Maximal antipodal subgroups in the automorphism groups of compact Lie algebras

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1 Antipodal sets

M : a Riem. sym. space

s_x : the geod. symmetry at $x \in M$

$S \subset M$: antipodal set

$\Leftrightarrow \forall x, y \in S \quad s_x y = y$

$|S|$: the cardinality of S

$\#_2 M$: 2-number of M

$= \sup\{|S| \mid S : \text{antipodal set}\}$

S : great antip. set $\Leftrightarrow \#_2 M = |S|$

$\{x, -x\}$: **great antipodal set of** S^n

$$\#_2 S^n = 2$$

$$\left\{ 1_3, \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \right\}$$

: **great antipodal set of** $SO(3)$

$$\#_2 SO(3) = 4 = 2^2$$

This is a subgroup

$$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

2 Symmetric R-spaces

Riem. sym. space which is an orbit of the isotropy action of Riem. sym. pair : symmetric R-space

transformed by $I_0(M)$: congruent

Thm 2.1(Tanaka-T.) **In a sym. R-space**

(A) $\forall$antipodal set $\subset$ $\exists$great antipodal set

(B) Great antipodal sets are congruent.

Great antipodal set is an orbit of the Weyl group of the symmetric pair.

3 Comact Lie groups

Comp. Lie gr with bi-inv. Riem. met.

$\Rightarrow$ compact Riemannian symmetric space

Maximal antipodal set $\ni e \Rightarrow$ subgroup

$\cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$

Many clas, comp. Lie group : sym. R-sp.

Their quotient group : not sym. R-sp.

We consider max. antip. subgr. in

$U(n)/\mathbb{Z}_\mu$, $SU(n)/\mathbb{Z}_\mu$, $O(n)/\{\pm 1_n\}$,

$SO(n)/\{\pm 1_n\}$, $Sp(n)/\{\pm 1_n\}$ and the

auto. gr. of their Lie alg.

$$\boldsymbol{\Delta}_n = \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n),$$

$$\boldsymbol{\Delta}_n^{\pm} = \{ g \in \boldsymbol{\Delta}_n \mid \det g = \pm 1 \}$$

$\boldsymbol{\Delta}_n$ and $\boldsymbol{\Delta}_n^{+}$ are unique max. antip. subgr. of $O(n)$, $U(n)$, $Sp(n)$ and $SO(n)$, $SU(n)$. These are great antip. and Thm. 2.1 holds for these.

$$D[4] = \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \subset O(2),$$

$$D^{\pm}[4] = \{g \in D[4] \mid \det g = \pm 1\}.$$

$D[4]$: dihedral group preserving the square

$n = 2^k \cdot l$ (l : odd). For $0 \leq s \leq k$ we define the tensor product $D(s,n)$ of s copies of $D[4]$ and $\Delta_{n/2^s}$:

$$D(s,n) = D[4] \otimes \cdots \otimes D[4] \otimes \Delta_{n/2^s} \subset O(n)$$

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1q}B \\ \vdots & & \vdots \\ a_{p1}B & \cdots & a_{pq}B \end{bmatrix}$$

$G_1 \subset GL(\mathbb{R}^{n_1}), G_2 \subset GL(\mathbb{R}^{n_2})$

$$G_1 \otimes G_2 = \{g_1 \otimes g_2 \mid g_1 \in G_1,\ g_2 \in G_2\}$$

G_1, G_2 : subgroups

$\Rightarrow G_1 \otimes G_2$: subgroup of $GL(\mathbb{R}^{n_1 n_2})$

Thm 3.1(Tanaka-T.) $\mu \in \mathbb{N}$

$\mathbb{Z}_\mu$: **cycl. gr. of ord.** μ **in cent. of** $U(n)$

θ : **a primitive** 2μ**-th root of 1**

$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$ **projection**

Maxi. antip. subgr. of $U(n)/\mathbb{Z}_\mu$ **is conjugate with one of the followings :**

(1) n **or** μ **is odd**

$$\pi_n(\{1, \theta\}D(0, n)) = \pi_n(\{1, \theta\}\Delta_n)$$

(2) n **and** μ **are even**

$$\pi_n(\{1, \theta\}D(s, n)) \quad (0 \leq s \leq k),$$

where $(s, n) = (k - 1, 2^k)$ **is excluded.**

Remark 3.2 $\boldsymbol{\Delta_2 \subsetneq D[4]}$ **implies**

$$\boldsymbol{D[4] \otimes \cdots \otimes D[4] \otimes \Delta_2}$$
$$\boldsymbol{\subsetneq D[4] \otimes \cdots \otimes D[4] \otimes D[4].}$$

That is $\boldsymbol{D(k-1, 2^k) \subsetneq D(k, 2^k)}$**, so** $\boldsymbol{D(k-1, 2^k)}$ **is not maximal.**

Thm 3.3(Tanaka-T.) $\quad \mu | n$

$\mathbb{Z}_\mu$ **: cycl. gr. of ord.** μ **in cent. of** $SU(n)$

θ **: a primitive** 2μ**-th root of 1**

Maxi. antip. subgr. of $SU(n)/\mathbb{Z}_\mu$ **is conjugate with one of the followings :**

(1) n **or** μ **is odd**

$$\pi_n(\Delta_n^+)$$

(2) n **and** μ **are even**

(a) $k = 1$

$\pi_n(\Delta_n^+ \cup \theta\Delta_n^-), \pi_n((D^+[4] \cup \theta D^-[4]) \otimes \Delta_l),$

where $\pi_2(\Delta_2^+ \cup \theta\Delta_2^-)$ **is excluded if**

$n = \mu = 2$

(b) $k \geq 2,\ \mu = 2^{k'} \cdot l'$

(b1) $k' = k$

$\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$,

$\pi_n(D(s,n)) \quad (1 \leq s \leq k)$,

where $(s,n) = (k-1, 2^k)$ is excluded.

(b2) $1 \leq k' < k$

$\pi_n(\{1,\theta\}\Delta_n^+)$,

$\pi_n(\{1,\theta\}D(s,n)) \quad (1 \leq s \leq k)$,

where $(s,n) = (k-1, 2^k)$ is excluded and $\pi_4(\{1,\theta\}\Delta_4^+)$ is excluded if $n = 4$.

Remark 3.4

$\boldsymbol{\Delta}_4^+ = \boldsymbol{\Delta}_2 \otimes \boldsymbol{\Delta}_2 \subsetneq \boldsymbol{D}[4] \otimes \boldsymbol{D}[4] = \boldsymbol{D}(2, 4)$,
so $\boldsymbol{\Delta}_4^+$ is excluded.

Outline of Proof of Thm 3.3

$\boldsymbol{A}$: Max.antip subgr of $\boldsymbol{SU(n)/\mathbb{Z}_\mu}$
$\boldsymbol{A'}$: Max.antip subgr of
$\boldsymbol{U(n)/\mathbb{Z}_\mu}$containing $\boldsymbol{A}$
We apply Thm 3.1 to $\boldsymbol{A'}$
$\Rightarrow$ Proof of Thm 3.3

$Q[8] = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$ is the quaternion group, where $\mathrm{i}, \mathrm{j}, \mathrm{k}$ are standard basis of the quaternions.

Thm 3.5(Tanaka-T.) $\quad n = 2^k \cdot l$, l : odd
(I) Maxi. antip. subgr. of $O(n)/\{\pm 1_n\}$ is conjugate with one of the followings :

$$\pi_n(D(s,n)) \quad (0 \leq s \leq k),$$

where $(s,n) = (k-1, 2^k)$ is excluded.

(II) n : even. Maxi. antip. subgr. of $SO(n)/\{\pm 1_n\}$ is conjugate with one of the followings :

(1) $k = 1$

$$\pi_n(\Delta_n^+), \quad \pi_n(D^+[4] \otimes \Delta_l),$$

where $\pi_2(\Delta_2^+)$ is excluded if $n = 2$.

(2) $k \geq 2$

$$\pi_n(\Delta_n^+), \quad \pi_n(D(s,n)) \quad (1 \leq s \leq k),$$

where $(s,n) = (k-1, 2^k)$ is excluded and $\pi_4(\Delta_4^+)$ is excluded if $n = 4$.

(III) Maxi. antip. subgr. of $Sp(n)/\{\pm 1_n\}$ is conjugate with one of the followings :

$$\pi_n(Q[8] \cdot D(s,n)) \quad (0 \leq s \leq k),$$

where $(s,n) = (k-1, 2^k)$ is excluded.

4 Automorphism groups

$\mathfrak{g}$: comp. Lie algebra

Max. antip. subgr. of Aut($\mathfrak{g}$)

$\leftrightarrow$ max. set consisting of involutions of $\mathfrak{g}$ which are mutually commutative

Mutually comm. involutions are useful for geometry

- **symmetric triad (Ikawa)**
- **two real forms in HSS of comp. type**
- **two real forms in complex flag mfd**

$\mathrm{Aut}(\mathfrak{o}(n)) = \mathrm{Ad}(O(n)) \cong O(n)/\{\pm 1_n\}$,

if $n \neq 8$.

We have already classified max. antip.

subgr. : $\mathrm{Ad}(D(s,n))$

$\mathrm{Aut}(\mathfrak{o}(8))/\mathrm{Ad}(SO(8)) \cong \mathfrak{S}_3$

Max antip. subgr of $\mathfrak{S}_3 \cong \mathbb{Z}_2$

$\Rightarrow$ Max antip. subgr of $\mathrm{Aut}(\mathfrak{o}(8))$:

$\mathrm{Ad}(D(s,8))$

$\mathbf{Aut}(\mathfrak{sp}(n)) = \mathbf{Ad}(Sp(n)) \cong Sp(n)/\{\pm 1_n\}$

We have already classified max. antip. subgr. : $\mathbf{Ad}(Q[8] \cdot D(s, n))$

$\boldsymbol{G}$ **: conn. comp. Lie gr.**

$\boldsymbol{T}$ **: max. torus of** $\boldsymbol{G}$

$$\boldsymbol{G = \bigcup_{g \in G} gTg^{-1}}$$

G : comp. Lie gr.

G_0 : identity comp. of G

$g_1 \in G \backslash G_0$ $I_{g_1} : G_0 \to G_0$; $g \mapsto g_1 g g_1^{-1}$

T_1 : max. torus of $\mathrm{Fix}(G_0, I_{g_1})$

$$(*) \quad g_1 G_0 = \bigcup_{g \in G_0} g g_1 T_1 g^{-1}$$

$\boldsymbol{\tau} : \mathbb{C}^n \to \mathbb{C}^n \ \boldsymbol{z} \mapsto \bar{\boldsymbol{z}}$: **complex conj.**

$\mathbf{Aut}(\mathfrak{su}(\boldsymbol{n})) = \{1, \boldsymbol{\tau}\}\mathbf{Ad}(\boldsymbol{SU}(\boldsymbol{n})) \cong$

$\{1, \boldsymbol{\tau}\}\boldsymbol{SU}(\boldsymbol{n})/\mathbb{Z}_{\boldsymbol{n}}$

$\{1, \boldsymbol{\tau}\}\boldsymbol{SU}(\boldsymbol{n})/\mathbb{Z}_{\boldsymbol{n}}$ **has two conn. comp.**

$(*) +$ **results on** $\boldsymbol{SU}(\boldsymbol{n})/\mathbb{Z}_{\boldsymbol{n}}$

$\Rightarrow$ **results on** $\{1, \boldsymbol{\tau}\}\boldsymbol{SU}(\boldsymbol{n})/\mathbb{Z}_{\boldsymbol{n}}$