The intersection of two real forms in Hermitian symmetric spaces of compact type

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Contents

1 Introduction

$M = S^2$

L_1, L_2 : great circles intersecting transversally

$$\Longrightarrow \quad L_1 \cap L_2 = \{o, \bar{o}\}$$

(o and $\bar{o}$ are antipodal)

$S^2 \cong \mathbb{C}P^1$

$L_1, L_2 \cong \mathbb{R}P^1$: totally geodesic

R. Howard

$M = \mathbb{C}P^n$

$L = \mathbb{R}P^n$: totally geodesic

$$\Longrightarrow \quad \#(L \cap g \cdot L) = n + 1$$

($^\forall g \in I(M)$ s.t. $g \cdot L$ intersects L transversally)

$^\forall x, y \in L \cap g \cdot L$ are antipodal

$\mathbb{R}P^n$ is a "real form" of $\mathbb{C}P^n$.

"2-number" (B.-Y. Chen and T. Nagano)

$$\#_2(\mathbb{R}P^n) = \underline{n+1}$$

M. Takeuchi

N : symmetric R-space

$$\Longrightarrow \quad \#_2(N) = SB(N, \mathbb{Z}_2)$$

(=the sum of $\mathbb{Z}_2$-Betti numbers of N)

Remark Every real form in a Hermitian symmetric space of compact type is a symmetric R-space.

"globally tight" (Y.-G. Oh)

M : Hermitian symmetric space

L : Lagrangian submanifold

L : **globally tight** $\overset{\text{def}}{\Longleftrightarrow}$ $\#(L \cap g \cdot L) = SB(M, \mathbb{Z}_2)$ for ${}^\forall g \in I(M)$

s.t. $g \cdot L$ intersects L transversally

Remark : $\mathbb{R}P^n$ is globally tight.

Problem 1

Does the intersection of two real forms of a Hermitian symmetric space of compact type consist of antipodal points if they intersect transversally? Moreover, does the number of such points coincide with the 2-number of the real form?

Problem 2

Is every real form of any Hermitian symmetric space of compact type globally tight?

2 Preliminaries

Definition

M : connected Riemannian manifold

M : **Riemannian symmetric space**

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}x \in M$, ${}^{\exists}s_x$: involutive isometry

s.t. x is an isolated fixed point of s_x

s_x is called the **symmetry** at x

Definition

M : connected Hermitian manifold

M : **Hermitian symmetric space**

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}x \in M$, ${}^{\exists}s_x$: involutive holomorphic isometry

s.t. x is an isolated fixed point of s_x

If M is a Riemannian (resp. Hermitian) symmetric space, $M = G/K$ with the identity component of the isometry group (resp. holomorphic isometry group) G and the isotropy subgroup K at some point in M.

Definition

M : Hermitian manifold

L : **real form** of M

$\overset{\text{def}}{\Longleftrightarrow}$ $^\exists\sigma : M \to M$: anti-holomorphic involutive isometry

s.t. $L = \{x \in M \mid \sigma(x) = x\}$

Remark Real forms are connected. (M. Takeuchi)

M : Hermitian symmetric space of compact type

(i.e., $M = G/K$, G : compact, semisimple)

$\Longrightarrow$ Every real form of M is a totally geodesic Lagrangian submanifold.

M : irreducible Hermitian symmetric space of compact type

L : real form of M

L	M
$UI(n) = U(n)/SO(n)$	$CI(n) = Sp(n)/U(n)$
$Sp(n) = Sp(n) \times Sp(n)/\Delta$	$CI(2n)$
$G_r(\mathbb{R}^{n+r}) = SO(n+r)/S(O(r) \times O(n))$	$G_r(\mathbb{C}^{n+r}) = SU(n+r)/S(U(r) \times U(n))$
$G_r(\mathbb{H}^{n+r}) = Sp(n+r)/Sp(r) \times Sp(n)$	$G_{2r}(\mathbb{C}^{2n+2r})$
$U(n) = U(n) \times U(n)/\Delta$	$G_n(\mathbb{C}^{2n})$
$SO(n) = SO(n) \times SO(n)/\Delta$	$DIII(n) = SO(2n)/U(n)$
$UII(n) = U(2n)/Sp(n)$	$DIII(2n)$
$S^k \times S^l/\mathbb{Z}_2$	$G_2^o(\mathbb{R}^{k+l}) = SO(k+l)/SO(2) \times SO(k+l-2)$
$FII = F_4/Spin(9)$	$EIII = E_6/T \cdot Spin(10)$
$G_2(\mathbb{H}^4)/\mathbb{Z}_2$	$EIII$
$T \cdot EIV = T \cdot E_6/F_4$	$EVII = E_7/T \cdot E_6$
$AII(4)/\mathbb{Z}_2$	$EVII$

(Δ : the diagonal subgroup , $T \cong U(1)$)

M. Takeuchi, D.P.S. Leung

Definition (B.-Y. Chen - T. Nagano)

M : Riemannian symmetric space

$S \subset M$: subset

S : **antipodal set** $\overset{\text{def}}{\iff}$ ${}^{\forall}x, y \in S, \quad s_x y = y$

(s_x : the symmetry at x)

The 2-number of M

$$\#_2(M) := \sup\{\#S \mid S \subset M : \text{antipodal set}\}$$

If an antipodal set S satisfies $\#(S) = \#_2(M)$, S is called a **great antipodal set**.

Remark $\#_2(M) < \infty$

Remark $s_x y = y \iff {}^{\exists}c$: closed geodesic

s.t. x and y are antipodal on c

Hence we can define the "2-number" for any Riemannian manifold.

Examples

$\#_2(S^n) = 2$

$\#_2(T^n) = 2^n$

$\#_2(\mathbb{K}P^n) = n + 1 \quad (\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$

$\qquad e_1, \ldots, e_{n+1}$: o. n. b. of $\mathbb{K}^{n+1}$

$\qquad \Longrightarrow \ \{\mathbb{K}e_1, \ldots, \mathbb{K}e_{n+1}\}$: great antipodal set

$\#_2(G_r(\mathbb{K}^n)) = \begin{pmatrix} n \\ r \end{pmatrix}$, the binomial coefficient $\quad (\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$

$\qquad e_1, \ldots, e_n$: o. n. b. of $\mathbb{K}^n$

$\qquad \Longrightarrow \ \{\langle e_{i_1}, \ldots, e_{i_r}\rangle_{\mathbb{K}} \mid 1 \leq i_1 < \cdots < i_r \leq n\}$: great antipodal set

$\#_2(U(n)) = 2^n$

$$\left\{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \right\} : \text{great antipodal set}$$

Definition

M : compact Riemannian symmetric space, $o \in M$

$$F(s_o, M) = \{x \in M \mid s_o(x) = x\} = \bigcup_{j=0}^{k} M_j^+, \quad \text{where } M_0^+ = \{o\}$$

Each connected component M_j^+ is called **a polar** of M w.r.t. o.

Examples

(1) $M = S^n$, $o = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$

$$F(s_o, M) = \{o\} \cup \{-o\}$$

(2) $M = \mathbb{K}P^n \quad (\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$

$e_1, \ldots, e_{n+1}$: o.n.b. of $\mathbb{K}^{n+1}$, $\quad o = \mathbb{K}e_1$

$$\begin{aligned} F(s_o, M) &= \{\mathbb{K}e_1\} \cup \{\text{1-dim subspaces in } \langle e_2, \ldots, e_{n+1} \rangle_{\mathbb{K}}\} \\ &\cong \{o\} \cup \mathbb{K}P^{n-1} \end{aligned}$$

(3) $M = U(n)$, $o = I$: the identity matrix

$$\begin{aligned} F(s_o, M) &= \{X \in U(n) \mid X^2 = I\} \\ &= [I] \cup [I_1] \cup \cdots \cup [I_{n-1}] \cup [I_n] \\ &\cong \{I\} \cup G_1(\mathbb{C}^n) \cup \cdots \cup G_{n-1}(\mathbb{C}^n) \cup \{-I\} \end{aligned}$$

$[X]$: conjugacy class of X

$$I_k = \begin{pmatrix} -1 & & & & & 0 \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$$

(the cardinality of -1's is k and that of 1's is $n-k$)

Lemma 2.1

M : compact Riemannian symemtric space

S : antipodal set of M, $x \in S$

$\Longrightarrow\ S \subset F(s_x, M)$

Lemma 2.2

M : Hermitian symmetric space of compact type

$\Longrightarrow\ {}^\forall M_j^+$: Hermitian symmetric space of compact type

Lemma 2.3

M : Hermitian symmetric space of compact type

L : real form of M, $o \in L$

$\Longrightarrow\ L \cap M_j^+$: real form of M_j^+ if $L \cap M_j^+ \neq \emptyset$

(M_j^+ : polar of M w.r.t. o)

3 Main results

Theorem 1

M: Hermitian symmetric space of compact type

L_1, L_2: real forms of M, intersect transversally

$\Longrightarrow \quad L_1 \cap L_2$: antipodal set of L_1, L_2

Remark : We do **not** assume that L_1 and L_2 are congruent to each other.

L_1 and L_2 are **congruent** if they are transformed to each other by some holomorphic isometry of M.

Theorem 2

M: Hermitian symmetric space of compact type

L_1, L_2: real forms of M, **congruent**, intersect transversally

$\Longrightarrow$ $L_1 \cap L_2$: **great** antipodal set of L_1, L_2
i.e., $\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$

Corollary

Any real form of a Hermitian symmetric space of compact type is globally tight.

Theorem 3

M : irreducible Hermitian symmetric space of compact type

L_1, L_2 : real forms of M, intersect transversally

(1) $(M, L_1, L_2) = (G_{2m}(\mathbb{C}^{4m}), G_m(\mathbb{H}^{2m}), U(2m))$

$\implies \quad \#(L_1 \cap L_2) = 2^m < \#_2 L_1 < \#_2 L_2$

(2) $(M, L_1, L_2) \neq (G_{2m}(\mathbb{C}^{4m}), G_m(\mathbb{H}^{2m}), U(2m))$

$\implies \quad \#(L_1 \cap L_2) = \min\{\#_2 L_1, \#_2 L_2\}$

4 Outline of proofs

Theorem 1

M: Hermitian symmetric space of compact type
L_1, L_2: real forms of M, intersect transversally
$\Longrightarrow \quad L_1 \cap L_2$: antipodal set of L_1, L_2

Proof of Theorem 1

Lemma (H. Tasaki)

M : compact Kähler manifold with positive holomorphic sectional curvature
L_1, L_2 : totally geodesic compact Lagrangian submanifolds in M
$\Longrightarrow \quad L_1 \cap L_2 \neq \emptyset$

We can choose $o \in L_1 \cap L_2$.
We prove that o and p are antipodal for ${}^{\forall}p \in L_1 \cap L_2 - \{o\}$ by using the properties of maximal tori.

Lemma

M : compact Riemannian symmetric space, $\quad o \in M$

A, A_1 : maximal tori of M, $\quad o \in A \cap A_1$

S : fundamental cell of A, $\quad \overline{S} = \bigcup_i S_i$

$A_1 \cap A \cap \mathrm{Exp}\ S_i \neq \emptyset \quad \Longrightarrow \quad \mathrm{Exp}\ S_i \subset A_1 \cap A$

Theorem 2

M: Hermitian symmetric space of compact type

L_1, L_2: real forms of M, **congruent**, intersect transversally

$\Longrightarrow \quad L_1 \cap L_2$: **great** antipodal set of L_1, L_2

i.e., $\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$

Proof of Theorem 2

We can choose $o \in L_1 \cap L_2$.

L_1, L_2 : congruent $\Longrightarrow$ $\#_2 L_1 = \#_2 L_2$

Let

$$F(s_o, M) = \bigcup_{j=0}^{r} M_j^+,$$

then

$$F(s_o, L_i) = \bigcup_{j=0}^{r} (L_i \cap M_j^+) \qquad (i = 1, 2)$$

and $L_i \cap M_j^+$ is a real form of M_j^+ by Lemma 2.3.

By Theorem 1 we have

$$L_1 \cap L_2 = \bigcup_{j=0}^{r} \{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\}$$

and

$$\#(L_1 \cap L_2) = \sum_{j=0}^{r} \#\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\}.$$

Thus we obtain :

(∗)

$$\forall j, \quad \#\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\} = \#_2(L_1 \cap M_j^+) = \#_2(L_2 \cap M_j^+)$$

$$\Longrightarrow \quad \#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$$

$$F(s_o, M) = \bigcup_{j_1=0}^{r} M^+_{j_1}$$

$$o_{j_1} \in M^+_{j_1}, \qquad F(s_{o_{j_1}}, M^+_{j_1}) = \bigcup_{j_2} M^+_{j_1,j_2}$$

$$\vdots$$

$$\vdots$$

$$o_{j_1,\dots,j_k} \in M^+_{j_1,\dots,j_k}, \qquad F(s_{o_{j_1,\dots,j_k}}, M^+_{j_1,\dots,j_k}) = \bigcup_{j_{k+1}} M^+_{j_1,\dots,j_k,j_{k+1}}$$

$$\Longrightarrow \quad \text{dim } M > \text{dim } M^+_{j_1} > \text{dim } M^+_{j_1,j_2} > \cdots > \text{dim } M^+_{j_1,\dots,j_l} = 0$$
$$\text{(i.e., } M^+_{j_1,\dots,j_l} = \{\text{a point}\})$$

$$L_i \cap M^+_{j_1,\dots,j_l} \neq \emptyset \quad (i = 1, 2)$$
$$\Longrightarrow$$
$$\#\{(L_1 \cap M^+_{j_1,\dots,j_l}) \cap (L_2 \cap M^+_{j_1,\dots,j_l})\} = \#_2(L_i \cap M^+_{j_1,\dots,j_l}) = 1 \quad (i = 1, 2)$$

Now Theorem 2 is proved by induction due to $(*)$.

Theorem 3

M : irreducible Hermitian symmetric space of compact type

L_1, L_2 : real forms of M, intersect transversally

(1) $(M, L_1, L_2) = (G_{2m}(\mathbb{C}^{4m}), G_m(\mathbb{H}^{2m}), U(2m))$

$\Longrightarrow \quad \#(L_1 \cap L_2) = 2^m < \#_2 L_1 < \#_2 L_2$

(2) $(M, L_1, L_2) \neq (G_{2m}(\mathbb{C}^{4m}), G_m(\mathbb{H}^{2m}), U(2m))$

$\Longrightarrow \quad \#(L_1 \cap L_2) = \min\{\#_2 L_1, \#_2 L_2\}$

Proof of Theorem 3

We prove it case by case due to the classification of real forms of irreducible Hermitian symmetric spaces of compact type by D. P. S. Leung and M. Takeuchi.

We may consider only the cases where L_1 and L_2 are not congruent.

M	L_1	L_2
$Q_n(\mathbb{C})$	$S^{k,n-k}$	$S^{l,n-l}$
$G^{\mathbb{C}}_{2q}(\mathbb{C}^{2m+2q})$	$G^{\mathbb{H}}_{q}(\mathbb{H}^{m+q})$	$G^{\mathbb{R}}_{2q}(\mathbb{R}^{2m+2q})$
$G^{\mathbb{C}}_{n}(\mathbb{C}^{2n})$	$U(n)$	$G^{\mathbb{R}}_{n}(\mathbb{R}^{2n})$
$G^{\mathbb{C}}_{2m}(\mathbb{C}^{4m})$	$G^{\mathbb{H}}_{m}(\mathbb{H}^{2m})$	$U(2m)$
$Sp(2m)/U(2m)$	$Sp(m)$	$U(2m)/O(2m)$
$SO(4m)/U(2m)$	$U(2m)/Sp(m)$	$SO(2m)$
$E_6/T \cdot Spin(10)$	$F_4/Spin(9)$	$G^{\mathbb{H}}_{2}(\mathbb{H}^{4})/\mathbb{Z}_2$
$E_7/T \cdot E_6$	$T \cdot (E_6/F_4)$	$(SU(8)/Sp(4))/\mathbb{Z}_2$