L_2) \not\cong$

$(G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m}), G_m^{\mathbb{H}}(\mathbb{H}^{2m}), U(2m))$ ($m \geq 2$)

$\Rightarrow L_1 \cap L_2$: great antipodal set of L_1

$$\begin{aligned}
(2) \quad & (M, L_1, L_2) \cong \\
& (G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m}), G_m^{\mathbb{H}}(\mathbb{H}^{2m}), U(2m)) \quad (m \geq 2) \\
& \Rightarrow
\end{aligned}$$

$$\#(L_1 \cap L_2) = 2^m$$

$$< \binom{2m}{m} = \#_2 G_m^{\mathbb{H}}(\mathbb{H}^{2m})$$

$$< 2^{2m} = \#_2 U(2m)$$

M : Hermitian symmetric space

L : Lagrangian submanifold in M

L : globally tight (Y.-G. Oh)

$\Leftrightarrow L, g \cdot L$ transversally intersect

$L \cap g \cdot L \neq \emptyset$

(g : holomorphic isometry)

$$\#(L \cap g \cdot L) = \dim H_*(L, \mathbb{Z}_2)$$

Corollary 1.4

Any real form of a Hermitian symmetric space of compact type is globally tight.

It follows from Theorem 1.2.

2. Outline of the proofs

Lemma 2.1

M : compact Kähler manifold

holom. sect. curvature > 0

L_1, L_2 : compact totally geodesic

Lagrangian submanifolds of M

$\Rightarrow L_1 \cap L_2 \neq \emptyset$

Results of
maximal torus by Takeuchi
cut locus by Sakai

Lemma 2.2

The intersection of two maximal
tori of a compact symmetric space
has a special shape.

Lemma 2.2 $\Rightarrow$ Theorem 1.1

M : compact symmetric space

Fixed point set of s_o

$$F(s_o, M) = \bigcup_{j=0}^r M_j^+$$

each connected component M_j^+
: polar

polar : totally geodesic submanifold,
compact symmetric space

(This is introduced by Chen-Nagano)

Examples of polars

$$F(s_x, S^n) = \{x, -x\}$$

$$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \text{Cay}(n = 2)$$

$$F(s_o, \mathbb{K}P^n) = \{o\} \cup \mathbb{K}P^{n-1}$$

$$F(s_o, G_r^{\mathbb{K}}(\mathbb{K}^{n+r}))$$

$$= \bigcup_{j=0}^r G_j^{\mathbb{K}}(\mathbb{K}^r) \times G_{r-j}^{\mathbb{K}}(\mathbb{K}^n)$$

M : Hermitian symmetric space
of compact type

$\Rightarrow$ each polar M^+ : same as above

L : real form of M , $L \cap M^+ \neq \emptyset$

$\Rightarrow L \cap M^+$: real form of M^+

M, L_1, L_2 : in Thms. 1.2 and 1.3

$$F(s_o, M) = \bigcup_{j=0}^r M_j^+$$

We can suppose $o \in L_1 \cap L_2$

$L_1 \cap L_2$: antipodal by Th. 1.1

$$L_1 \cap L_2 \subset F(s_o, M)$$

$$L_1 \cap L_2 =$$

$$\bigcup_{j=0}^r \{ (L_1 \cap M_j^+) \cap (L_2 \cap M_j^+) \}$$

Proofs : induction on polars

$$Q_n(\mathbb{C}) = \left\{ [z_i] \in \mathbb{C}P^{n+1} \mid \sum_{i=1}^{n+2} z_i^2 = 0 \right\}$$

$$z_i = x_i + \sqrt{-1}y_i$$

$$[z_i] \in Q_n(\mathbb{C}) \Leftrightarrow (x_i), (y_i) : \text{orthogonal}$$

$$Q_n(\mathbb{C}) \rightarrow \tilde{G}_2(\mathbb{R}^{n+2}) : \text{isometry}$$

$$[z_i] \mapsto \{(x_i), (y_i)\}_{\mathbb{R}}$$

$$\tilde{G}_2(\mathbb{R}^{n+2}) \rightarrow \wedge^2 \mathbb{R}^{n+2}$$

$$\{u, v\}_{\mathbb{R}} \mapsto u \wedge v,$$

where u, v are orthonormal.

$$0 \leq k \leq [n/2]$$

$$\begin{aligned} S^{k,n-k} = & S^k(\mathbb{R}e_1 + \cdots + \mathbb{R}e_{k+1}) \\ & \wedge S^{n-k}(\mathbb{R}e_{k+2} + \cdots + \mathbb{R}e_{n+2}) \\ & (\cong S^k \times S^{n-k} / \mathbb{Z}_2) \end{aligned}$$

: real forms of $\tilde{G}_2(\mathbb{R}^{n+2})$

Any real form is congruent to $S^{k,n-k}$.

$$0 \leq k \leq l \leq [n/2]$$

L_1 : congruent to $S^{k,n-k}$,

L_2 : congruent to $S^{l,n-l}$

transversally intersect

$\Rightarrow L_1 \cap L_2$: congruent to

$$\{\pm e_1 \wedge e_{k+2}, \dots, \pm e_{k+1} \wedge e_{2k+2}\}$$

$$M = G_r^{\mathbb{C}}(\mathbb{C}^{n+r})$$

$$L_1 = G_r^{\mathbb{R}}(\mathbb{R}^{n+r}), \quad u \in U(n+r)$$

L_1, uL_1 transversally intersect in
 M

$\Rightarrow \exists$ an orthonormal basis of $\mathbb{R}^{n+r}$

$$v_1, \dots, v_{n+r}$$

$$L_1 \cap uL_1 = \{ \{v_{i_1}, \dots, v_{i_r}\}_{\mathbb{C}} \mid \\ 1 \leq i_1 < \dots < i_r \leq n+r \}.$$