

# The intersection of two real forms in a complex flag manifold

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First we explain two concepts in the title, a complex flag manifold and a real form. We call an orbit of the adjoint action of a compact semisimple Lie group a *complex flag manifold*. This is almost equivalent to a simply connected compact homogeneous Kähler manifold. We take an involutive anti-holomorphic isometry of a Kähler manifold. Each connected component of the fixed point set  $\text{Fix}(\tau)$  of  $\tau$  is called a *real form*. We can see that a real form is a totally geodesic Lagrangian submanifold.

We assume that  $M$  is a complex flag manifold and that  $L_0$  and  $L_1$  are two real forms of  $M$ . If the intersection  $L_0 \cap L_1$  is discrete, then we expect that  $L_0 \cap L_1$  has a *good shape*, which may lead a calculation of the Floer homology  $HF(L_0, L_1)$ . In this note a subset of good shape is an antipodal subset.

**Definition** (Chen-Nagano[1]) Let  $M$  be a Riemannian symmetric space and denote by  $s_x$  the geodesic symmetry at  $x$  for  $x \in M$ . A subset  $S \subset M$  is called *antipodal*, if  $s_x(y) = y$  for any  $x, y \in S$ . We define the *2-number*  $\#_2 M$  of  $M$  by

$$\#_2 M = \max\{\#S \mid S : \text{antipodal in } M\}.$$

We say that an antipodal set  $S$  of  $M$  is *great*, if  $\#S = \#_2 M$ .

We show some examples of compact Riemannian symmetric spaces and their great antipodal sets. We assume that  $\mathbb{K}$  is one of the real numbers  $\mathbb{R}$ , the complex numbers  $\mathbb{C}$  and the quaternionic numbers  $\mathbb{H}$ . We denote by  $G_k(\mathbb{K}^n)$  the Grassmann manifold consisting of  $\mathbb{K}$ -subspaces of  $\mathbb{K}$ -dimension  $k$  in  $\mathbb{K}^n$ . We take a  $\mathbb{K}$ -orthonormal basis  $\{v_i\}$  of  $\mathbb{K}^n$ . We can see that

$$\{\langle v_{i_1}, \dots, v_{i_k} \rangle_{\mathbb{K}} \mid 1 \leq i_1 < \dots < i_k \leq n\} = \{V \in G_k(\mathbb{K}^n) \mid V : \text{spanned by } v_j\}$$

is a great antipodal set of  $G_k(\mathbb{K}^n)$ . Thus we obtain the 2-number of the Grassmann manifold:

$$\#_2 G_k(\mathbb{K}^n) = \binom{n}{k},$$

which does not depend on the choice of  $\mathbb{K}$ .

We have the following theorem on the intersection of two real forms in a Hermitian symmetric space of compact type.

**Theorem 1** (T.[12], Tanaka-T.[9], [10], [11]) Let  $M$  be a Hermitian symmetric space of compact type and  $L_0, L_1$  be two real forms in  $M$ . If  $L_0 \cap L_1$  is discrete, then  $L_0 \cap L_1$  is antipodal in  $L_0$  and  $L_1$ .

Tanaka and I first proved this theorem by some results on maximal tori of compact symmetric spaces. Recently we obtained another proof of Theorem 1 and an explicit necessary and sufficient condition for the intersection of two real forms to be discrete in [4] and [5]. Using Theorem 1 Iriyeh, Sakai and I proved the following theorem.

**Theorem 2** (Iriyeh-Sakai-T.[6]) Let  $M$  be a Hermitian symmetric space of compact type and  $L_0, L_1$  be two real forms in  $M$ . If  $L_0 \cap L_1$  is discrete, the Floer homology satisfies

$$HF(L_0, L_1) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2 p.$$

This result has some applications:

- (1) generalized Arnold-Givental inequality
- (2) Hamilton volume minimizing

(1) means estimates of the intersection numbers of two Hamiltonian deformations of real forms in a Hermitian symmetric space of compact type. We proved Hamiltonian volume minimizing property of certain real forms by the use of (1) and integral formulas of integral geometry for (2).

After the paper [6] we have tried to generalize the results mentioned above to the case of complex flag manifolds and obtained some results of some special cases in [8] and [7].

We assume that  $M$  is a complex flag manifold. A complex flag manifold is an orbit of the adjoint action of a compact semisimple Lie group. Using the root system of its Lie algebra, we can construct a  $k$ -symmetric structure on  $M$ . The  $k$ -symmetric structure is a generalization of the usual symmetric

structure, where  $k$  is a natural number greater than 1. This  $k$ -symmetric structure on  $M$  leads a definition of an antipodal set in  $M$  and a great antipodal set.

We show some examples of complex flag manifolds and their great antipodal sets. For  $n_1 + \cdots + n_r < n$  we denote by  $F_{n_1, \dots, n_r}(\mathbb{K}^n)$  the flag manifold consisting of ascending sequences of  $\mathbb{K}$ -subspaces of certain  $\mathbb{K}$ -dimensions in  $\mathbb{K}^n$ :

$$F_{n_1, \dots, n_r}(\mathbb{K}^n) = \left\{ (V_1, \dots, V_r) \left| \begin{array}{l} V_i : \mathbb{K}\text{-subspace in } \mathbb{K}^n \\ \dim V_i = n_1 + \cdots + n_i \\ V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{K}^n \end{array} \right. \right\}.$$

Let  $\mathbb{K} = \mathbb{C}$ ,  $G = SU(n)$  and  $\mathfrak{g}$  be its Lie algebra. There exists an element  $Z \in \mathfrak{g}$  which satisfies  $F_{n_1, \dots, n_r}(\mathbb{C}^n) \cong \text{Ad}(G)Z \subset \mathfrak{g}$ . In this case we can see that  $\{(V_1, \dots, V_r) \in F_{n_1, \dots, n_r}(\mathbb{C}^n) \mid V_i : \text{spanned by } v_j\}$  is a great antipodal set of  $F_{n_1, \dots, n_r}(\mathbb{C}^n)$ .

From now on I explain about a joint work with Ikawa, Iriyeh, Okuda and Sakai. Let  $G$  be a compact semisimple Lie group and  $\mathfrak{g}$  be its Lie algebra. We take a nonzero element  $Z \in \mathfrak{g}$ .  $M = \text{Ad}(G)Z \subset \mathfrak{g}$  is a complex flag manifold. If  $\mathfrak{t}$  is a maximal abelian subalgebra of  $\mathfrak{g}$ , then  $M \cap \mathfrak{t}$  is a great antipodal set of  $M$ .

In order to construct a real form we take a symmetric pair  $(G, K)$  such that the canonical decomposition of  $\mathfrak{g}$ :  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  satisfies  $Z \in \mathfrak{p}$ . In this case  $L = \text{Ad}(K)Z \subset M$  is a real form of  $M$ . We show examples of real forms. The real flag manifold  $F_{n_1, \dots, n_r}(\mathbb{R}^n)$  is a real form of  $F_{n_1, \dots, n_r}(\mathbb{C}^n)$ , which is obtained from the symmetric pair  $(SU(n), SO(n))$ . The quaternionic flag manifold  $F_{n_1, \dots, n_r}(\mathbb{H}^n)$  is a real form of  $F_{2n_1, \dots, 2n_r}(\mathbb{C}^{2n})$ , which is obtained from the symmetric pair  $(SU(2n), Sp(n))$ .

We take a maximal abelian subspace  $\mathfrak{a}$  in  $\mathfrak{p}$  and put  $A = \exp \mathfrak{a}$ . The conjugacy of maximal tori of compact symmetric space implies

$$G/K = \bigcup_{k \in K} kA \cdot o$$

and  $G = KAK$ . Using this decomposition of  $G$  we consider the intersection  $L \cap \text{Ad}(g)L$  for  $g \in G$ . There exist  $k_i \in K$  ( $i = 0, 1$ ) and  $a \in A$  such that  $g = k_0 a k_1$ . Since  $\text{Ad}(k_1)L = L$ , we have

$$L \cap \text{Ad}(g)L = L \cap \text{Ad}(k_0 a k_1)L = \text{Ad}(k_0)(L \cap \text{Ad}(a)L).$$

In order to investigate the intersection of  $L$  and  $\text{Ad}(g)L$  for  $g \in G$  it is sufficient to investigate the intersection of  $L$  and  $\text{Ad}(a)L$  for  $a \in A$ . The adjoint action of  $a \in A$  on the Lie algebra  $\mathfrak{g}$  is well described by the restricted

root system of the symmetric pair  $(G, K)$ , because  $\mathfrak{a}$  is a maximal abelian subspace. Using this we can obtain the following theorem.

**Theorem 3** A necessary and sufficient condition for  $L \cap \text{Ad}(a)L$  to be discrete is explicitly described by the restricted root system of  $(G, K)$ . In the case  $L \cap \text{Ad}(a)L$  is an antipodal set of  $M$ , which is an orbit of the Weyl group  $W(G, K)$  of  $(G, K)$ .

Next we consider two real forms. Let  $(G, K_i)$  be symmetric pairs for  $i = 0, 1$  satisfying the following conditions. We take the canonical decompositions of  $\mathfrak{g} : \mathfrak{g} = \mathfrak{k}_i + \mathfrak{p}_i$  such that  $Z \in \mathfrak{p}_0 \cap \mathfrak{p}_1$ . In this case  $L_i = \text{Ad}(K_i)Z \subset M$  are two real forms of  $M$ . We take a maximal abelian subspace  $\mathfrak{a}$  in  $\mathfrak{p}_0 \cap \mathfrak{p}_1$  and put  $A = \exp \mathfrak{a}$ . A result of Heintze-Palais-Terng-Thorbergsson [2] implies

$$G/K_1 = \bigcup_{k \in K_0} kA \cdot o$$

and  $G = K_0AK_1$ . Using this decomposition of  $G$  we consider the intersection  $L_0 \cap \text{Ad}(g)L_1$  for  $g \in G$ . There exist  $k_i \in K$  ( $i = 0, 1$ ) and  $a \in A$  such that  $g = k_0ak_1$ . Since  $\text{Ad}(k_1)L_1 = L_1$ , we have

$$L_0 \cap \text{Ad}(g)L_1 = L_0 \cap \text{Ad}(k_0ak_1)L_1 = \text{Ad}(k_0)(L_0 \cap \text{Ad}(a)L_1).$$

In order to investigate the intersection of  $L_0$  and  $\text{Ad}(g)L_1$  for  $g \in G$  it is sufficient to investigate the intersection of  $L_0$  and  $\text{Ad}(a)L_1$  for  $a \in A$ . The adjoint action of  $a \in A$  on the Lie algebra  $\mathfrak{g}$  is well described by the symmetric triad introduced by Ikawa [3]. Using this we can obtain the following theorem.

**Theorem 4** We assume that two involutions of  $(G, K_0)$  and  $(G, K_1)$  are commutative. A necessary and sufficient condition for  $L_0 \cap \text{Ad}(a)L_1$  to be discrete is explicitly described by the symmetric triad obtained from  $(G, K_0, K_1)$ . In the case  $L_0 \cap \text{Ad}(a)L_1$  is an antipodal set of  $M$ , which is an orbit of a certain Weyl group.

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