The intersection of two real forms in a complex flag manifold

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Complex flag manifold

: orbit of the adjoint rep.

of a cpt. semisimple Lie group,

a simply conn. cpt. homog.

Kähler mfd

$\boldsymbol{\tau}$ **: invol. anti-holo. isometry**

Fix($\boldsymbol{\tau}$) : real form

tot. geodesic Lagrangian submfd.

M : a complex flag manifold

L_0, L_1 : two real forms of M

$L_0 \cap L_1$: discrete

$\Rightarrow L_0 \cap L_1$ has a good shape,

which may lead a calculation of

the Floer homology $HF(L_0, L_1)$.

Def.(Chen-Nagano)

M : a Riemann. symmetric space

s_x : the geod. symmetry at $x \in M$

$S \subset M$: **antipodal**

$\Leftrightarrow \forall x, y \in S \ s_x(y) = y$

2-number $\#_2 M$ of M

$\#_2 M = \max\{\#S \mid S : \text{antipodal in } M\}$

S : **great** $\Leftrightarrow \#S = \#_2 M$

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$G_k(\mathbb{K}^n)$: **Grassmann manifold**

$\{v_i\}$: $\mathbb{K}$-**orthonormal basis of** $\mathbb{K}^n$

$\Rightarrow \{\langle v_{i_1}, \ldots, v_{i_k} \rangle_{\mathbb{K}} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$

$= \{V \in G_k(\mathbb{K}^n) \mid V$: **spanned by** $v_j\}$

: **a great antipodal set of** $G_k(\mathbb{K}^n)$

$$\#_2 G_k(\mathbb{K}^n) = \binom{n}{k}$$

M : a complex flag manifold

$\exists$ k-symmetric structure on M

$\longrightarrow$ an antipodal set in M,

great antipodal

For $n_1 + \cdots + n_r < n$

$$F_{n_1,\dots,n_r}(\mathbb{K}^n)$$

$$= \left\{ (V_1, \dots, V_r) \;\middle|\; \begin{array}{l} V_i : \mathbb{K}\text{-subspace in } \mathbb{K}^n \\ \dim V_i = n_1 + \cdots + n_i \\ V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{K}^n \end{array} \right\}$$

$\mathbb{K} = \mathbb{C}$, $G = SU(n)$

$\mathfrak{g}$: its Lie algebra

$\exists\ Z \in \mathfrak{g}\ F_{n_1,\dots,n_r}(\mathbb{C}^n) \cong \mathrm{Ad}(G)Z \subset \mathfrak{g}$

$\{(V_1, \dots, V_r) \in F_{n_1,\dots,n_r}(\mathbb{C}^n) \mid V_i : \text{spanned by } v_j\}$

: a great antipodal set of $F_{n_1,\dots,n_r}(\mathbb{C}^n)$

Thm 1(Tanaka-T.)

M **: Herm. sym. space of cpt type**

L_0, L_1 **: real forms of** M

$L_0 \cap L_1$ **: discrete**

$\Rightarrow L_0 \cap L_1$ **: antipodal**

Thm 2(Iriyeh-Sakai-T.)

M **: Herm. sym. space of cpt type**

L_0, L_1 **: real forms of** M

$L_0 \cap L_1$ **: discrete**

$$\Rightarrow HF(L_0, L_1) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2 p$$

This result has some applications.

Applications

- generalized Arnold-Givental inequality
- Hamilton volume minimizing

Generalization

- complex flag manifolds

Joint work with Ikawa, Iriyeh, Okuda and Sakai

G **: a cpt semisimple Lie group**

$\mathfrak{g}$ **: its Lie algebra,** $Z \in \mathfrak{g}$

$M = \mathrm{Ad}(G)Z \subset \mathfrak{g}$

: complex flag manifold

$\mathfrak{t}$ **: maximal abelian subalgebra**

$M \cap \mathfrak{t}$ **: great antipodal set**

(G, K) : symmetric pair

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $Z \in \mathfrak{p}$

$L = \mathrm{Ad}(K)Z \subset M$: real form

Examples of real forms

$$F_{n_1,\ldots,n_r}(\mathbb{R}^n) \subset F_{n_1,\ldots,n_r}(\mathbb{C}^n),$$

$$F_{n_1,\ldots,n_r}(\mathbb{H}^n) \subset F_{2n_1,\ldots,2n_r}(\mathbb{C}^{2n})$$

Thm 1 L_0, L_1 : real forms of M

$\Rightarrow L_0 \cap L_1 \neq \emptyset$

(G, K) : symmetric pair

$\mathfrak{a}$: maximal abelian subspace in $\mathfrak{p}$

$A = \exp \mathfrak{a}$

Conjugacy of maximal tori

$$\Rightarrow G/K = \bigcup_{k \in K} kA \cdot o$$

$G = KAK$

$L = \mathrm{Ad}(K)Z$: real form of M

$\forall k \in K \qquad \mathrm{Ad}(k)L = L$

$g \in G \qquad L \cap \mathrm{Ad}(g)L$

$\exists k_i \in K,\ \exists a \in A \qquad g = k_0 a k_1$

$$\begin{aligned} L \cap \mathrm{Ad}(g)L &= L \cap \mathrm{Ad}(k_0 a k_1)L \\ &= \mathrm{Ad}(k_0)(L \cap \mathrm{Ad}(a)L) \end{aligned}$$

$$L \cap \mathrm{Ad}(g)L \quad \longrightarrow \quad L \cap \mathrm{Ad}(a)L$$

Thm 2 $\quad L \cap \mathrm{Ad}(a)L$ **: discrete**

$\Leftrightarrow$ **a certain condition for** a

In this case

$L \cap \mathrm{Ad}(a)L$ **: antipodal set of** M,

orbit of the Weyl group $W(G, K)$

(G, K_i) : symmetric pairs $(i = 0, 1)$

$\mathfrak{g} = \mathfrak{k}_i + \mathfrak{p}_i$, $Z \in \mathfrak{p}_0 \cap \mathfrak{p}_1$

$\mathfrak{a}$: max. abelian subsp. in $\mathfrak{p}_0 \cap \mathfrak{p}_1$

$A = \exp \mathfrak{a}$

Heintze-Palais-Terng-Thorbergsson

$$\Rightarrow G/K_1 = \bigcup_{k \in K_0} kA \cdot o$$

$G = K_0 A K_1$

$L_i = \mathrm{Ad}(K_i)Z$: real form of M

$\forall k_i \in K_i \quad \mathrm{Ad}(k_i)L_i = L_i$

$g \in G \quad L_0 \cap \mathrm{Ad}(g)L_1$

$\exists k_i \in K_i,\ \exists a \in A \quad g = k_0 a k_1$

$$\begin{aligned} L_0 \cap \mathrm{Ad}(g)L_1 &= L_0 \cap \mathrm{Ad}(k_0 a k_1)L_1 \\ &= \mathrm{Ad}(k_0)(L_0 \cap \mathrm{Ad}(a)L_1) \end{aligned}$$

$$L_0 \cap \mathrm{Ad}(g)L_1 \quad \longrightarrow \quad L_0 \cap \mathrm{Ad}(a)L_1$$

$\boldsymbol{\sigma}_i$: **the involution of** (G, K_i)

We assume $\boldsymbol{\sigma}_0\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_0$

Thm 3 $\quad L_0 \cap \mathrm{Ad}(a)L_1$: **discrete**

$\Leftrightarrow$ $\quad$ **a certain condition for** a

In this case

$L_0 \cap \mathrm{Ad}(a)L_1$: **antipodal set of** M,

orbit of a certain Weyl group