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Antipodal sets of compact Riemannian symmetric spaces and their applications

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Abstract. This is a joint work with Hiroyuki Tasaki ([10], [11]). We investigate fundamental properties of antipodal sets of symmetric R-spaces. We also investigate the intersection of two real forms in a Hermitian symmetric space of compact type. We obtained that the intersection is an antipodal set and that its cardinality is equal to the 2-number of the real form if two real forms are congruent. As a consequence we obtained that every real form of a Hermitian symmetric space of compact type is a globally tight Lagrangian submanifold.

1 Introduction

A subset S in a Riemannian symmetric space is called an *antipodal set* if $s_x(y) = y$ for any x and y in S, where s_x denotes the symmetry at x. The *2-number* $\#_2 M$ of a compact Riemannian symmetric space M is the supremum of the cardinality of antipodal sets in M. It is known that $\#_2 M$ is finite. If the cardinality of an antipodal set S in M attains $\#_2 M$, S is called a *great* antipodal set. These notions were introduced by Chen and Nagano [3]. Takeuchi [9] proved that the 2-number of a symmetric R-space is coincides with the sum of the Betti numbers with $\mathbb{Z}_2$-coefficient. A compact Riemannian symmetric space is called a *symmetric R-space* if it is an orbit of the linear isotropy action of a Riemannian symmetric pair of semisimple type.

Hermitian symmetric spaces of compact type have realizations as orbits of the

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adjoint representations of compact semisimple Lie groups. We proved that the following properties of antipodal sets of Hermitian symmetric spaces of compact type (Theorem 3.1).

(A) Any antipodal set is induced in a great antipodal set.

(B) Any two great antipodal sets are congruent.

Here we say that two subsets in a Hermitian symmetric space of compact type M are *congruent*, if one is transformed to another by an element of $I_0(M)$, the identity component of the group of all isometries on M. We also proved that antipodal sets of symmetric R-spaces satisfy the properties (A) and (B) (Theorem 3.3).

The fixed point set of an involutive anti-holomorphic isometry of Hermitian symmetric space is called a *real form*. It is known that a real form is connected. Every real form is a totally geodesic Lagrangian submanifold ([5], [8]). Leung [5] and Takeuchi [8] classified real forms of Hermitian symmetric spaces of compact type. A simple example of real form is a great circle in the 2-sphere S^2. It is easily seen that any two great circles in S^2 intersects in two points, which are antipodal to each other. Roughly speaking, we have the similar situation for two real forms in a Hermitian symmetric space of compact type. Precisely, we proved that the intersection of two real forms is an antipodal set if they intersect transversally (Theorem 4.1) and that furthermore if two real forms are congruent to each other, the intersection is a great antipodal set of the real form. Hence the cardinality of the intersection is equal to the 2-number of the real form (Theorem 4.5). They are generalizations of the earlier results of Tasaki [12].

Takeuchi [8] proved that every real form of Hermitian symmetric spaces of compact type is a symmetric R-space. As a consequence, every real form of Hermitian symmetric spaces of compact type is a globally tight Lagrangian submanifold in the meaning of Oh [6] (Corollary 4.6).

By applying Theorems 4.1 and 4.7, Iriyeh, Sakai and Tasaki [4] computed the Lagrangian Floer homology $HF(L_0, L_1 : \mathbb{Z}_2)$ of a pair of real forms (L_0, L_1) in a monotone Hermitian symmetric space M of compact type (Theorems 4.8 and 4.9).

2 Preliminaries

Let M be a compact connected Riemannian symmetric space and $p \in M$. We decompose the fixed point set $F(s_p, M)$ of s_p to the disjoint union of its connected components:

$$F(s_p, M) = \bigcup_{j=0}^{r} M_j^+,$$

where $M_0^+ = \{p\}$. We call each connected component M_j^+ a *polar* of M with respect to p ([1], [2], [3]). Every polar is a totally geodesic submanifold. Since the symmetry of a Hermitian symmetric space at any point is an involutive holomorphic isometry, every polar of a Hermitian symmetric space of compact type is a Hermitian symmetric space of compact type. For example, let M be the complex projective space $\mathbb{C}P^n$ and $e_1, \dots, e_{n+1}$ be a unitary basis of $\mathbb{C}^{n+1}$. The symmetry s_o at $o = \langle e_1 \rangle_{\mathbb{C}}$, the subspace spanned by e_1 is induced by the reflection with respect to the hyperplane $\langle e_2, \dots, e_{n+1} \rangle_{\mathbb{C}}$. Then

$$\begin{aligned} F(s_o, \mathbb{C}P^n) &= \{o\} \cup \{V \mid V : \text{one-dimensional subspace in} \langle e_2, \dots, e_{n+1} \rangle_{\mathbb{C}}\} \\ &\cong \{o\} \cup \mathbb{C}P^{n-1}. \end{aligned}$$

Lemma 2.1 ([10]). *Let M be a Hermitian symmetric space of compact type and L be a real form of M through o. If a polar M^+ with respect to o satisfies $L \cap M^+ \neq \emptyset$, then $L \cap M^+$ is a real form of M^+.*

Lemma 2.2 ([10]). *Let M be a Hermitian symmetric space of compact type, and denote by*

$$F(s_o, M) = \bigcup_{j=0}^{r} M_j^+$$

the polars of M with respect to a point $o \in M$.

(1) *If L is a real form of M through o, then the polars of L with respect to o is described by*

$$F(s_o, L) = \bigcup_{j=0}^{r} L \cap M_j^+,$$

and the following equality holds.

$$\#_2 L = \sum_{j=0}^{r} \#_2 (L \cap M_j^+).$$

(2) *If L_1, L_2 are real forms of M through o, then we have*

$$L_1 \cap L_2 = \bigcup_{j=0}^{r} \left\{ (L_1 \cap M_j^+) \cap (L_2 \cap M_j^+) \right\},$$

$$\#(L_1 \cap L_2) = \sum_{j=0}^{r} \# \left\{ (L_1 \cap M_j^+) \cap (L_2 \cap M_j^+) \right\}.$$

3 Antipodal sets of symmetric R-spaces

Hermitian symmetric spaces of compact type have realizations as orbits of the adjoint representations of compact semisimple Lie groups.

Let $\mathfrak{g}$ be a compact semisimple Lie algebra and let $G = \mathrm{Int}(\mathfrak{g})$. We take a G-invariant innner product $\langle \ , \ \rangle$ on $\mathfrak{g}$. Let $J \in \mathfrak{g}$ be an nonzero element which satisfies $(\mathrm{ad}J)^3 = -\mathrm{ad}J$. Then the G-orbit $M = G \cdot J$ is a Hermitian symmetric space of compact type with respect to the induced metric from $\langle\ ,\ \rangle$. Let K be the isotropy subgroup at J. Then the Lie algebra $\mathfrak{k}$ of K is $\mathfrak{k} = \{X \in \mathfrak{g} \mid [J, X] = 0\}$. Let $\mathfrak{m} = \{[J, X] \mid X \in \mathfrak{g}\}$, then we have an orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. $\mathfrak{k}$ is the $(+1)$-eigenspace and $\mathfrak{m}$ is the (-1)-eigenspace of the involutive automorphism $e^{\pi \mathrm{ad}J}$ of $\mathfrak{g}$ respectively. $\mathrm{ad}J$ is a complex structure of $\mathfrak{m}$ which can be identified with the tangent space of M at J. Conversely, every Hermitian symmetric space of compact type is obtained like this.

Theorem 3.1 ([7], [11]). *Let M be a Hermitian symmetric space of compact type and take $X, Y \in M$. $s_X(Y) = Y$ if and only if $[X, Y] = 0$. Moreover the following conditions* (A) *and* (B) *hold.*

(A) *Any antipodal set is included in a great antipodal set.*

(B) *Any two great antipodal sets are congruent.*

A great antipodal set of M is represented as $M \cap \mathfrak{t}$ for a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. In particular, a great antipodal set of M is an orbit of the Weyl group of $\mathfrak{g}$.

Remark 3.2. Sánchez [7] obtained a similar result for complex flag manifolds.

Theorem 3.3 ([11]). *Let M be a Hermitian symmetric space of compact type and $\tau : M \to M$ be an involutive anti-holomorphic isometry determining a real form $L = F(\tau, M)$. We define an automorphism I_τ of G by*

$$I_\tau : G \to G\ ;\ g \mapsto \tau g \tau^{-1}.$$

We assume that L contains J. Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{p}$ be the canonical direct sum decomposition determined by I_τ. We have $L = M \cap \mathfrak{p}$. Moreover the following conditions (A) *and* (B) *hold.*

(A) *Any antipodal set is included in a great antipodal set.*

(B) *Any two great antipodal sets are congruent.*

A great antipodal set of L is represented as $M \cap \mathfrak{a}$ for a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. In particular, a great antipodal set of L is an orbit of the Weyl group of the symmetric pair determined by I_τ.

Since every symmetric R-space is a real form of some Hermitian symmetric space of compact type by a result of Takeuchi [8], we have the following.

Corollary 3.4 ([11])**.** *For a symmetric R-space the following conditions* (A) *and* (B) *hold.*

(A) *Any antipodal set is included in a great antipodal set.*

(B) *Any two great antipodal sets are congruent.*

In [11] we proved that the adjoint group $\mathrm{Ad}(SU(4))$ does not satisfy (A).

4 Intersections of two real forms

Theorem 4.1 ([10])**.** *Let M be a Hermitian symmetric space of compact type. If two real forms L_1 and L_2 of M intersect transversally, then $L_1 \cap L_2$ is an antipodal set of L_1 and L_2.*

To prove it, we use some properties of maximal tori. One of them is the following.

Lemma 4.2 ([10])**.** *Let A_1, A_2 be two maximal tori of a compact Riemannian symmetric space through the origin o. We define the root system from A_2 and determine $S \subset \mathfrak{a}_2$. If $A_1 \cap A_2 \cap \mathrm{Exp} S^{\Delta} \neq \emptyset$ for a subset $\Delta \subset \Pi^{\#}$, then $\mathrm{Exp} S^{\Delta} \subset A_1 \cap A_2$.*

Here we omit the detailed definitions of notations in this lemma. It is known that a maximal torus has a fundamental domain which has a stratification. The lemma says that if two maximal tori through the origin intersect at least in a point, then the intersection includes the image of Exp of the whole cell which contains the point.

Theorem 4.3 ([10])**.** *Let M be a Hermitian symmetric space of compact type and let L_1, L_2, L_1', L_2' be real forms of M. We assume that L_1, L_1' are congruent and that L_2, L_2' are congruent. If L_1, L_2 intersect transversally and if L_1', L_2' intersect transversally, then $\#(L_1 \cap L_2) = \#(L_1' \cap L_2')$.*

Corollary 4.4 ([11])**.** *Under the assumption of Thoerem 4.3, if $\#(L_1 \cap L_2) = min\{\#_2 L_1, \#_2 L_2\}$, then $L_1 \cap L_2$ is congruent to $L_1' \cap L_2'$.*

Theorem 4.5 ([10]). *Let M be a Hermitian symmetric space of compact type and let L_1 and L_2 be real forms of M which are congruent to each other and intersect transversally. Then $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2. That is, $\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$.*

The idea of the proof is to use induction on polars. Every polar of a Hermitian symmetric space M of compact type is also a Hermitian symmetric space of compact type. And every polar of a real form L of M is a real form of some polar of M by Lemma 2.1. These facts and Lemma 2.2 make us reduce the problem to lower dimensional cases, since the dimension of a polar is less than that of the ambient space.

Oh [6] introduced the notion of global tightness of Lagrangian submanifolds in a Hermitian symmetric space. We call a Lagrangian submanifold L of a Hermitian symmetric space M *globally tight*, if L satisfies

$$\#(L \cap g \cdot L) = \dim H_*(L, \mathbb{Z}_2)$$

for any $g \in I_0(M)$ with property that L intersects transversally with $g \cdot L$. By Takeuchi [8] and Theorem 4.5 we have the following.

Corollary 4.6 ([11]). *Any real form of a Hermitian symmetric space of compact type is a globally tight Lagrangian submanifold.*

Let $G_k^{\mathbb{K}}(\mathbb{K}^n)$ denote the Grassmann manifold of the k dimensional $\mathbb{K}$-subspaces in $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Theorem 4.7 ([10]). *Let M be an irreducible Hermitian symmetric space of compact type and let L_1 and L_2 be two real forms of M which intersect transversally.*

(1) *If $M = G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m})$ $(m \geq 2)$, L_1 is congruent to $G_m^{\mathbb{H}}(\mathbb{H}^{2m})$ and L_2 is congruent to $U(2m)$, then*

$$\#(L_1 \cap L_2) = 2^m < \binom{2m}{m} = \#_2 L_1 < 2^{2m} = \#_2 L_2.$$

(2) *Otherwise, $L_1 \cap L_2$ is a great antipodal set of one of L_i's whose 2-number is less than or equal to another and we have*

$$\#(L_1 \cap L_2) = \min\{\#_2 L_1, \#_2 L_2\}.$$

The list of irreducible Hermitian symmetric spaces of compact type and their real forms which we have to show the statements of the theorem is, according to the results of Leung [5] or Takeuchi [8], as follows:

M	L_1	L_2
$Q_n(\mathbb{C})$	$S^{k,n-k}$	$S^{l,n-l}$
$G_{2q}^{\mathbb{C}}(\mathbb{C}^{2m+2q})$	$G_q^{\mathbb{H}}(\mathbb{H}^{m+q})$	$G_{2q}^{\mathbb{R}}(\mathbb{R}^{2m+2q})$
$G_n^{\mathbb{C}}(\mathbb{C}^{2n})$	$U(n)$	$G_n^{\mathbb{R}}(\mathbb{R}^{2n})$
$G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m})$	$G_m^{\mathbb{H}}(\mathbb{H}^{2m})$	$U(2m)$
$Sp(2m)/U(2m)$	$Sp(m)$	$U(2m)/O(2m)$
$SO(4m)/U(2m)$	$U(2m)/Sp(m)$	$SO(2m)$
$E_6/T \cdot Spin(10)$	$F_4/Spin(9)$	$G_2^{\mathbb{H}}(\mathbb{H}^4)/\mathbb{Z}_2$
$E_7/T \cdot E_6$	$T \cdot (E_6/F_4)$	$(SU(8)/Sp(4))/\mathbb{Z}_2$

The following results which was obtained by Iriyeh, Sakai and Tasaki [4] is an application of Theorems 4.1 and 4.7.

Theorem 4.8 ([4]). *Let (M, J_0, ω) be a Hermitian symmetric space of compact type which is monotone as a symplectic manifold. Let L_0, L_1 be real forms of M such that L_0 intersects L_1 transversally. Assume that the minimal Maslov numbers of L_0 and L_1 are greater than or equal to 3. Then we have*

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2[p].$$

That is, the intersection $L_0 \cap L_1$ itself becomes a basis of the Floer homology $HF(L_0, L_1 : \mathbb{Z}_2)$.

Theorem 4.9 ([4]). *Let M be an irreducible Hermitian symmetric space of compact type and L_0, L_1 be real forms of M which intersect transversally. Then the following results hold.*

(1) *If $M = G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m}) (m \geq 2)$, L_0 is congruent to $G_m^{\mathbb{H}}(\mathbb{H}^{2m})$ and L_1 is congruent to $U(2m)$, then we have*

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{2^m},$$

where $2^m < \binom{2m}{m} = \#_2 L_0 < 2^{2m} = \#_2 L_1$. Here $\#_2 L$ denotes the 2-number of L.

(2) *Otherwise, we have*

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{\min\{\#_2 L_0, \#_2 L_1\}}.$$

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