Volume estimates of Lagrangian submanifolds in complex hyperquadrics

joint work with Iriyeh, Sakai and Tanaka

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$\mathbb{R}^2 \supset S^1$: **a round circle**

$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ **is area preserving**

$\Rightarrow \text{Length}(S^1) \leq \text{Length}(\phi(S^1))$

$S^2 \supset S^1$: **a great circle**

$\phi : S^2 \longrightarrow S^2$ **is area preserving**

$\Rightarrow \text{Length}(S^1) \leq \text{Length}(\phi(S^1))$

consequences of isoperimetric inequalities

Generalizations

M : a Kähler manifold

Ham(M) : all Hamiltonian diffeo.

L : a compact Lagrangian submanifold

L : Hamiltonian volume minimizing

$\Leftrightarrow \qquad \mathrm{vol}(L) \leq \mathrm{vol}(\phi L)$

for any $\phi \in \mathrm{Ham}(M)$ (Oh)

Theorem 1 (Oh)

$\mathbb{R}P^n$ in $\mathbb{C}P^n$ are Hamiltonian volume minimizing.

M : Kähler manifold
σ : an involutive antiholomorphic isometry
$L = \mathrm{Fix}(\sigma)$: real form, if not empty
Real form : totally geodesic Lagrangian submfd

$\mathbb{R}P^n$ in $\mathbb{C}P^n$ is a typical example of real forms.

Real forms in Hermitian symmetric spaces are classified. (Leung, Takeuchi)

Theorem 2 (Iriyeh-Sakai-T.)
Real forms S^n in $Q_n(\mathbb{C})$ are Hamiltonian volume minimizing.

$Q_n(\mathbb{C}) = \{[z] \in \mathbb{C}P^{n+1} \mid z_1^2 + \cdots + z_{n+2}^2 = 0\}$
: the complex hyperquadric
$G_2(\mathbb{R}^{n+2})$ **: the oriented Grassmann mfd**
$G_2(\mathbb{R}^{n+2}) \subset \bigwedge^2 \mathbb{R}^{n+2}$
$G_2(\mathbb{R}^{n+2}) \ni x \wedge y \mapsto [x + \sqrt{-1}y] \in Q_n(\mathbb{C})$
: diffeomorphism

$$\mathbb{R}^{n+2} = E^{k+1} \oplus E^{n-k+1} \text{ : orthogonal } (0 \leq k \leq n)$$

$$\begin{aligned} S^{k,n-k} &= S^k(E^{k+1}) \wedge S^{n-k}(E^{n-k+1}) \\ &\cong S^k \times S^{n-k}/\mathbb{Z}_2 \\ S^{0,n} &\cong S^0 \times S^n/\mathbb{Z}_2 \cong S^n \\ S^{k,n-k} &\cong S^{n-k,k} \end{aligned}$$

$X, Y \subset Q_n(\mathbb{C})$ are congrunet
$\Leftrightarrow \exists g \in I_0(Q_n(\mathbb{C}))$: $Y = gX$
All real forms in $Q_n(\mathbb{C})$ are congruent to one of

$$\{S^{k,n-k} \mid 0 \leq k \leq [n/2]\}.$$

Some of $S^{k,n-k}$ are Hamiltonian unstable.

M : a compact Riemannian symmetric space
s_x : the symmetry at $x \in M$

(1) s_x : isometry

(2) $s_x^2 = 1_M$

(3) x is an isolated fixed point of s_x

$S \subset M$: antipodal set (Chen-Nagano)
$\Leftrightarrow s_x y = y$ for any $x, y \in S$

$\{x, -x\}$: antipodal in S^n for $x \in S^n$

$\#_2 M = \max\{\#S \mid S : \text{antipodal in } M\}$

S : great antipodal set of M (Chen-Nagano)

$\Leftrightarrow \#S = \#_2 M$

$\{x, -x\}$: great antipodal in S^n, $\#_2 S^n = 2$

$\{\pm e_1 \wedge e_2, \ldots, \pm e_{2[n/2]+1} \wedge e_{2[n/2]+2}\}$

: great antipodal in $Q_n(\mathbb{C})$

$\#_2(Q_n(\mathbb{C})) = 2[n/2] + 2 = \dim H_*(Q_n(\mathbb{C}) : \mathbb{Z}_2)$

Theorem 3 (T.) $0 \leq k_0 \leq k_1 \leq [n/2]$

L_i :congruent to $S^{k_i, n-k_i}$ in $Q_n(\mathbb{C})$, $L_0 \pitchfork L_1$

$\Rightarrow L_0 \cap L_1$: congruent to

$$\{\pm e_1 \wedge e_2, \ldots, \pm e_{2k_0+1} \wedge e_{2k_0+2}\}$$

$L_0 \cap L_1$: great antipodal in L_0

Theorem 4 (Tanaka-T.)

M : **Hermitian symmetric space of compact type**

L_0, L_1 : **real forms of** M, $L_0 \pitchfork L_1$

$\Rightarrow L_0 \cap L_1$: **antipodal in** L_0 **and** L_1

Theorem 5 (Iriyeh-Sakai-T.)

M : **irreducible Herm. sym. sp. of cpt type**

L_0, L_1 : **real forms of** M, $L_0 \pitchfork L_1$

$\Rightarrow$ **The Floer homology of** L_0 **and** L_1 :

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2 p$$

The chain of the Floer homology

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2 p$$

The boundary operator

$\partial : CF(L_0, L_1) \to CF(L_0, L_1)$

$$\partial(p) = \sum_{q \in L_0 \cap L_1} n(p, q) q \quad (p \in L_0 \cap L_1)$$

$HS(p, q) =$
$\{$**isolated hol. strip from** p **to** q **with boundary** L_0 **and** $L_1\}$

$n(p, q) \equiv \# HS(p, q) \pmod 2$

∂ is well-defined

$\partial^2 = 0$

$HF(L_0, L_1) = \ker \partial / \mathrm{im} \partial$

Floer homology of (L_0, L_1) with coefficient $\mathbb{Z}_2$

$\phi \in \mathrm{Ham}(M)$ and $L_0 \pitchfork L_1$

$$\Rightarrow \quad HF(L_0, L_1 : \mathbb{Z}_2) \cong HF(L_0, \phi L_1 : \mathbb{Z}_2)$$

By the definition of $HF(L_0, L_1 : \mathbb{Z}_2)$

$$\begin{aligned} \dim HF(L_0, L_1 : \mathbb{Z}_2) &= \dim HF(L_0, \phi L_1 : \mathbb{Z}_2) \\ &\leq \#(L_0 \cap \phi L_1) \end{aligned}$$

For $p \in L_0 \cap L_1$, the symmetry s_p induces

$$s_p : HS(p,q) \to HS(p,q) \; ; \; u \mapsto s_p \circ u$$

This action has no fixed point and $s_p^2 = 1$.
$\# HS(p,q)$ is even, so $n(p,q) = 0$ and $\partial = 0$.

$$HF(L_0, L_1 : \mathbb{Z}_2) = \ker \partial = \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2 p$$

Moreover for $\phi \in \mathrm{Ham}(M)$

$$\#(L_0 \cap L_1) = \dim HF(L_0, L_1 : \mathbb{Z}_2)$$
$$= \dim HF(L_0, \phi L_1) \leq \#(L_0 \cap \phi L_1)$$

generalization of Arnold-Givental inequality

Theorem 6 (Lê)

N : n-dim submanifold in $Q_n(\mathbb{C})$, $G = SO(n+2)$

$$\frac{2\mathrm{vol}(G)}{\mathrm{vol}(S^n)}\mathrm{vol}(N) \geq \int_G \#(N \cap gS^n) dg$$

For $\phi \in \mathrm{Ham}(Q_n(\mathbb{C}))$ and $N = S^{k,n-k}$

$$\frac{2\mathrm{vol}(G)}{\mathrm{vol}(S^n)}\mathrm{vol}(\phi S^{k,n-k}) \geq \int_G \#(\phi S^{k,n-k} \cap gS^n) dg$$

$$\geq \int_G \#(S^{k,n-k} \cap gS^n) dg = 2\mathrm{vol}(G).$$

Hence $\mathrm{vol}(\phi S^{k,n-k}) \geq \mathrm{vol}(S^n)$.

Theorem 7 (Iriyeh-Sakai-T.)
$\phi \in \mathrm{Ham}(Q_n(\mathbb{C}))$

$$\Rightarrow \quad \mathrm{vol}(S^n) \leq \mathrm{vol}(\phi S^{k,n-k}).$$

In particular S^n is Ham. volume minimizing.