Antipodal sets of compact Riemannian symmetric spaces and their applications

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#### Makiko Sumi Tanaka

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Joint with Hiroyuki Tasaki

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- *s*<sub>*x*</sub> : the geodesic symmetry at  $x \in M$

**Introduction** 

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- *M* : a Riemannian symmetric space
- *s<sup>x</sup>* : the geodesic symmetry at *x ∈ M*
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 $\mathsf{Example 1.}$   $\forall p \in S^n (\subset \mathbb{R}^{n+1}), \ s_p = 1_{\langle p \rangle_{\mathbb{R}}} - 1_{p^{\perp}}$ =*⇒ {p, −p}* : an antipodal set

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Chen-Nagano gave #2*M* for compact irreducible Riemannian symmetric spaces *M* with some exceptions.

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Takeuchi, 1989  $M:$  a symmetric *R*-space  $\implies$   $\#_2M = \dim H_*(M, \mathbb{Z}_2)$ 

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A compact Riemannian symmetric space is called a **symmetric** *R***-space** if it is an orbit under the linear isotropy action of a semisimple Riemannian symmetric space.

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- **Sánchez, 1993, 1997** generalization to *k*-symmetric spaces and flag manifolds
- **•** Berndt, Console and Fino, 2001

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- A real form is connected.
- A real form *L* is totally geodesic Lagrangian submanifold of *M*.
- Every real form is a symmetric *R*-space, and vice versa (Takeuchi).

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**Introduction** 

 $\bullet$  to investigate the intersection of two real forms in a Hermitian symmetric space of compact type and to clarify the relation to antipodal sets.

damental properties of antipodal sets

Fundamental properties of antipodal sets

*M* : a Hermitian symmetric space of compact type

 $M = \text{Ad}(G)J \subset \mathfrak{g} = \text{Lie}(G),$ 

where *G* : a compact semisimple Lie group,

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 $S_1, S_2 \subset M$  $S_1$  and  $S_2$  are congruent  $\iff \exists g \in I_0(M)$ , s.t.  $g(S_1) = S_2$ 

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. Theorem 2 (T.-Tasaki) the symmetric pair determined by  $I_{\tau}$ . *M : a Hermitian symmetric space of compact type, M* =  $Ad(G)J ⊂ g$ *τ : an involutive anti-holomorphic isometry of M*  $L = F(\tau, M)$  *: a real form, assume*  $J \in L$  $I_{\tau}: G \to G, \quad I_{\tau}(g) = \tau g \tau^{-1} (g \in G)$  $g = I + p$  *: the decomposition w.r.t. dl<sub><i>τ*</sub> *Then,*  $L = M \cap \mathfrak{p}$ . *Moreover, (A) and (B) in Theorem 1 hold. <sup>∀</sup>S : a great antipodal set of L*  $\exists$ a *: a maximal abelian subspace of*  $\mathfrak{p}$  *s.t.*  $S = M \cap \mathfrak{a}$ *In particular, a great antipodal set is an orbit of the Weyl group of the symmetric pair determined by*  $I_τ$ *.* 

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Fundamental properties of antipodal sets

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Remark.  $Ad(SU(4)) = SU(4)/\mathbb{Z}_4$  does not satisfy (A), that is, there exists a maximal antipodal set which is not great.

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**Polars** 

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*M* : a compact Riemannian symmetric space

**Polars** 

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Remark. *S* : an antipodal set,  $p \in S \implies S \subset F(s_p, M)$ 

Polars

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- *M : a compact Riemannian symmetric space*  $\implies$   $\#_2 M \ge \chi(M)$
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.. . =*⇒ M*<sup>+</sup> *∩ L is a real form of M*<sup>+</sup> .. . Makiko Sumi Tanaka (The 15th International Workshop on Differential Geometry) Antipodal sets of compact Riemannian symmetric spaces and their applications November 4, 2011 15 / 24*M : a Hermitian symmetric space of compact type L* : a real form of  $M$ ,  $o \in L$ *M*<sup>+</sup> *: a polar of M w.r.t. o, M*<sup>+</sup> ∩ *L*  $\neq$  *Ø* 

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Polars

Intersections of two real forms

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Remark. The classification of real forms is obtained by D. P. S. Leung (1979) and M. Takeuchi (1984).

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Example.  $M = G_k^{\mathbb{C}}(\mathbb{C}^n)$ 

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Example.  $M = G_k^{\mathbb{C}}(\mathbb{C}^n)$ 

$$
L \cong \left\{\begin{array}{l} G_k^{\mathbb{R}}(\mathbb{R}^n) \\ G_l^{\mathbb{H}}(\mathbb{H}^m) \text{ if } k = 2l, n = 2m \\ U(k) \text{ if } n = 2k \end{array}\right.
$$

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Intersections of two real forms

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Intersections of two real forms

Example (non-irreducible case).

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Intersections of two real forms

Example (non-irreducible case).

 $M = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$  $\tau_1,\tau_2:{\mathbb C} P^1\to{\mathbb C} P^1$  : involutive anti-holomorphic isometries s.t. real forms determined by  $\tau_1, \tau_2$  intersect transversally  $L_1 = \{(x, y, \tau_1(x), \tau_1(y)) \mid x, y \in \mathbb{C}P^1\}$  $L_2 = \{(x, \tau_2(x), y, \tau_2(y)) \mid x, y \in \mathbb{C}P^1\}$  $\implies$   $L_1, L_2$ : real forms of *M*,  $L_1 \pitchfork L_2$  $#(L_1 ∩ L_2) = 2 < 4 = #_2L_1 = #_2L_2$ 

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Application.

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 $\implies$  *HF***(***L***<sub>1</sub>***, L***<sub>2</sub> : Z<sub>2</sub>) =**  $\bigoplus_{p \in L_1 \cap L_2}$  **Z<sub>2</sub>p** 

*homology HF* $(L_1, L_2 : \mathbb{Z}_2)$ . .. . *i.e., the intersection L*<sup>1</sup> *∩ L*<sup>2</sup> *itself becomes a basis of the Floer*