Antipodal sets of compact Riemannian symmetric spaces and their applications

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Joint with Hiroyuki Tasaki

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$\Longrightarrow \{p, -p\}$: an antipodal set

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Example 2. For $x \in \mathbb{R}P^n$, s_x is induced by $1_x - 1_{x^{\perp}}$ on $\mathbb{R}^{n+1}$
$y \subset x^{\perp}$: 1-dim subspace $\Longrightarrow \{x, y\}$: an antipodal set
More generally,
$e_1, e_2, \ldots, e_{n+1}$: o.n.b. of $\mathbb{R}^{n+1}$
$\Longrightarrow \{\langle e_1 \rangle_{\mathbb{R}}, \ldots, \langle e_{n+1} \rangle_{\mathbb{R}}\}$: a (maximal) antipodal set

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Chen-Nagano gave $\#_2 M$ for compact irreducible Riemannian symmetric spaces M with some exceptions.

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$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$
$G_k^{\mathbb{K}}(\mathbb{K}^n) = \{V \subset \mathbb{K}^n \mid V:$ $\mathbb{K}$-subspace, $\dim_{\mathbb{K}} V = k\}$

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$\{\langle e_{i_1}, \ldots, e_{i_r} \rangle_{\mathbb{K}} \in G_k^{\mathbb{K}}(\mathbb{K}^n) \mid 1 \leq i_1 < \cdots < i_r \leq n\}$

where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{K}^n$

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  A compact Riemannian symmetric space is called a **symmetric R-space** if it is an orbit under the linear isotropy action of a semisimple Riemannian symmetric space.

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- Berndt, Console and Fino, 2001

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Remarks.

- A real form is connected.
- A real form L is totally geodesic Lagrangian submanifold of M.
- Every real form is a symmetric R-space, and vice versa (Takeuchi).

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- to investigate fundamental properties of antipodal sets of a Hermitian symmetric space of compact type and those of a real form,
- to investigate the intersection of two real forms in a Hermitian symmetric space of compact type and to clarify the relation to antipodal sets.

Fundamental properties of antipodal sets

M : a Hermitian symmetric space of compact type

$M = \mathrm{Ad}(G)J \subset \mathfrak{g} = \mathrm{Lie}(G)$,

where G : a compact semisimple Lie group,

$J(\neq 0) \in \mathfrak{g},\ (\mathrm{ad}J)^3 = -\mathrm{ad}J$

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$S_1, S_2 \subset M$

S_1 and S_2 are **congruent** $\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}g \in I_0(M),$ s.t. $g(S_1) = S_2$

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Theorem 1 (Sánchez(1997), T.-Tasaki)

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(A) Any antipodal set is included in a great antipodal set.

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$^{\forall}S$ *: a great antipodal set of* M

$^{\exists}\mathfrak{t}$ *: a maximal abelian subalgebra of* $\mathfrak{g}$ *s.t.* $S = M \cap \mathfrak{t}$

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$^\exists \mathfrak{t}$ *: a maximal abelian subalgebra of $\mathfrak{g}$ s.t. $S = M \cap \mathfrak{t}$*
In particular, a great antipodal set is an orbit of the Weyl group of $\mathfrak{g}$.

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$L = F(\tau, M)$ *: a real form,* *assume $J \in L$*
$I_\tau : G \to G, \quad I_\tau(g) = \tau g \tau^{-1} \ (g \in G)$
$\mathfrak{g} = \mathfrak{l} + \mathfrak{p}$ *: the decomposition w.r.t. dI_τ*

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${}^\exists \mathfrak{a}$: a maximal abelian subspace of $\mathfrak{p}$ s.t. $S = M \cap \mathfrak{a}$
In particular, a great antipodal set is an orbit of the Weyl group of the symmetric pair determined by I_τ.

Fundamental properties of antipodal sets

Corollary 3 (T.-Tasaki)

M : a symmetric R-space

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Corollary 3 (T.-Tasaki)

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(A) Any antipodal set is included in a great antipodal set.
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Remark. $\mathrm{Ad}(SU(4)) = SU(4)/\mathbb{Z}_4$ does not satisfy (A), that is, there exists a maximal antipodal set which is not great.

Polars

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M_j^+ is called **a polar** of M w.r.t. p.

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$F(s_p, \mathbb{C}P^n) = \{p\} \cup \{V \subset \langle e_2, \ldots, e_{n+1} \rangle_{\mathbb{C}} \mid \dim V = 1\} (\cong \mathbb{C}P^{n-1})$

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Theorem 5 (Takeuchi,1989)

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Lemma 6

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Lemma 6

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Lemma 6

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$\Longrightarrow M_j^+$: a Hermitian symmetric space of compact type if $\dim M_j^+ > 0$

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(2) L_1, L_2 *: real forms of M,* $o \in L_1 \cap L_2$

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$L \cap g(L) = \{\langle u_1 \rangle_{\mathbb{C}}, \ldots, \langle u_{n+1} \rangle_{\mathbb{C}}\}$ (Howard, 1993)

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In particular, $L \cap g(L)$ is a great antipodal set of L.

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Theorem 8 (T.-Tasaki)

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Corollary 12 (T.-Tasaki)

Any real form of a Hermitian symmetric space of compact type is a globally tight Lagrangian submanifold.

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Remark. The classification of real forms is obtained by D. P. S. Leung (1979) and M. Takeuchi (1984).

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$$L \cong \begin{cases} G_k^{\mathbb{R}}(\mathbb{R}^n) \\ G_l^{\mathbb{H}}(\mathbb{H}^m) \text{ if } k = 2l,\ n = 2m \\ U(k) \text{ if } n = 2k \end{cases}$$

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M : an irreducible Hermitian symmetric space of compact type
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In particular, $L_1 \cap L_2$ is not a great antipodal set of L_1 (and not of L_2).

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(2) Otherwise, $\#(L_1 \cap L_2) = \#_2 L_1$
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$$\#(L_1 \cap L_2) = 2 < 4 = \#_2 L_1 = \#_2 L_2$$

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$$\Longrightarrow \; HF(L_1, L_2 : \mathbb{Z}_2) = \bigoplus_{p \in L_1 \cap L_2} \mathbb{Z}_2 p$$

i.e., the intersection $L_1 \cap L_2$ itself becomes a basis of the Floer homology $HF(L_1, L_2 : \mathbb{Z}_2)$.