The intersection of two real forms in the complex Grassmann manifold

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Contents

1 Introduction

$M = S^2$

L_1, L_2: great circles intersecting transversally

$\Longrightarrow \quad L_1 \cap L_2 = \{o, \bar{o}\}$

(o and $\bar{o}$ are antipodal to each other)

$S^2 \cong \mathbb{C}P^1$

$L_1, L_2 \cong \mathbb{R}P^1$: totally geodesic

Known fact (R. Howard):

$M = \mathbb{C}P^n$

$L = \mathbb{R}P^n$: totally geodesic

$\Longrightarrow \quad \sharp(L \cap g \cdot L) = n + 1$

($^\forall g \in I(M)$ such that L intersects $g \cdot L$ transversally)

$^\forall x, y \in L \cap g \cdot L$ are antipodal to each other

“2-number” (B.-Y. Chen and T. Nagano)

$$\#_2(\mathbb{R}P^n) = n + 1$$

(M. Takeuchi) $\#_2(M) = SB(M, \mathbb{Z}_2)$

(:=the sum of $\mathbb{Z}_2$-Betti numbers of M)

if M is a symmetric R-space

“globally tight” (Y.-G. Oh)

M : Hermitian symmetric space

L : Lagrangian submanifold

L : *globally tight* $\overset{\text{def}}{\Longleftrightarrow}$ $\sharp(L \cap g \cdot L) = SB(M, \mathbb{Z}_2)$ for ${}^\forall g \in I(M)$

such that L intersects $g \cdot L$ transversally

Remark : $\mathbb{R}P^n$ is globally tight.

Problem :
Does the intersection of two real forms of a compact Hermitian symmetric space consist of antipodal points if they intersect transversally? Moreover, does the number of such points coincide with the 2-number of the real form?

Problem :
Is every real form of any compact Hermitian symmetric space globally tight?

2 Preliminaries

Definition

M : Hermitian manifold

L : *real form* of M $\overset{\text{def}}{\iff}$ ${}^{\exists}\sigma : M \to M$: anti-holomorphic involutive isometry s.t. $L = \{x \in M \mid \sigma(x) = x\}$

M : Hermitian symmetric space of compact type

$\Longrightarrow$ Every real form of M is a totally geodesic Lagrangian submanifold.

Remark : A real forms of a Hermitian symmetric space of compact type is so-called a symmetric R-space.

M : Hermitian symmetric space of compact type
L : real form of M

L	M
$UI(n) = U(n)/SO(n)$	$CI(n) = Sp(n)/U(n)$
$Sp(n) = Sp(n) \times Sp(n)/\Delta$	$CI(2n)$
$G_r(\mathbb{R}^{n+r}) = SO(n+r)/S(O(r) \times O(n))$	$G_r(\mathbb{C}^{n+r}) = SU(n+r)/S(U(r) \times U(n))$
$G_r(\mathbb{H}^{n+r}) = Sp(n+r)/Sp(r) \times Sp(n)$	$G_{2r}(\mathbb{C}^{2n+2r})$
$U(n) = U(n) \times U(n)/\Delta$	$G_n(\mathbb{C}^{2n})$
$SO(n) = SO(n) \times SO(n)/\Delta$	$DIII(n) = SO(2n)/U(n)$
$UII(n) = U(2n)/Sp(n)$	$DIII(2n)$
$S^k \times S^l/\mathbb{Z}_2$	$G_2^o(\mathbb{R}^{k+l}) = SO(k+l)/SO(2) \times SO(k+l-2)$
$FII = F_4/Spin(9)$	$EIII = E_6/T \cdot Spin(10)$
$G_2(\mathbb{H}^4)/\mathbb{Z}_2$	$EIII$
$T \cdot EIV = T \cdot E_6/F_4$	$EVII = E_7/T \cdot E_6$
$AII(4)/\mathbb{Z}_2$	$EVII$

Δ : the diagonal subgroup, $T \cong U(1)$

Definition

M : Riemannian symmetric space

$S \subset M$: subset

S : *antipodal set* $\overset{\text{def}}{\Longleftrightarrow}$ ${}^\forall x, y \in S, \quad s_x y = y$

(s_x : the symmetry at x)

The 2-number of M

$$\#_2(M) := \sup\{\#S \mid S \subset M : \text{antipodal set}\}$$

B.-Y. Chen and T. Nagano (Trans. Amer. Math. Soc.,**308** (1988), 273–297)

Examples :

$\#_2(S^n) = 2$

$\#_2(T^n) = 2^n$

$\#_2(\mathbb{R}P^n) = n + 1$

$\#_2(G_r(\mathbb{K}^n)) = {}_nC_r$, the binomial coefficient ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$)

$\#_2(U(n)) = 2^n$

3 Intersection of two real forms in $Q_n(\mathbb{C})$

$$Q_n(\mathbb{C}) = \{[z_1, \ldots, z_{n+2}] \in \mathbb{C}P^{n+1} \mid z_1^2 + \cdots + z_{n+2}^2 = 0\}$$

: complex hyperquadric

$\cong G_2^o(\mathbb{R}^{n+2})$: oriented Grassmann manifold

$= SO(n+2)/SO(2) \times SO(n)$

L : real form of $G_2^o(\mathbb{R}^{n+2}) \iff L \cong S^k \times S^{n-k}/\mathbb{Z}_2 \quad (0 \leq k \leq [n/2])$

Theorem (H. Tasaki)

L_1, L_2: real forms of $G_2^o(\mathbb{R}^{n+2})$ intersecting transversally

$\Longrightarrow \quad L_1 \cap L_2$: antipodal set

$\#(L_1 \cap L_2) = \min\{\#_2(L_1), \#_2(L_2)\}$

Remark ; We do **not** assume that L_1 and L_2 are congruent to each other.

Corollary

Every real form of $G_2^o(\mathbb{R}^{n+2})$ is globally tight.

Remark : H. Iriyeh and T. Sakai proved that

$$S^2,\ S^1 \times S^1/\mathbb{Z}_2 \text{ in } G_2^o(\mathbb{R}^4) \cong S^2 \times S^2$$
$$S^n,\ S^1 \times S^{n-1}/\mathbb{Z}_2 \text{ in } G_2^o(\mathbb{R}^{n+2})$$

are globally tight in a different way.

Lemma 1

M : compact Kähler manifold with HS > 0

L_1, L_2 : totally geodesic compact Lagrangian submanifolds

$$\Longrightarrow \quad L_1 \cap L_2 \neq \emptyset$$

HS : holomorphic sectional curvature

X : compact Riemannian manifold, $p \in X$

$C_p(X)$: the cut locus of X w.r.t. p

$\tilde{C}_p(X)$: the tangent cut locus of X w.r.t. p

Lemma 2

M : Hermitian symmetric space of compact type

L : real form, $o \in L$

$$\Longrightarrow \quad \tilde{C}_o(L) = T_oL \cap \tilde{C}_o(M)$$

$$C_o(L) = L \cap C_o(M)$$

In particular, every minimal geodesic in L is minimal in M.

Idea of Proof :

(a) $^\exists o \in L_1 \cap L_2$ (from Lemma 1)

(b) $L_1 \cap L_2 - \{o\} \subset C_o(M)$ (by using Lemma 2)

(c) Proving $L_1 \cap L_2 \subset F(s_o, M)$ ($\subset C_o(M)$), the fixed point set of s_o, we reduce the problem to the case of real forms of $F(s_o, M)$ and use the induction.

(Since the symmetry s_o is holomorphic involutive isometry of M, each connected componet of $F(s_o, M)$ is a compact Hermitian symmetric space.)

Remark : In the step (c), we use the properties peculiar to $G_2^o(\mathbb{R}^{n+2})$.

Remark :

M : compact Riemannian symmetric space, $o \in M$

Each connected component of $F(s_o, M)$ is called *a polar* and its detailed studies were done in a series of papers by Nagano and Nagano -T. (Tokyo J. Math. **11**(1988), 57–79, **15**(1992), 39–82, **18**(1995), 193–212, **22**(1999), 193–211, **23**(2000), 403–416).

4 Intersection of two real forms in $G_r(\mathbb{C}^n)$

Theorem (Tasaki–T.)

L_1, L_2: real forms of $G_r(\mathbb{C}^{n+r})$ intersecting transversally as one the following three cases:

(i) $L_1, L_2 \cong G_r(\mathbb{R}^{n+r})$

(ii) $L_1, L_2 \cong \mathbb{H}P^m$ for $r = 2, n = 2m$

(iii) $L_1 \cong \mathbb{H}P^m$ and $L_2 \cong G_2(\mathbb{R}^{2m+2})$ for $r = 2, n = 2m$

$\Longrightarrow$ $L_1 \cap L_2$: antipodal set

$\#(L_1 \cap L_2) = \min\{\#_2(L_1), \#_2(L_2)\}$

The real forms of $G_r(\mathbb{C}^{n+r})$ are :

$$
\begin{array}{ll}
G_r(\mathbb{R}^{n+r}) & \\
G_s(\mathbb{H}^{m+s}) & \text{when } n = 2m \text{ and } r = 2s \\
U(m) & \text{when } n = 2m \text{ and } r = m
\end{array}
$$

Conjecture

L_1, L_2: real forms of $G_r(\mathbb{C}^{n+r})$ intersecting transversally

$\Longrightarrow$ $L_1 \cap L_2$: antipodal set

$\#(L_1 \cap L_2) = \min\{\#_2(L_1), \#_2(L_2)\}$