Maximal antipodal subgroups and covering homomorphisms with odd degree

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Symmetric spaces and antipodal sets

M : a symmetric space

$\Leftrightarrow$ (1) M : a connected Riemannian manifold

(2) $\forall x \in M \ \exists s_x$: a symmetry at x

(a) s_x : an isometry of M, $s_x^2 = 1_M$

(b) x : an isolated fixed point of s_x

A : an antipodal set in M

$\Leftrightarrow$ (1) A : a subset of M

(2) $\forall x, y \in A \ s_x(y) = y$

Maximal antipodal sets (w. r. t. inclusions)

$\#_2 M = \max\{|A| \mid A : \text{antipodal in } M\}$

: 2-number of M

A : a great antipodal set if $|A| = \#_2 M$

$$f : X \to X \quad F(f, X) = \{x \in X \mid f(x) = x\}$$

Examples

$M = S^n \ni x \quad F(s_x, S^n) = \{\pm x\}$

$\{\pm x\}$: **a great antipodal set**

$\#_2 S^n = 2 = \dim H_*(S^n; \mathbb{Z}_2)$

$M = \mathbb{R}P^n \quad e_1, \ldots, e_{n+1}$: **o. n. basis of** $\mathbb{R}^{n+1}$

$\mathbb{R}P^n \ni x \quad F(s_x, \mathbb{R}P^n) = \{x\} \cup P(x^\perp)$

$\{\mathbb{R}e_1, \ldots, \mathbb{R}e_{n+1}\}$: **a great antipodal set**

$\#_2 \mathbb{R}P^n = n + 1 = \dim H_*(\mathbb{R}P^n, \mathbb{Z}_2)$

Polars

M : a compact symmetric space

$M \ni x$ Each connected compornent of $F(s_x, M)$

: a polar

Symmetric R-spaces

special compact symmetric spaces which have high symmetry

Examples of symmetric R-spaces

S^n, $G_k(\mathbb{K}^n)$ $(\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$, $U(n)$, $SO(n)$, $Sp(n)$

Theorem (M. Takeuchi 1989) M : a symmetric R-space

$$\#_2 M = \dim H_*(M; \mathrm{Z}_2).$$

Teorem (M. Takeuchi 1965) M **: a symmetric** R**-space,**

$M_0, M_1, \ldots, M_s$ **: all of the polars of** M

$$\dim H_*(M; \mathrm{Z}_2) = \sum_{i=0}^{s} \dim H_*(M_i; \mathrm{Z}_2).$$

Corollary M **: symmetric** R**-space,**

$M_0, M_1, \ldots, M_s$ **: all of the polars of** M

$$\#_2 M = \sum_{i=0}^{s} \#_2 M_i.$$

Group theory

G : a group, $X, Y \subset G$

$$XY = \{xy \mid x \in X,\ y \in Y\}$$

Lemma G : a group, e : the unit element of G

H, K : subgroups of G

The following conditions are equivalent.

(1) $\forall x \in HK\ \exists! h \in H, \exists! k \in K\ \ x = hk.$

(2) $hk = e\,(h \in H,\ k \in K) \Rightarrow h = k = e.$

(3) $H \cap K = \{e\}.$

Theorem (Lagrange) G : a finite group

H : a subgroup of $G \Rightarrow |H|$ devides $|G|$.

Corollary G : a finite group, $g \in G$
$\Rightarrow \min\{n \in \mathbb{N} \mid g^n = e\}$ devides $|G|$.

Theorem (Sylow) G : a finite group,
$|G| = p^n m$, p : prime, p and m : mutually prime
$\Rightarrow \exists H$: a subgroup of G, $|H| = p^n$
: p-Sylow subgroup

p-Sylow subgroups are conjugate.
K : a subgroup with $|K| = p^l$
$\Rightarrow \exists H$: a p-Sylow subgroup $K \subset H$

Proposition G : a group, $\forall g \in G\ g^2 = e$
$\Rightarrow G$: abelian
Moreover G : finite $\Rightarrow G \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$
$(\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z})$

Proof For $g \in G\ g = g^{-1}$. For $x, y \in G$

$$xy = (xy)^{-1} = y^{-1}x^{-1} = yx.$$

We construct an ascending sequence of subgroups $A_1, A_2, \ldots$. Take $a_1 \in G \setminus \{e\}$ and set $A_1 = \langle a_1 \rangle$. If $G \setminus A_1 = \emptyset$, then $G = A_1 \cong \mathbb{Z}_2$. If $G \setminus A_1 \neq \emptyset$, we can take $a_2 \in G \setminus A_1$ and set $A_2 = A_1 \langle a_2 \rangle$. If $G \setminus A_2 = \emptyset$, then $G = A_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. When G is finite, $\exists k$ such that $G = A_k \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

Compact Lie groups

Theorem G **: a connected compact Lie group with a bi-invariant Riemannian metric**
$\Rightarrow G$ **: a symmetric space with**

$$s_g(x) = gx^{-1}g \quad (g, x \in G).$$

Remark **Even if** G **is not connected, we can define the symmetry by**

$$s_g(x) = gx^{-1}g \quad (g, x \in G).$$

We always equip compact Lie groups with bi-invariant Riemannian metrics and regard them as Riemannian symmetric spaces.

Lemma G : a compact Lie group

A : a maximal antipodal set containing e

$\Rightarrow A$: a subgroup $\cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$

In this case A is a mximal antipodal subgroup.

We mainly treat maximal antipodal subgroups.

Example We define

$$\Delta_n = \left\{ \begin{bmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_n \end{bmatrix} \middle| \epsilon_1, \dots, \epsilon_n = \pm 1 \right\}.$$

Any max. antip. subgroup of $U(n)$ is conjugate to Δ_n. Δ_n is a great antipodal subgroup of $U(n)$.

Proof A : a maximal antipodal subgroup of $U(n)$
By simultaneous diagonalization we can take g in $U(n)$ such that $gAg^{-1} \subset \Delta_n$. The maximality of A implies $gAg^{-1} = \Delta_n$.

Lemma G, G' : compact Lie groups
$\pi : G \to G'$: a covering homo. with odd degree
$\forall A'$: an antipodal subgroup of G'
$\exists A$: an antipodal subgroup of G satisfying

(1) A is a 2-Sylow subgroup of $\pi^{-1}(A')$ with $|A| = |A'|$

(2) $\pi|_A : A \to A'$ is an isomorphism.

Proof We apply Sylow's theorem to $\pi^{-1}(A')$.

Theorem $\pi : G \to G'$: as above,

G_0, G'_0 : the identity components of G, G'

(1) A : an antip. subgroup of G

$\Rightarrow \pi(A)$: an antip. subgroup of G'

A : a max. antip. subgroup of G

$\Rightarrow \pi(A)$: a max. antip. subgroup of G'

max. antip. subgr. A_1, A_2 of G : conjugate

$\Rightarrow$ max. antip. subgroups $\pi(A_1), \pi(A_2)$ of G'

: conjugate

maxi. antip. subgroups A_1, A_2 of G :

G_0-conjugate

$\Rightarrow$ max. antip. subgroups $\pi(A_1), \pi(A_2)$ of G'

: G'_0-conjugate

(2) A' **: an antip. subgroup of** G'

$\Rightarrow \exists A$ **: an antip. subgroup** A **of** G **such that**
$\pi|_A : A \to A'$ **: an isomorphism**

A' **: a max. antip. subgroup of** G'

$\Rightarrow \exists A$ **: a max. antip. subgroup of** G **such that** $\pi|_A : A \to A'$ **: an isomorphism**

A'_1, A'_2 **: max. antip. subgroups of** G' **: conjugate**

$\Rightarrow$ **max. antip. subgroups** A_1, A_2 **of** G **: conjugate, where** $\pi|_{A_i} : A_i \overset{\cong}{\to} A'_i \ (i = 1, 2)$

Furthermore, $\ker\pi \subset G_0$**, the following holds:**
max. antip. subgroups A'_1, A'_2 **of** G' **:**
G'_0**-conjugate**

$\Rightarrow$ **max. antip. subgroups** A_1, A_2 **of** G **:**
G_0**-conjugate, where** $\pi|_{A_i} : A_i \overset{\cong}{\to} A'_i \ (i = 1, 2)$

In short, a covering homomorphism with odd degree does not change antipodal subgroups in compact Lie groups.