WEAKLY REFLECTIVE SUBMANIFOLDS AND AUSTERE SUBMANIFOLDS

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Abstract. An austere submanifold is a minimal submanifold where for each normal vector, the set of eigenvalues of its shape operator is invariant under the multiplication by -1. In the present paper, we introduce the notion of weakly reflective submanifold, which is an austere submanifold with a reflection for each normal direction, and study its fundamental properties. Using these, we determine weakly reflective orbits and austere orbits of linear isotropy representations of Riemannian symmetric spaces.

1. Introduction

Orbits of an s-representation, that is a linear isotropy representation of a Riemannian symmetric pair, are important examples of homogeneous submanifolds in the hypersphere of a Euclidean space. For example, a homogeneous isoparametric hypersurface in the hypersphere, which many mathematicians have investigated, can be obtained as a principal orbit of an s-representation of a Riemannian symmetric pair of rank two. It is known that there exists a unique minimal isoparametric hypersurface in each parallel family of isoparametric hypersurfaces. Furthermore, typical examples of minimal submanifolds in the hypersphere are given as orbits of s-representations. Hirohashi-Song-Takagi-Tasaki [7] showed that there exists a unique minimal orbit in each strata of the stratification of orbit types. However, in general we can not explicitly point out which orbit among each strata is a minimal submanifold.

Harvey-Lawson [4] introduced the notion of austere submanifold, which is a minimal submanifold whose second fundamental form has a certain symmetry. They showed that one can construct a special Lagrangian cone, therefore absolutely area-minimizing, in a complex Euclidean space as the twisted normal bundle of an austere submanifold in a sphere (see [4], [2]). As we mentioned above, the complete list of minimal orbits of s-representations in the hypersphere is unknown at the moment. Therefore we first attempt to determine all austere orbits. We give a necessary and sufficient condition for an orbit to be an austere submanifold in the hypersphere in terms of the restricted root system of a Riemannian symmetric pair. By this criterion, we can determine all orbits which are austere submanifolds in the hypersphere. Since the definition is focused on a symmetry of its second fundamental form, the notion of austere submanifold is an infinitesimal property of a submanifold. However, we observe that some of austere orbits, which we classified, have a certain

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global symmetry. This symmetry is a globalization of the notion of austere submanifold and a weakened condition of a reflective submanifold. Therefore we shall call them weakly reflective submanifolds, and study some fundamental properties of them. Finally we determine all weakly reflective orbits of s-representations.

The organization of this paper is as follows. In Section 2, we will give the definition of weakly reflective submanifold (Definition 2.1), and recall some related notions. We study their relationship and fundamental properties. In Section 3, we summarize the geometry of orbits of s-representations of Riemannian symmetric pairs. This will be a preliminary for the sections below. In Section 4, we shall give the list of orbits of s-representations which are weakly reflective submanifolds in the hypersphere (Theorem 4.1). We show that these orbits are weakly reflective submanifold in the hypersphere there, however, we will show that the list gives all weakly reflective orbits later. In Section 5, we will give a criterion of austere orbits (Lemma 5.3), and determine all orbits which are austere submanifolds in the hypersphere (Theorem 5.1). Furthermore we show that austere orbits which are not enumerated in the list of weakly reflective orbits are not weakly reflective submanifolds. Then we will complete the proof of the list of weakly reflective orbits. In Section 6, we will study relationships between weakly reflective submanifolds in a sphere and those in Euclidean spaces or complex projective spaces.

The authors are profoundly grateful to Makoto Kimura and Osami Yasukura for their helpful suggestion on Proposition 4.4. Before we wrote this paper, Kimura, Yasukura and the third named author showed a previous version of Proposition 4.4 which is unpublished, that is, the orbit of the highest root of a compact Lie group under the adjoint action is an austere submanifold in the hypersphere. The authors would also like to thank Reiko Miyaoka for her valuable comments. In fact, Proposition 2.9 was essentially suggested by her. Finally the authors are grateful to the referee who gave them many useful comments.

2. Definitions and fundamental results

We begin with recalling the definition of reflective submanifold given by Leung [8]. Let $\tilde{M}$ be a complete Riemannian manifold. A connected component of the fixed point set of an involutive isometry of $\tilde{M}$ is called a *reflective submanifold*. A reflective submanifold is a complete totally geodesic submanifold. The involutive isometry which defines a reflective submanifold M can be determined uniquely. We call it the *reflection* of M and denote by σ_M. If M is a reflective submanifold in $\tilde{M}$ and σ_M is its reflection, then for any normal vector $\xi \in T^{\perp}_x M$

$$\sigma_M(x) = x, \qquad (d\sigma_M)_x\xi = -\xi, \qquad \sigma_M(M) = M$$

hold. Taking notice of these properties, we define a weakly reflective submanifold as follows.

Definition 2.1. Let M be a submanifold of a Riemannian manifold $\tilde{M}$. For each normal vector $\xi \in T^{\perp}_x M$ at each point $x \in M$, if there exists an isometry σ_ξ of $\tilde{M}$ which satisfies

$$\sigma_\xi(x) = x, \qquad (d\sigma_\xi)_x\xi = -\xi, \qquad \sigma_\xi(M) = M,$$

then we call M a *weakly reflective submanifold* and σ_ξ a *reflection* of M with respect to ξ.

In the case where M is a hypersurface, σ_ξ is independent of the choice of ξ at each point x. In this paper mainly we deal with orbits of some isometric actions of compact Lie groups. We note that if M is an extrinsic homogeneous submanifold in $\tilde{M}$, that is an orbit of an isometric action of a Lie group on $\tilde{M}$, then it suffices to ascertain that at one point of M it satisfies the condition to be a weakly reflective submanifold.

Remark 2.2. For a reflective submanifold, there exists a reflection which is independent of the choice of a normal vector. So it is clear that a reflective submanifold is always a weakly reflective submanifold.

Example 2.3.

$$S^{n-1}(1) \times S^{n-1}(1) = \big\{(x,y) \mid x, y \in S^{n-1}(1)\big\}$$

is a weakly reflective submanifold in $(2n-1)$*-dimensional sphere* $S^{2n-1}(\sqrt{2})$ *of radius* $\sqrt{2}$*.*

Proof. Since $S^{n-1}(1) \times S^{n-1}(1)$ is a homogeneous submanifold of $S^{2n-1}(\sqrt{2})$, it suffices to ascertain that at one point of $S^{n-1}(1) \times S^{n-1}(1)$ it satisfies the condition to be a weakly reflective submanifold. The tangent space of $S^{n-1}(1) \times S^{n-1}(1)$ at

$$x = (1, 0, \ldots, 0, \overset{n+1}{\breve{1}}, 0, \ldots, 0) \in S^{n-1}(1) \times S^{n-1}(1)$$

is given by

$$T_x(S^{n-1}(1) \times S^{n-1}(1)) = \{(0, x_2, \ldots, x_n, 0, y_2, \ldots, y_n) \mid x_i, y_j \in \mathbf{R}\},$$

and the normal space in $S^{2n-1}(\sqrt{2})$ is

$$T_x^\perp(S^{n-1}(1) \times S^{n-1}(1)) = \mathbf{R}(1, 0, \ldots, 0, \overset{n+1}{\breve{-1}}, 0, \ldots, 0).$$

Now we define an isometry σ of $S^{2n-1}(\sqrt{2})$ by

$$\sigma(x_1, \ldots, x_n, y_1, \ldots, y_n) = (y_1, \ldots, y_n, x_1, \ldots, x_n)$$

for $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in S^{2n-1}(\sqrt{2})$. Then

$$\sigma(x) = x, \qquad \sigma(S^{n-1}(1) \times S^{n-1}(1)) = S^{n-1}(1) \times S^{n-1}(1)$$

and $d\sigma_x$ acts on $T_x^\perp(S^{n-1}(1) \times S^{n-1}(1))$ as $-\mathrm{id}$. Thus $S^{n-1}(1) \times S^{n-1}(1)$ is a weakly reflective submanifold in $S^{2n-1}(\sqrt{2})$. □

Definition 2.4. Let M be a submanifold of a Riemannian manifold $\tilde{M}$. We denote the shape operator of M by A. M is called an *austere submanifold* if for each normal vector $\xi \in T_x^\perp M$, the set of eigenvalues with their multiplicities of A_ξ is invariant under the multiplication by -1. It is obvious that an austere submanifold is a minimal submanifold.

The notion of austere submanifold was first given by Harvey-Lawson [4].

Proposition 2.5. *A weakly reflective submanifold is an austere submanifold.*

Proof. Let M be a weakly reflective submanifold in a Riemannian manifold $\tilde{M}$. Then for each normal vector $\xi \in T_x^\perp M$, there exists an isometry σ_ξ of $\tilde{M}$ which satisfies

$$\sigma_\xi(x) = x, \qquad (d\sigma_\xi)_x \xi = -\xi, \qquad \sigma_\xi(M) = M.$$

For a normal vector $\xi \in T_x^{\perp}M$, we denote by A_ξ the shape operator of M with respect to ξ and by h the second fundamental form of M. For $X, Y \in T_xM$, we take vector fields $\tilde{X}$ and $\tilde{Y}$ defined on a neighborhood of x in $\tilde{M}$ which are tangent to M and $\tilde{X}_x = X$ and $\tilde{Y}_x = Y$. Since σ_ξ satisfies $\sigma_\xi(M) = M$, vector fields $d\sigma_\xi \tilde{X}$ and $d\sigma_\xi \tilde{Y}$ are tangent to M. Let $\bar{\nabla}$ denote the covariant derivative of $\tilde{M}$. Then we have

$$\begin{aligned} h((d\sigma_\xi)_x X, (d\sigma_\xi)_x Y) &= (\bar{\nabla}_{d\sigma_\xi \tilde{X}} d\sigma_\xi \tilde{Y})_x^{\perp} = ((d\sigma_\xi)_x \bar{\nabla}_{\tilde{X}} \tilde{Y})^{\perp} \\ &= (d\sigma_\xi)_x (\bar{\nabla}_{\tilde{X}} \tilde{Y})^{\perp} = (d\sigma_\xi)_x h(X, Y). \end{aligned}$$

From the following calculation

$$\begin{aligned} \langle A_\xi (d\sigma_\xi)_x X, (d\sigma_\xi)_x Y \rangle &= \langle h((d\sigma_\xi)_x X, (d\sigma_\xi)_x Y), \xi \rangle \\ &= \langle (d\sigma_\xi)_x h(X, Y), \xi \rangle = \langle h(X, Y), (d\sigma_\xi)_x^{-1} \xi \rangle \\ &= \langle h(X, Y), -\xi \rangle = -\langle A_\xi X, Y \rangle, \end{aligned}$$

we have $(d\sigma_\xi)_x^{-1} A_\xi (d\sigma_\xi)_x = -A_\xi$. This implies that $(d\sigma_\xi)_x$ provides an isomorphism between eigenspaces of A_ξ for eigenvalues λ and $-\lambda$. Thus M is an austere submanifold. □

In the rest of this section, we shall study weakly reflective orbits of isometric actions of Lie groups on Riemannian manifolds. First we shall provide some preliminaries. Let G be a Lie group acting isometrically on a Riemannian manifold $\tilde{M}$ and G_x be the isotropy subgroup at x, that is, $G_x = \{g \in G \mid gx = x\}$. Then the orbit $G(x)$ is diffeomorphic to the coset manifold G/G_x. An orbit $G(x)$ is a *principal orbit* if, for any $y \in \tilde{M}$, there exists $g \in G$ such that $G_x \subset gG_y g^{-1}$. It is known that there exists a principal orbit. The codimension of a principal orbit is called the *cohomogeneity* of the action of G on $\tilde{M}$. An orbit which is not principal is called a *singular orbit*. The differential of the action of G_x defines a linear representation of G_x on $T_x\tilde{M}$ called the *linear isotropy representation*. The tangent space $T_x(G(x))$ and the normal space $T_x^{\perp}(G(x))$ of $G(x)$ at x are invariant subspaces of the linear isotropy representation. The restriction of the linear isotropy representation to $T_x^{\perp}(G(x))$ is called the *slice representation* at x.

Theorem 2.6. (Slice representation theorem [9, Theorem 1.1], [11, Theorem 4.6], [12, Proposition 5.4.7]). *The cohomogeneity of a slice representation equals the cohomogeneity of the action of G on $\tilde{M}$. Moreover, $G(x)$ is a principal orbit if and only if the slice representation at x is trivial.*

Proposition 2.7. *Any singular orbit of a cohomogeneity one action on a Riemannian manifold is a weakly reflective submanifold.*

Proof. Suppose that the isometric action of a Lie group G on a Riemannian manifold $\tilde{M}$ is cohomogeneity one. Let $G(x)$ be a singular orbit.

First we consider the case where the codimension of $G(x)$ is greater than or equal to 2. From the slice representation theorem, the isotropy subgroup G_x acts transitively on the hypersphere in $T_x^{\perp}(G(x))$. In particular, for any $\xi \in T_x^{\perp}(G(x))$ there exists $g \in G_x$ such that $dg_x(\xi) = -\xi$. Therefore g becomes a reflection of $G(x)$ at x with respect to ξ. Since $G(x)$ is a homogeneous submanifold, $G(x)$ has a reflection with respect to any normal vector at any point. Thus $G(x)$ is a weakly reflective submanifold in $\tilde{M}$.

When the codimension of $G(x)$ is 1, $\dim T_x^\perp(G(x)) = 1$ and the dimension of a nontrivial orbit of the slice representation is equal to 0 by Theorem 2.6. Moreover the slice representation at x is not trivial, because $G(x)$ is a singular orbit. Therefore the image of the slice representation is not $SO(1)$ but $O(1)$ and for any $\xi \in T_x^\perp(G(x))$ there exists $g \in G_x$ such that $dg_x(\xi) = -\xi$. Thus, by the same discussion with above, $G(x)$ is a weakly reflective submanifold in $\tilde{M}$. $\square$

Remark 2.8. Podestá [13] proved that any singular orbit of a cohomogeneity one action is an austere submanifold. However, essentially he showed Proposition 2.7.

Proposition 2.9. *Let G be a connected Lie group acting isometrically on a complete, connected Riemannian manifold $\tilde{M}$. Suppose that the action of G on $\tilde{M}$ is cohomogeneity one with two singular orbits. If there exists a principal orbit which is a weakly reflective submanifold in $\tilde{M}$, then it has a same distance from two singular orbits and two singular orbits are isometric.*

Proof. Since there exist two singular orbits, the orbit space $\tilde{M}/G$ is homeomorphic to a closed interval (Mostert [10], Bergery [1]). Orbits of interior points are principal and those of end points are singular. Moreover principal orbits are hypersurfaces in $\tilde{M}$, because the cohomogeneity of the action of G is one. Suppose that $G(x)$ is a principal orbit which is a weakly reflective submanifold. Then, by the slice representation theorem, there exists a unit normal vector field ξ on $G(x)$, which is invariant under the action of G. We take a geodesic $\gamma(t)$ of $\tilde{M}$ which satisfies an initial condition

$$\gamma(0) = x, \qquad \gamma'(0) = \xi_x.$$

Then $\gamma(t)$ is a section of the action of G on $\tilde{M}$, namely $\gamma(t)$ meets all orbits perpendicularly ([12] Proposition 5.6.2). Since $dg_x(\xi_x) = \xi_{gx}$ for any $g \in G$, $g\gamma(t)$ is a geodesic of $\tilde{M}$ which satisfies an initial condition

$$g\gamma(0) = gx, \qquad (g\gamma)'(0) = \xi_{gx}.$$

Since $G(x)$ is a weakly reflective submanifold of $\tilde{M}$, there exists an isometry σ of $\tilde{M}$ which satisfies

$$\sigma(x) = x, \quad d\sigma_x(\xi_x) = -\xi_x, \quad \sigma(G(x)) = G(x),$$

that is a reflection of $G(x)$ with respect to ξ_x. We set

$$G(x)_\pm = \{y \in G(x) \mid d\sigma_y(\xi_y) = \pm\xi_{\sigma(y)}\}.$$

The sets $G(x)_+$ and $G(x)_-$ are closed subsets of $G(x)$, and $G(x)$ is a disjoint union of $G(x)_+$ and $G(x)_-$ because $G(x)$ is a hypersurface in $\tilde{M}$. Since $G(x)$ is connected and $x \in G(x)_-$, we have $G(x) = G(x)_-$. This implies that $d\sigma_y(\xi_y) = -\xi_{\sigma(y)}$ for any $y \in G(x)$. For any $g \in G$, $\sigma g\gamma(t)$ is a geodesic which satisfies an initial condition

$$\sigma g\gamma(0) = \sigma(gx), \qquad (\sigma g\gamma)'(0) = d\sigma_{gx}(g\gamma)'(0) = d\sigma_{gx}(\xi_{gx}) = -\xi_{\sigma gx}.$$

Now we take $g_1 \in G$ such that $g_1 x = \sigma(gx)$. Then $\sigma g\gamma(t)$ and $g_1\gamma(-t)$ are geodesics of same initial conditions, hence $\sigma g\gamma(t) = g_1\gamma(-t) \in G(\gamma(-t))$. Therefore we have $\sigma(G(\gamma(t))) \subset G(\gamma(-t))$ for each t. Since σ^{-1} is also a reflection of $G(x)$ at x, we also have $\sigma^{-1}(G(\gamma(-t))) \subset G(\gamma(t))$ by the same discussion for σ^{-1} and $\gamma(-t)$. Thus $\sigma(G(\gamma(t))) = G(\gamma(-t))$ and σ induces a homeomorphism of $\tilde{M}/G$. This implies that σ maps one singular orbit to the other one. Hence two singular orbits can be expressed as $G(\gamma(t_1))$ and $G(\gamma(-t_1))$ for some t_1. Consequently we have the conclusion. $\square$

3. Orbits of s-representations

A linear isotropy representation of a Riemannian symmetric pair is called an s-representation as we mentioned in Introduction. In the following sections, we will study orbits of s-representations which are austere submanifolds or weakly reflective submanifolds. For this purpose, we shall provide some fundamental notions of orbits of s-representations in this section.

Let G be a compact, connected Lie group and K a closed subgroup of G. Assume that θ is an involutive automorphism of G and $G_\theta^0 \subset K \subset G_\theta$, where

$$G_\theta = \{g \in G \mid \theta(g) = g\}$$

and G_θ^0 is the identity component of G_θ. Then (G, K) is a symmetric pair with respect to θ. We denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. The involutive automorphism of $\mathfrak{g}$ induced from θ will be also denoted by θ. Then we have

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}.$$

Take an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ which is invariant under θ and the adjoint representation of G. Set

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid \theta(X) = -X\},$$

then we have a canonical orthogonal direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

Henceforth we assume that the symmetric pair (G, K) is irreducible, namely K acts irreducibly on $\mathfrak{m}$.

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$ and a maximal abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ containing $\mathfrak{a}$. For $\alpha \in \mathfrak{t}$ we set

$$\tilde{\mathfrak{g}}_\alpha = \{X \in \mathfrak{g}^{\mathbf{C}} \mid [H, X] = \sqrt{-1}\langle \alpha, H\rangle X\ (H \in \mathfrak{t})\}$$

and define the root system $\tilde{R}$ of $\mathfrak{g}$ by

$$\tilde{R} = \{\alpha \in \mathfrak{t} - \{0\} \mid \tilde{\mathfrak{g}}_\alpha \neq \{0\}\}.$$

For $\alpha \in \mathfrak{a}$ we set

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^{\mathbf{C}} \mid [H, X] = \sqrt{-1}\langle \alpha, H\rangle X\ (H \in \mathfrak{a})\}$$

and define the restricted root system R of $(\mathfrak{g}, \mathfrak{k})$ by

$$R = \{\alpha \in \mathfrak{a} - \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}.$$

Set

$$\tilde{R}_0 = \tilde{R} \cap \mathfrak{k}$$

and denote the orthogonal projection from $\mathfrak{t}$ to $\mathfrak{a}$ by $H \mapsto \bar{H}$. Then we have

$$R = \{\bar{\alpha} \mid \alpha \in \tilde{R} - \tilde{R}_0\}.$$

We take a basis of $\mathfrak{t}$ extended from a basis of $\mathfrak{a}$ and define the lexicographic orderings $>$ on $\mathfrak{a}$ and $\mathfrak{t}$ with respect to these bases. Then for $H \in \mathfrak{t}$, $\bar{H} > 0$ implies $H > 0$. We denote by $\tilde{F}$ the fundamental system of $\tilde{R}$ with respect to the ordering $>$. Set

$$\tilde{F}_0 = \tilde{F} \cap \tilde{R}_0,$$

then the fundamental system F of R with respect to the ordering $>$ is given by

$$F = \{\bar{\alpha} \mid \alpha \in \tilde{F} - \tilde{F}_0\}.$$

We set
$$\tilde{R}_+ = \{\alpha \in \tilde{R} \mid \alpha > 0\}, \qquad R_+ = \{\alpha \in R \mid \alpha > 0\}.$$
Then we have
$$R_+ = \{\bar{\alpha} \mid \alpha \in \tilde{R}_+ - \tilde{R}_0\}.$$
We also set
$$\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [X, H] = 0 \ (H \in \mathfrak{a})\},$$
and define
$$\mathfrak{k}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}), \qquad \mathfrak{m}_\alpha = \mathfrak{m} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$
for $\alpha \in R_+$. Under these notations, we have the following lemma.

Lemma 3.1 ([14]). (1) *We have orthogonal direct sum decompositions*
$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in R_+} \mathfrak{k}_\alpha, \qquad \mathfrak{m} = \mathfrak{a} + \sum_{\alpha \in R_+} \mathfrak{m}_\alpha.$$

(2) *For each $\alpha \in \tilde{R}_+ - \tilde{R}_0$, there exist $S_\alpha \in \mathfrak{k}$ and $T_\alpha \in \mathfrak{m}$ such that*
$$\{S_\alpha \mid \alpha \in \tilde{R}_+, \ \bar{\alpha} = \lambda\}, \quad \{T_\alpha \mid \alpha \in \tilde{R}_+, \ \bar{\alpha} = \lambda\}$$
are respectively orthonormal bases of $\mathfrak{k}_\lambda$ and $\mathfrak{m}_\lambda$ and that for $H \in \mathfrak{a}$
$$\begin{aligned}
&[H, S_\alpha] = \langle \alpha, H\rangle T_\alpha, \quad [H, T_\alpha] = -\langle \alpha, H\rangle S_\alpha, \quad [S_\alpha, T_\alpha] = \bar{\alpha},\\
&\mathrm{Ad}(\exp H) S_\alpha = \cos\langle \alpha, H\rangle S_\alpha + \sin\langle \alpha, H\rangle T_\alpha,\\
&\mathrm{Ad}(\exp H) T_\alpha = -\sin\langle \alpha, H\rangle S_\alpha + \cos\langle \alpha, H\rangle T_\alpha.
\end{aligned}$$

We define a subset D of $\mathfrak{a}$ by
$$D = \bigcup_{\alpha \in R} \{H \in \mathfrak{a} \mid \langle \alpha, H\rangle = 0\}.$$
A connected component of $\mathfrak{a} - D$ is a Weyl chamber. We set
$$C = \{H \in \mathfrak{a} \mid \langle \alpha, H\rangle > 0 \ (\alpha \in F)\}.$$
Then C is an open convex subset of $\mathfrak{a}$ and the closure of C is given by
$$\bar{C} = \{H \in \mathfrak{a} \mid \langle \alpha, H\rangle \geq 0 \ (\alpha \in F)\}.$$
For a subset $\Delta \subset F$, we define
$$C^\Delta = \{H \in \bar{C} \mid \langle \alpha, H\rangle > 0 \ (\alpha \in \Delta), \ \langle \beta, H\rangle = 0 \ (\beta \in F - \Delta)\}.$$

Lemma 3.2. (1) *For $\Delta_1 \subset F$, the decomposition*
$$\overline{C^{\Delta_1}} = \bigcup_{\Delta \subset \Delta_1} C^\Delta$$
is a disjoint union. In particular, $\bar{C} = \bigcup_{\Delta \subset F} C^\Delta$ is a disjoint union.

(2) *For $\Delta_1, \Delta_2 \subset F$, $\Delta_1 \subset \Delta_2$ if and only if $C^{\Delta_1} \subset \overline{C^{\Delta_2}}$.*

For each $\alpha \in F$, we take $H_\alpha \in \mathfrak{a}$ such that
$$\langle H_\alpha, \beta\rangle = \begin{cases} 1 & (\beta = \alpha), \\ 0 & (\beta \neq \alpha) \end{cases} \qquad (\beta \in F).$$
Then we have
$$\bar{C} = \left\{ \sum_{\alpha \in F} t_\alpha H_\alpha \ \middle| \ t_\alpha \geq 0 \right\},$$

and for $\Delta \subset F$

$$C^\Delta = \left\{ \sum_{\alpha \in \Delta} t_\alpha H_\alpha \;\middle|\; t_\alpha > 0 \right\}.$$

We set

$$\begin{aligned} R^\Delta &= R \cap (F - \Delta)_{\mathbf{Z}}, \\ R^\Delta_+ &= R^\Delta \cap R_+, \\ \mathfrak{g}^\Delta &= \mathfrak{k}_0 + \mathfrak{a} + \sum_{\alpha \in R^\Delta_+} (\mathfrak{k}_\alpha + \mathfrak{m}_\alpha). \end{aligned}$$

We also set

$$\begin{aligned} \mathfrak{k}^\Delta &= \mathfrak{g}^\Delta \cap \mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in R^\Delta_+} \mathfrak{k}_\alpha, \\ \mathfrak{m}^\Delta &= \mathfrak{g}^\Delta \cap \mathfrak{m} = \mathfrak{a} + \sum_{\alpha \in R^\Delta_+} \mathfrak{m}_\alpha. \end{aligned}$$

Then we have an orthogonal direct sum decomposition

$$\mathfrak{g}^\Delta = \mathfrak{k}^\Delta + \mathfrak{m}^\Delta.$$

For $H \in \mathfrak{m}$ we set

$$Z^H_K = \{k \in K \mid \mathrm{Ad}(k)H = H\}.$$

Then Z^H_K is a closed subgroup of K and the orbit $\mathrm{Ad}(K)H$ is diffeomorphic to the coset manifold K/Z^H_K.

Under these notations, we have the following lemma.

Lemma 3.3 ([6]). *Fix a subset $\Delta \subset F$. For $H \in C^\Delta$ we have the following:*

(1) $R^\Delta_+ = \{\alpha \in R_+ \mid \langle \alpha, H \rangle = 0\}$,
(2) $R^\Delta = \{\alpha \in R \mid \langle \alpha, H \rangle = 0\}$,
(3) $\mathfrak{g}^\Delta = \{X \in \mathfrak{g} \mid [H, X] = 0\}$,
(4) $(\mathfrak{g}^\Delta, \mathfrak{k}^\Delta)$ *is a symmetric pair and its canonical decomposition is given by*

$$\mathfrak{g}^\Delta = \mathfrak{k}^\Delta + \mathfrak{m}^\Delta,$$

(5) $\mathfrak{k}^\Delta$ *is the Lie algebra of* Z^H_K.

Now we shall study an orbit $\mathrm{Ad}(K)H$ of the linear isotropy representation of (G, K) through $H \in \mathfrak{m}$. An orbit $\mathrm{Ad}(K)H$ is a submanifold of the hypersphere S of radius $\|H\|$ in $\mathfrak{m}$. From [6], $\mathrm{Ad}(K)H$ is connected. Since

$$\mathfrak{m} = \bigcup_{k \in K} \mathrm{Ad}(k)\bar{C},$$

without loss of generalities we may assume $H \in \bar{C}$. Moreover, from Lemma 3.2, there exists $\Delta \subset F$ such that $H \in C^\Delta$. For $X \in \mathfrak{k}$ we define a vector field X^* on $\mathfrak{m}$ by

$$X^*_x = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp tX)x = [X, x]$$

at $x \in \mathfrak{m}$. Then $X^*|_{\mathrm{Ad}(K)H}$ is a tangent vector field on $\mathrm{Ad}(K)H$. From Lemma 3.1 we have the following lemma.

Lemma 3.4. *For $\Delta \subset F$ and $H \in C^{\Delta}$, the tangent space $T_H(\mathrm{Ad}(K)H)$ of the orbit $\mathrm{Ad}(K)H$ at H and the normal space $T_H^{\perp}(\mathrm{Ad}(K)H)$ in the hypersphere can be expressed as*

$$\begin{aligned} T_H(\mathrm{Ad}(K)H) &= \sum_{\alpha \in R_+ - R_+^{\Delta}} \mathfrak{m}_\alpha, \\ T_H^{\perp}(\mathrm{Ad}(K)H) &= H^{\perp} \cap \mathfrak{a} + \sum_{\alpha \in R_+^{\Delta}} \mathfrak{m}_\alpha = \bigcup_{k \in Z_K^H} \mathrm{Ad}(k)(H^{\perp} \cap \mathfrak{a}). \end{aligned}$$

Let h denote the second fundamental form of $\mathrm{Ad}(K)H$ at H in the hypersphere S. Then we have

$$h(X_H^*, Y_H^*) = [Y, [X, H]]^N,$$

where $[Y, [X, H]]^N$ is $T_H^{\perp}(\mathrm{Ad}(K)H)$-component of $[Y, [X, H]]$.

Proof. From Lemma 3.1 we have

$$\begin{aligned} T_H(\mathrm{Ad}(K)H) &= \sum_{\alpha \in R_+ - R_+^{\Delta}} \mathfrak{m}_\alpha, \\ T_H^{\perp}(\mathrm{Ad}(K)H) &= H^{\perp} \cap \mathfrak{a} + \sum_{\alpha \in R_+^{\Delta}} \mathfrak{m}_\alpha = H^{\perp} \cap \mathfrak{m}^{\Delta} \end{aligned}$$

Moreover, from Lemma 3.3

$$\mathfrak{m}^{\Delta} = \bigcup_{k \in Z_K^H} \mathrm{Ad}(k)\mathfrak{a}.$$

Since $\mathrm{Ad}(k)H = H$ for $k \in Z_K^H$, we have

$$H^{\perp} \cap \mathfrak{m}^{\Delta} = \bigcup_{k \in Z_K^H} \mathrm{Ad}(k)(H^{\perp} \cap \mathfrak{a}).$$

The calculation of X_x^* mentioned above shows the representation of the second fundamental form. □

For orbits of s-representations which are minimal submanifolds in the hypersphere, the following theorem is known.

Theorem 3.5 ([7])**.** *Fix a hypersphere S in $\mathfrak{m}$ centered at 0. For each subset $\Delta \subset F$, there exists a unique $H \in C^{\Delta} \cap S$ such that $\mathrm{Ad}(K)H$ is a minimal submanifold in S.*

However, in general we can not determine H where $\mathrm{Ad}(K)H$ is a minimal submanifold in S explicitly. In the following two sections, we will give the complete lists of H where $\mathrm{Ad}(K)H$ is an austere submanifold or a weakly submanifold in S.

4. Weakly reflective orbits of s-representations

In this section, we shall study orbits of irreducible s-representations which are weakly reflective submanifolds in the hypersphere. In the next section, we will study austere orbits. Since these two properties of orbits are invariant under scalar multiples on the vector spaces, we do not discriminate the difference of the length of a vector. The following theorem is the main result of this section. We shall follow the notations of root systems in [3].

Theorem 4.1. *An orbit of an irreducible s-representation which is a weakly reflective submanifold in the hypersphere is one of the following list:*

(1) *an orbit through a restricted root vector* (Proposition 4.4)*,*
(2) *the orbit through the vector $2e_1 - e_2 - e_3$ and the orbit through the vector $e_1 + e_2 - 2e_3$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i - e_j)\}$ of type A_2* (Proposition 4.5)*,*
(3) *the orbit through the vector $e_1+e_2-e_3-e_4$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i-e_j)\}$ of type A_3* (Proposition 4.6)*,*
(4) *the orbit through the vector e_1 of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i \pm e_j)\}$ of type D_l $(l \geq 4)$* (Proposition 4.7)*,*
(5) *the orbit through the vector $e_1+e_2+e_3+e_4$ and the orbit through the vector $e_1+e_2+e_3-e_4$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i \pm e_j)\}$ of type D_4* (Proposition 4.8).

Remark 4.2. *In the case of the restricted root system of type A_2, the orbit through a restricted root is a principal orbit called a Cartan hypersurface, that is a minimal isoparametric hypersurface with three distinct principal curvatures. And the orbits through the vectors $2e_1 - e_2 - e_3$ and $e_1 + e_2 - 2e_3$ are its focal submanifolds, called Veronese surfaces. These two singular orbits are not equal but isometric with each other.*

In the case of the restricted root system of type D_4, from the proof of Proposition 4.8 we can see that three orbits through e_1, $e_1 + e_2 + e_3 + e_4$ and $e_1 + e_2 + e_3 - e_4$ are isometric with each other by the "triality".

Here we prove that orbits listed above are weakly reflective submanifolds. In the next section, we will classify all austere orbits of irreducible s-representations and show that all weakly reflective orbits can be obtained in Theorem 4.1. For this purpose, we shall first give the following lemma.

Lemma 4.3. *For $H \in \mathfrak{a}$, the orbit $\mathrm{Ad}(K)H$ is a weakly reflective submanifold in the hypersphere S if and only if, for any $\xi \in H^\perp \cap \mathfrak{a}$, there exists a linear isometry σ_ξ of $\mathfrak{m}$ which satisfies*

$$\sigma_\xi(H) = H, \qquad \sigma_\xi(\xi) = -\xi, \qquad \sigma_\xi(\mathrm{Ad}(K)H) = \mathrm{Ad}(K)H. \tag{4.1}$$

Proof. From Lemma 3.4, the normal space of the orbit $\mathrm{Ad}(K)H$ at H in S is given by

$$T_H^\perp(\mathrm{Ad}(K)H) = H^\perp \cap \mathfrak{a} + \sum_{\alpha \in R_+^\Delta} \mathfrak{m}_\alpha = \bigcup_{k \in Z_K^H} \mathrm{Ad}(k)(H^\perp \cap \mathfrak{a}).$$

If $\mathrm{Ad}(K)H$ is a weakly reflective submanifold in S, then for $\xi \in H^\perp \cap \mathfrak{a}$ there exists a linear isometry σ_ξ of $\mathfrak{m}$ which satisfies

$$\sigma_\xi(H) = H, \qquad (d\sigma_\xi)_H \xi = -\xi, \qquad \sigma_\xi(\mathrm{Ad}(K)H) = \mathrm{Ad}(K)H.$$

Here we have $(d\sigma_\xi)_H = \sigma_\xi$, since σ_ξ is a linear isometry.

Conversely, assume that $\mathrm{Ad}(K)H$ satisfies the condition (4.1). We take an arbitrary normal vector $\xi \in T_H^\perp(\mathrm{Ad}(K)H)$. From Lemma 3.4, there exists $k_0 \in Z_K^H$ such that $\mathrm{Ad}(k_0)\xi \in H^\perp \cap \mathfrak{a}$. Then, from the assumption, there exists a linear isometry σ which satisfies

$$\sigma(H) = H, \qquad \sigma \mathrm{Ad}(k_0)\xi = -\mathrm{Ad}(k_0)\xi, \qquad \sigma(\mathrm{Ad}(K)H) = \mathrm{Ad}(K)H.$$

We now define $\sigma_\xi = \mathrm{Ad}(k_0)^{-1}\sigma\mathrm{Ad}(k_0)$. Then σ_ξ satisfies

$$\sigma_\xi(H) = H, \qquad \sigma_\xi(\xi) = -\xi, \qquad \sigma_\xi(\mathrm{Ad}(K)H) = \mathrm{Ad}(K)H.$$

Thus σ_ξ is a reflection of $\mathrm{Ad}(K)H$ with respect to a normal vector ξ at H. Since $\mathrm{Ad}(K)H$ is a homogeneous submanifold, we have a reflection with respect to any normal vector at an arbitrary point. Consequently $\mathrm{Ad}(K)H$ is a weakly reflective submanifold in S. □

Proposition 4.4. *An orbit through a restricted root vector of the linear isotropy representation of an irreducible compact symmetric pair is a weakly reflective submanifold in the hypersphere S.*

Proof. Let α_0 be a restricted root vector and put $H = \alpha_0$. The reflection s_{α_0} on $\mathfrak{a}$ with respect to α_0 is given by

$$s_{\alpha_0}(X) = X - \frac{2\langle \alpha_0, X\rangle}{\langle \alpha_0, \alpha_0\rangle}\alpha_0 \qquad (X \in \mathfrak{a})$$

and satisfies

$$s_{\alpha_0}(H) = -H, \qquad s_{\alpha_0}|_{\mathfrak{a}\cap H^\perp} = 1_{\mathfrak{a}\cap H^\perp}.$$

The reflection s_{α_0} is an element of the Weyl group, hence there exists $k_0 \in N_K$ such that $\mathrm{Ad}(k_0)|_{\mathfrak{a}} = s_{\alpha_0}$, where

$$N_K = \{k \in K \mid \mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}\}.$$

Therefore

$$-H = \mathrm{Ad}(k_0)H \in \mathrm{Ad}(K)H,$$

and we have $\mathrm{Ad}(K)(-H) = \mathrm{Ad}(K)H$. We define a linear isometry σ of $\mathfrak{m}$ by

$$\sigma = -\mathrm{Ad}(k_0)|_{\mathfrak{m}}.$$

Then σ satisfies

$$\sigma(H) = H, \qquad \sigma|_{\mathfrak{a}\cap H^\perp} = -1|_{\mathfrak{a}\cap H^\perp}, \qquad \sigma(\mathrm{Ad}(K)H) = \mathrm{Ad}(K)H.$$

Thus, from Lemma 4.3, $\mathrm{Ad}(K)H$ is a weakly reflective submanifold in S. □

Proposition 4.5. *The orbit through the vector $2e_1 - e_2 - e_3$ and the orbit through the vector $e_1 + e_2 - 2e_3$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i - e_j)\}$ of type A_2 is a weakly reflective submanifold in the hypersphere S.*

Proof. Since the symmetric pair (G, K) is of rank 2, the action of K on S is cohomogeneity one. The vector $2e_1 - e_2 - e_3$ (resp. $e_1 + e_2 - 2e_3$) is orthogonal to a restricted root $e_2 - e_3$ (resp. $e_1 - e_2$). Therefore the orbit of K through $2e_1 - e_2 - e_3$ (resp. $e_1 + e_2 - 2e_3$) is a singular orbit. Hence from Proposition 2.7, this orbit is a weakly reflective submanifold in S. □

Proposition 4.6. *The orbit through the vector $e_1 + e_2 - e_3 - e_4$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i - e_j)\}$ of type A_3 is a weakly reflective submanifold in the hypersphere S.*

Proof. Put $H = e_1 + e_2 - e_3 - e_4$. The set R_+^Δ of all positive restricted roots which are orthogonal to H is given by

$$R_+^\Delta = \{e_1 - e_2,\ e_3 - e_4\}.$$

Let $s_{e_1-e_2}$ and $s_{e_3-e_4}$ be the reflections with respect to restricted roots $e_1 - e_2$ and $e_3 - e_4$, respectively. Then $s_{e_1-e_2}$ and $s_{e_3-e_4}$ are elements of the Weyl group, hence there exist $k_0, k_1 \in N_K$ such that

$$s_{e_1-e_2} = \mathrm{Ad}(k_0)|_{\mathfrak{a}}, \qquad s_{e_3-e_4} = \mathrm{Ad}(k_1)|_{\mathfrak{a}}.$$

We now define a linear isometry of $\mathfrak{m}$ by

$$\sigma(X) = \mathrm{Ad}(k_0)\mathrm{Ad}(k_1)X \qquad (X \in \mathfrak{m}).$$

Then σ satisfies

$$\sigma(H) = H, \qquad \sigma|_{\mathfrak{a}\cap H^\perp} = -1_{\mathfrak{a}\cap H^\perp}, \qquad \sigma(\mathrm{Ad}(K)H) = \mathrm{Ad}(K)H.$$

Thus from Lemma 4.3, $\mathrm{Ad}(K)H$ is a weakly reflective submanifold in S. □

Proposition 4.7. *The orbit through the vector e_1 of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm e_i \pm e_j\}$ of type D_l is a weakly reflective submanifold in the hypersphere S.*

Proof. An irreducible compact symmetric pair with the restricted root system of type D_l is one of $(SO(2l) \times SO(2l), SO(2l)^*)$ and $(SO(2l), SO(l) \times SO(l))$.

First we consider the case of $(SO(2l) \times SO(2l), SO(2l)^*)$. In this case, $\mathfrak{m}$ can be identified with $\mathfrak{o}(2l)$ in a natural manner. We take a maximal abelian subalgebra

$$\mathfrak{a} = \left\{ \mathrm{diag}\left\{ \begin{bmatrix} 0 & -t_1 \\ t_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -t_l \\ t_l & 0 \end{bmatrix} \right\} \;\middle|\; t_1, \dots, t_l \in \mathbf{R} \right\}$$

of $\mathfrak{o}(2l)$, and put

$$H = e_1 = \mathrm{diag}\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, 0, \dots, 0 \right\}.$$

We define a linear isometry σ of $\mathfrak{o}(2l)$ by

$$\sigma(X) = sXs \qquad (X \in \mathfrak{o}(2l)),$$

where

$$s = \mathrm{diag}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \dots, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \in O(2l).$$

Then σ is an isometry of S and satisfies

$$\sigma(H) = H, \qquad \sigma|_{\mathfrak{a}\cap H^\perp} = -\mathrm{id}_{\mathfrak{a}\cap H^\perp}, \qquad \sigma(\mathrm{Ad}(K)H) = \mathrm{Ad}(K)H.$$

Hence from Lemma 4.3, $\mathrm{Ad}(K)H$ is a weakly reflective submanifold in S.

Second, we consider the case of $(SO(2l), SO(l) \times SO(l))$. We take a maximal abelian subspace

$$\mathfrak{a} = \left\{ \begin{bmatrix} 0 & X \\ -X & 0 \end{bmatrix} \;\middle|\; X = \mathrm{diag}(t_1, \dots, t_l),\ t_i \in \mathbf{R} \right\},$$

and put

$$H = e_1 = \begin{bmatrix} 0 & X_0 \\ -X_0 & 0 \end{bmatrix} \in \mathfrak{o}(2l),$$

where

$$X_0 = \mathrm{diag}\{1, 0, \dots, 0\} \in M_l(\mathbf{R}).$$

We define a linear isometry σ of $\mathfrak{m}$ by

$$\sigma(X) = \begin{bmatrix} s & 0 \\ 0 & I_l \end{bmatrix} X \begin{bmatrix} s & 0 \\ 0 & I_l \end{bmatrix} \qquad (X \in \mathfrak{m}),$$

where

$$s = \mathrm{diag}\{1, -1, \dots, -1\} \in O(l).$$

Then σ is an isometry of S and satisfies

$$\sigma(H) = H, \qquad \sigma|_{\mathfrak{a}\cap H^\perp} = -\mathrm{id}_{\mathfrak{a}\cap H^\perp}, \qquad \sigma(\mathrm{Ad}(K)H) = \mathrm{Ad}(K)H.$$

Hence from Lemma 4.3, $\mathrm{Ad}(K)H$ is a weakly reflective submanifold in S. $\square$

Proposition 4.8. *The orbit through the vector $e_1 + e_2 + e_3 + e_4$ and the orbit through the vector $e_1+e_2+e_3-e_4$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm e_i \pm e_j\}$ of type D_4 is a weakly reflective submanifold in the hypersphere S.*

Proof. We take a fundamental system of $\{\pm e_i \pm e_j\}$ of type D_4:

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_3 + e_4.$$

The automorphism group of the Dynkin diagram is the permutation group of $\{\alpha_1, \alpha_3, \alpha_4\}$. So there exists an automorphism of $\mathfrak{m}$ mapping α_1 to α_4 and fixing α_3, which gives an equivalence of the orbits through e_1 and $e_1 + e_2 + e_3 + e_4$. Thus the orbit through $e_1 + e_2 + e_3 + e_4$ is also a weakly reflective submanifold in the hypersphere. Similarly the permutation of α_1 and α_3 gives an equivalence of the orbits through e_1 and $e_1+e_2+e_3-e_4$. Thus the orbit through $e_1+e_2+e_3-e_4$ is also a weakly reflective submanifold in the hypersphere. $\square$

In the case of $(SO(8), SO(4)\times SO(4))$ we can explicitly represent a reflection of the orbit though $e_1 + e_2 + e_3 + e_4$. The linear isotropy representation is equivalent to

$$(g_1, g_2)\cdot X = g_1 X g_2^{-1} \qquad ((g_1, g_2) \in SO(4)\times SO(4),\ X \in M_4(\mathbf{R})).$$

Let e_i denotes an element of $M_4(\mathbf{R})$ whose (i,i) component is 1 and others are 0. Then the orbit through $e_1 + e_2 + e_3 + e_4$ is $SO(4)$ in $M_4(\mathbf{R})$.

For $z_1 \otimes z_2 \in \mathbf{H}\otimes\mathbf{H}$, we define

$$\phi_{z_1\otimes z_2} : \mathbf{H} \to \mathbf{H}\ ;\ z \mapsto z_1 z \bar{z}_2.$$

Since $\mathbf{H} \cong \mathbf{R}^4$, we can regard $\phi_{z_1\otimes z_2}$ as an element of $M_4(\mathbf{R})$ and ϕ induces an isomorphism $M_4(\mathbf{R}) \cong \mathbf{H}\otimes\mathbf{H}$ of real algebras. We define an involutive isometry σ of $\mathbf{H}\otimes\mathbf{H}$ by

$$\sigma : \mathbf{H}\otimes\mathbf{H} \to \mathbf{H}\otimes\mathbf{H}\ ;\ z_1\otimes z_2 \mapsto z_1 \otimes \bar{z}_2.$$

We also denote by σ the linear isometry of $M_4(\mathbf{R})$ induced from σ through ϕ. We note that

$$SO(4) = \{\phi_{z_1\otimes z_2} \mid z_1, z_2 \in Sp(1)\}.$$

Moreover

$$\{z_1\otimes z_2 \mid z_1, z_2 \in Sp(1)\} \subset \mathbf{H}\otimes\mathbf{H}$$

is invariant under σ. Therefore $SO(4)$ is invariant under σ. The identity element I is fixed by the action of σ. The normal space of $SO(4)$ at I in $S^{15}(2)$ is given by

$$T_I^\perp(SO(4)) = \{X \in M_4(\mathbf{R}) \mid X : \text{symmetric},\ \mathrm{tr}(X) = 0\}.$$

It is easy to see that $T_I^\perp(SO(4))$ is contained in the eigenspace of σ for an eigenvalue -1. Thus σ is a reflection of $SO(4)$ with respect to an arbitrary normal vector at I.

5. Austere orbits of s-representations

In this section we classify all orbits of irreducible s-representations which are austere submanifolds in the hypersphere S. In the previous section we showed that all orbits through restricted root vectors (or their scalar multiples) are weakly reflective, hence austere. Therefore, hereafter we shall concern with other orbits. We will also determine austere orbits which are not weakly reflective submanifolds. Then we will complete to prove Theorem 4.1.

The classification of austere orbits is following:

Theorem 5.1. *An orbit of an irreducible s-representation which is an austere submanifold in the hypersphere is one of the following list:*

(1) *an orbit through a restricted root vector,*
(2) *the orbit through the vector $2e_1 - e_2 - e_3$ and the orbit through the vector $e_1 + e_2 - 2e_3$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i - e_j)\}$ of type A_2* (Proposition 4.5 or 5.4)*,*
(3) *the orbit through the vector $e_1+e_2-e_3-e_4$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i-e_j)\}$ of type A_3* (Proposition 5.4)*,*
(4) *the orbit through the vector e_1 of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i \pm e_j)\}$ of type D_l $(l \geq 4)$* (Proposition 5.5)*,*
(5) *the orbit through the vector $e_1+e_2+e_3+e_4$ and the orbit through the vector $e_1+e_2+e_3-e_4$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm(e_i \pm e_j)\}$ of type D_4* (Proposition 5.5)*,*
(6) *the orbit through the vector $e_1 + \frac{e_1+e_2}{\sqrt{2}}$ of the linear isotropy representation of a compact symmetric pair with the restricted root system $\{\pm e_i, \pm e_i \pm e_j\}$ of type B_2 whose multiplicities are constant* (Proposition 5.6)*,*
(7) *the orbit through the vector $\alpha_1 + \frac{\alpha_2}{\sqrt{3}}$ of the linear isotropy representation of a compact symmetric pair with the restricted root system of type G_2, where $\alpha_1 = e_1 - e_2$ and $\alpha_2 = -2e_1 + e_2 + e_3$* (Proposition 5.8)*.*

Remark 5.2. *In the case where the rank of the symmetric pair is equal to two, any principal orbit of an s-representation is an isoparametric hypersurface in the hypersphere. It is known that the number g of distinct principal curvatures of an isoparametric hypersurface in the sphere is one of* 1, 2, 3, 4 *and* 6*. There exists a unique minimal isoparametric hypersurface in each parallel family of isoparametric hypersurfaces. Theorems 4.1 and 5.1 shows some of minimal isoparametric hypersurfaces and their focal submanifolds are austere, furthermore weakly reflective, and some of them are not. More precisely;*

- *When $g = 1$, a minimal isoparametric hypersurface is nothing but a great sphere.*
- *When $g = 2$, a minimal isoparametric hypersurface is weakly reflective (austere) if and only if the multiplicities of two distinct principal curvatures are constant as in Example 2.3.*

- *When $g = 3$, that corresponds to the case of type A_2, a minimal isoparametric hypersurface is weakly reflective. Moreover both of its two focal submanifolds are also weakly reflective.*
- *When $g = 4$, that corresponds to the case of types $B_2 = C_2$ or BC_2, a minimal homogeneous isoparametric hypersurface is austere, but not weakly reflective, if and only if the multiplicities of two distinct principal curvatures are constant. Both of focal submanifolds are always weakly reflective.*
- *When $g = 6$, that corresponds to the case of type G_2, a minimal homogeneous isoparametric hypersurface is austere, but not weakly reflective. Both of its two focal submanifolds are weakly reflective.*

Before giving a proof of Theorem 5.1 we shall provide some preliminaries. Let (G, K) be an irreducible compact symmetric pair. We shall use the notations in previous sections. From Lemma 3.4, for a normal vector $\xi \in T_H^\perp(\mathrm{Ad}(K)H)$, the shape operator A_ξ of $\mathrm{Ad}(K)H$ in the hypersphere S is given by

$$\langle A_\xi(X^*), Y^*\rangle = \langle h(X^*, Y^*), \xi\rangle = \langle [Y, [X, H]], \xi\rangle = -\langle [X, H], [Y, \xi]\rangle. \tag{5.2}$$

For simplification, we discuss a normalization of a normal vector ξ. From Lemma 3.4, there exists $k \in Z_K^H$ such that $\mathrm{Ad}(k)\xi \in H^\perp \cap \mathfrak{a}$. Then

$$\begin{aligned}
\langle A_\xi(X^*), Y^*\rangle &= \langle \mathrm{Ad}(k)h(X^*, Y^*), \mathrm{Ad}(k)\xi\rangle \\
&= \langle h(\mathrm{Ad}(k)X^*, \mathrm{Ad}(k)Y^*), \mathrm{Ad}(k)\xi\rangle \\
&= \langle A_{\mathrm{Ad}(k)\xi}\mathrm{Ad}(k)X^*, \mathrm{Ad}(k)Y^*\rangle \\
&= \langle \mathrm{Ad}(k)^{-1}A_{\mathrm{Ad}(k)\xi}(\mathrm{Ad}(k)X^*), Y^*\rangle.
\end{aligned}$$

Thus we have

$$A_\xi = \mathrm{Ad}(k)^{-1}A_{\mathrm{Ad}(k)\xi}\mathrm{Ad}(k).$$

This implies that eigenvalues of $A_{\mathrm{Ad}(k)\xi}$ and their multiplicities coincide with those of A_ξ. Hence, in order to show whether an orbit $\mathrm{Ad}(K)H$ is austere, it suffices to check eigenvalues of A_ξ for $\xi \in H^\perp \cap \mathfrak{a}$. Hereafter we assume that $\xi \in H^\perp \cap \mathfrak{a}$.

From Lemmas 3.1 and 3.4

$$\{T_\alpha \mid \alpha \in \tilde{R}_+ - \tilde{R}_+^\Delta\}$$

is an orthonormal basis of $T_H(\mathrm{Ad}(K)H)$. For $\alpha, \beta \in \tilde{R}_+ - \tilde{R}_+^\Delta$ we have

$$\langle A_\xi((S_\alpha^*)_H), (S_\beta^*)_H\rangle = \langle \alpha, H\rangle\langle \beta, H\rangle\langle A_\xi(T_\alpha), T_\beta\rangle.$$

On the other hand, from (5.2), we have

$$\langle A_\xi((S_\alpha^*)_H), (S_\beta^*)_H\rangle = -\langle [S_\alpha, H], [S_\beta, \xi]\rangle = -\langle \alpha, H\rangle\langle \beta, \xi\rangle\delta_{\alpha\beta}.$$

Therefore we have

$$A_\xi(T_\alpha) = -\frac{\langle \alpha, \xi\rangle}{\langle \alpha, H\rangle}T_\alpha.$$

This shows that T_α is an eigenvector of A_ξ and its eigenvalue is

$$-\frac{\langle \alpha, \xi\rangle}{\langle \alpha, H\rangle}.$$

Hence $\mathrm{Ad}(K)H$ is an austere submanifold in S if and only if, for any $\xi \in H^\perp \cap \mathfrak{a}$, the set

$$\left\{ \frac{\langle \alpha, \xi\rangle}{\langle \alpha, H\rangle} \;\middle|\; \alpha \in \tilde{R}_+ - \tilde{R}_+^\Delta \right\}$$

with multiplicities is symmetric by the multiplication of -1. We shall describe this condition in terms of a finite subset of a Euclidean space.

Let A be a finite subset of a Euclidean space V. We consider a condition that, for any $v \in V$, the set

$$\{\langle a, v\rangle \mid a \in A\}$$

with multiplicities is symmetric by the multiplication of -1. This condition is equivalent to a condition that A is symmetric by the multiplication of -1 on V. Indeed, it is obvious that $\{\langle a, v\rangle \mid a \in A\}$ is symmetric whenever A is symmetric. Conversely, fix an arbitrary $a \in A$. From the assumption we have

$$V = \bigcup_{b\in A} \{v \in V \mid \langle a, v\rangle = -\langle b, v\rangle\}.$$

If $-a \notin A$, then the right hand side consists of finite union of hyperplanes of V. This is a contradiction. Therefore $-a \in A$. Consequently A is symmetric by the multiplication of -1 on V.

Let $p_H : \mathfrak{a} \to H^\perp \cap \mathfrak{a}$ denote the orthogonal projection. An orbit $\mathrm{Ad}(K)H$ is austere in S if and only if the set

$$\left\{ \frac{p_H(\alpha)}{\langle \alpha, H\rangle} \;\middle|\; \alpha \in R_+ - R_+^\Delta \right\}$$

with multiplicities is symmetric by the multiplication of -1 on $H^\perp \cap \mathfrak{a}$. By this criterion, we can easily see that orbits listed in Theorem 5.1 are austere submanifolds in the hypersphere S. Hereafter we shall prove that all austere orbits can be obtained in Theorem 5.1.

We set $\boldsymbol{R}R = \{x\alpha \mid x \in \boldsymbol{R}, \alpha \in R\}$. We have already showed that the orbit through any element in $\boldsymbol{R}R$ is weakly reflective in the hypersphere, so we consider the orbits through elements in $\mathfrak{a} - \boldsymbol{R}R$

Lemma 5.3. *For $H \in \mathfrak{a} - \boldsymbol{R}R$, the orbit $\mathrm{Ad}(K)H$ is an austere submanifold in S if and only if there exist a mapping $f : R_+ - R_+^\Delta \to R_+ - R_+^\Delta$ without fixed points, and constants $n_\alpha \neq 0, \epsilon_\alpha = \pm 1$ for each $\alpha \in R_+ - R_+^\Delta$ such that*

$$H = n_\alpha \left(\frac{\alpha}{\|\alpha\|} + \epsilon_\alpha \frac{f(\alpha)}{\|f(\alpha)\|} \right), \tag{5.3}$$

and

$$\sum_{\substack{\mu \in R_+ - R_+^\Delta \\ \mu // \alpha}} m(\mu) = \sum_{\substack{\nu \in R_+ - R_+^\Delta \\ \nu // f(\alpha)}} m(\nu). \tag{5.4}$$

Here we denote by $m(\mu)$ the multiplicity of a restricted root μ, and $\mu // \alpha$ means that μ and α are linearly dependent.

Excepting the case where the restricted root system R is of type BC, the equality (5.4) is equivalent to $m(\alpha) = m(f(\alpha))$, moreover $\#(R_+ - R_+^\Delta)$ is even and $f^2 = 1$.

Proof. The orthogonal projection p_H is defined by

$$p_H(X) = X - \frac{\langle X, H\rangle}{\langle H, H\rangle} H \qquad (X \in \mathfrak{a}).$$

Therefore $\mathrm{Ad}(K)H$ is an austere submanifold in S if and only if the set

$$\left\{ \frac{\alpha}{\langle \alpha, H\rangle} - \frac{H}{\langle H, H\rangle} \;\middle|\; \alpha \in R_+ - R_+^\Delta \right\}$$

with multiplicities is symmetric by the multiplication of -1 on $H^\perp \cap \mathfrak{a}$. In other words, there exists a mapping $f : R_+ - R_+^\Delta \to R_+ - R_+^\Delta$ which satisfies

$$\frac{f(\alpha)}{\langle f(\alpha), H\rangle} - \frac{H}{\langle H, H\rangle} = -\frac{\alpha}{\langle \alpha, H\rangle} + \frac{H}{\langle H, H\rangle} \tag{5.5}$$

and

$$\begin{aligned} &\sum \left\{ m(\mu) \,\middle|\, \mu \in R_+ - R_+^\Delta,\ \frac{\mu}{\langle \mu, H\rangle} - \frac{H}{\langle H, H\rangle} = \frac{\alpha}{\langle \alpha, H\rangle} - \frac{H}{\langle H, H\rangle} \right\} \\ = &\sum \left\{ m(\nu) \,\middle|\, \nu \in R_+ - R_+^\Delta,\ \frac{\nu}{\langle \nu, H\rangle} - \frac{H}{\langle H, H\rangle} = \frac{f(\alpha)}{\langle f(\alpha), H\rangle} - \frac{H}{\langle H, H\rangle} \right\} \end{aligned}$$

for any $\alpha \in R_+ - R_+^\Delta$. This condition for the multiplicities is equivalent to (5.4). From (5.5), if f has a fixed point α, then $H \in \boldsymbol{R}R$. Thus f has no fixed points.

If we assume (5.5), then there exist non-zero real numbers x, y so that $H = x\alpha + yf(\alpha)$. Applying this to the equation (5.5), we have a quadratic equation

$$\|f(\alpha)\|^2 y^2 = \|\alpha\|^2 x^2$$

with respect to x and y. Thus we have

$$y = \pm \frac{\|\alpha\|}{\|f(\alpha)\|} x,$$

hence H can be expressed as

$$H = x\alpha \pm \frac{\|\alpha\|}{\|f(\alpha)\|} x f(\alpha) = x\|\alpha\| \left(\frac{\alpha}{\|\alpha\|} \pm \frac{f(\alpha)}{\|f(\alpha)\|} \right).$$

Since this equality holds for any $\alpha \in R_+ - R_+^\Delta$, we have the condition (5.3).

Replacing α in (5.5) by $f(\alpha)$, we have

$$\frac{f^2(\alpha)}{\langle f^2(\alpha), H\rangle} + \frac{f(\alpha)}{\langle f(\alpha), H\rangle} = \frac{2H}{\langle H, H\rangle}. \tag{5.6}$$

From equations (5.5) and (5.6), we have

$$\frac{\alpha}{\langle \alpha, H\rangle} = \frac{f^2(\alpha)}{\langle f^2(\alpha), H\rangle}.$$

The above discussion stands for any restricted root systems R, including of type BC. Henceforth we assume that R is not of type BC. Then α is the only element of $R_+ - R_+^\Delta$ which is a scalar multiple of α. Thus $f^2(\alpha) = \alpha$. Since f has no fixed points, $\#(R_+ - R_+^\Delta)$ is even. This completes the proof. □

Here we mention some results concerning with the Weyl group needed later. The action of the Weyl group maps the restricted root system R onto itself ([5]). We can see that, for any restricted roots α and β with the same length, there exists an element s in the Weyl group such that $\beta = s\alpha$ by the classification of the restricted root systems. In this case, $m(\beta) = m(\alpha)$ holds. In particular, if the restricted root system R is one of the types A_l, D_l, E_6, E_7 and E_8, then the Weyl group acts transitively on R and all restricted roots have constant multiplicities, since all restricted roots have the same length.

Proposition 5.4. *In the case where R is of type A_l, an austere orbit is one of the following except orbits through a restricted root vector:*

(1) *when* $l = 2$, *the orbit through* $H = 2e_1 - e_2 - e_3$ *and the orbit through* $e_1 + e_2 - 2e_3$,
(2) *when* $l = 3$, *the orbit through* $H = e_1 + e_2 - e_3 - e_4$.

Proof. In the case of $R = A_l$, R_+ is given by $R_+ = \{e_i - e_j \mid i < j\}$. Since all restricted roots have constant multiplicities, the condition (5.4) of Lemma 5.3 is always satisfied. From Lemma 5.3, without loss of generalities, we may assume that $H = \pm$(a positive root) $\pm$ (a positive root), since all restricted roots have the same length. Moreover since any root can be translated to $e_1 - e_2$ by the action of the Weyl group, we may assume that $H = (e_1 - e_2) \pm$(a positive root). The positive root in the second term of H is one of $e_1 - e_i$ $(3 \leq i), e_2 - e_j$ $(3 \leq i), e_i - e_j$ $(3 \leq i < j)$.

In the case of $H = (e_1 - e_2) \pm (e_1 - e_i)$ $(3 \leq i)$, e_i can be translated to e_3 by the action of an element of the Weyl group which fixes both e_1 and e_2. Therefore we can put

$$H = (e_1 - e_2) \pm (e_1 - e_3) = \begin{cases} (e_1 - e_2) + (e_1 - e_3) = 2e_1 - e_2 - e_3 \\ (e_1 - e_2) - (e_1 - e_3) = -e_2 + e_3 \quad \text{(root)} \end{cases}$$

Similarly, in the case of $H = (e_1 - e_2) \pm (e_2 - e_i)$ $(3 \leq i)$, we can put

$$H = (e_1 - e_2) \pm (e_2 - e_3) = \begin{cases} e_1 - 2e_2 + e_3 \sim e_1 + e_2 - 2e_3 \\ e_1 - e_3 \quad \text{(root)} \end{cases}$$

Here, for $H_1, H_2 \in \mathfrak{a}$, we express $H_1 \sim H_2$ when H_1 can be translated to H_2 by some element of K. In other words, H_1 is equivalent to H_2 under the action of the Weyl group.

In the case of $H = (e_1 - e_2) \pm (e_i - e_j)$ $(3 \leq i < j)$, there exists an element of the Weyl group which fixes e_1, e_2 and translates e_i to e_3 and e_j to e_4. Therefore we can put

$$H = (e_1 - e_2) \pm (e_3 - e_4) = \begin{cases} e_1 + e_3 - e_2 - e_4 \\ e_1 + e_4 - e_2 - e_3 \end{cases}$$

By the action of the Weyl group, these vectors are equivalent to $e_1 + e_2 - e_3 - e_4$. Consequently, it suffices to consider orbits through

$$H = 2e_1 - e_2 - e_3,\ e_1 + e_2 - 2e_3,\ e_1 + e_2 - e_3 - e_4,$$

which have a possibility to be austere.

In the case of $H = 2e_1 - e_2 - e_3$, the only possibility to be the form $H =$ (a positive root) $\pm$ (a positive root) is $H = (e_1 - e_2) + (e_1 - e_3)$. Thus, from Lemma 5.3, the set $R_+ - R_+^{\Delta}$ must be

$$R_+ - R_+^{\Delta} = \{e_1 - e_2,\ e_1 - e_3\}.$$

When $l \geq 3$, since $\langle e_3 - e_4, H \rangle \neq 0$, we have $e_3 - e_4 \in R_+ - R_+^{\Delta}$. This is a contradiction. Hence $l = 2$ and then $\mathrm{Ad}(K)H$ is austere in S. Similarly, the orbit through $H = e_1 + e_2 - 2e_3$ is also austere.

In the case of $H = e_1 + e_2 - e_3 - e_4$, possibilities to be the form $H =$ (a positive root)$\pm$ (a positive root) are

$$H = (e_1 - e_3) + (e_2 - e_4) = (e_1 - e_4) + (e_2 - e_3).$$

Thus $R_+ - R_+^{\Delta}$ must satisfy

$$R_+ - R_+^{\Delta} \subset \{e_1 - e_3,\ e_2 - e_4,\ e_1 - e_4,\ e_2 - e_3\}$$

When $l \geq 4$, since $\langle e_4 - e_5, H\rangle \neq 0$, we have $e_4 - e_5 \in R_+ - R_+^\Delta$. This is a contradiction. Hence $l = 3$, and then $\mathrm{Ad}(K)H$ is austere in S. □

Proposition 5.5. *In the case where R is of type D_l, an austere orbit is one of the following except orbits through a restricted root vector:*

(1) *the orbit through $H = e_1$,*
(2) *when $l = 4$, the orbits through $H = e_1 + e_2 + e_3 + e_4$, $e_1 + e_2 + e_3 - e_4$.*

Proof. In the case of $R = D_l$, R_+ is given by $R_+ = \{e_i \pm e_j \mid i < j\}$. Since all restricted roots have constant multiplicities, the condition (5.4) of Lemma 5.3 is always satisfied. It is easy to see that the orbit through e_1 (or its scalar multiple) is austere. Therefore we consider other orbits. From Lemma 5.3, we can assume $H = \pm$(a positive root) $\pm$ (a positive root), since all restricted roots have the same length. Since any root can be translated to $e_1 + e_2$ by the action of the Weyl group, we can assume $H = (e_1 + e_2) \pm$ (a positive root). Furthermore any root can be translated to one of

$$e_1 \pm e_2,\ e_1 + e_3,\ e_2 + e_4,\ e_3 + e_4,\ e_3 - e_4$$

by the action of elements of the Weyl group which fix e_1, e_2. Therefore H is one of the following:

$$\begin{aligned}
H &= (e_1 + e_2) \pm (e_1 - e_2) = 2e_1,\ 2e_2 \sim 2e_1,\\
H &= (e_1 + e_2) \pm (e_1 + e_3) = \begin{cases} 2e_1 + e_2 + e_3, \\ e_2 - e_3 \text{ (root)}, \end{cases}\\
H &= (e_1 + e_2) \pm (e_2 + e_4) = \begin{cases} 2e_2 + e_1 + e_4 \sim 2e_1 + e_2 + e_3, \\ e_1 - e_4 \text{ (root)}, \end{cases}\\
H &= (e_1 + e_2) \pm (e_3 + e_4) \sim e_1 + e_2 + e_3 + e_4,\\
H &= (e_1 + e_2) \pm (e_3 - e_4) \sim e_1 + e_2 + e_3 - e_4.
\end{aligned}$$

Consequently, it suffices to consider orbits through

$$H = 2e_1 + e_2 + e_3,\ e_1 + e_2 + e_3 + e_4,\ e_1 + e_2 + e_3 - e_4$$

which have a possibility to be austere.

In the case of $H = 2e_1 + e_2 + e_3$, the only possibility to be the form $H =$ (a positive root) $\pm$ (a positive root) is

$$H = (e_1 + e_2) + (e_1 + e_3).$$

Thus $R_+ - R_+^\Delta$ must be

$$R_+ - R_+^\Delta = \{e_1 + e_2,\ e_1 + e_3\}.$$

Since $\langle e_1 - e_2, H\rangle \neq 0$, we have $e_1 - e_2 \in R_+ - R_+^\Delta$. This is a contradiction. Hence this orbit is not austere.

In the case of $H = e_1+e_2+e_3+e_4$, possibilities of the form $H =$ (a positive root)$\pm$ (a positive root) are

$$H = (e_1 + e_2) + (e_3 + e_4) = (e_1 + e_3) + (e_2 + e_4) = (e_1 + e_4) + (e_2 + e_3).$$

Thus $R_+ - R_+^\Delta$ must satisfy

$$R_+ - R_+^\Delta \subset \{e_1 + e_2,\ e_3 + e_4,\ e_1 + e_3,\ e_2 + e_4,\ e_1 + e_4,\ e_2 + e_3\}.$$

When $l \geq 5$, since $\langle e_4 + e_5, H\rangle \neq 0$, we have $e_4 + e_5 \in R_+ - R_+^\Delta$. This is a contradiction. Hence $l = 4$, and then the orbit $\mathrm{Ad}(K)H$ is austere in S. In the case of $H = e_1 + e_2 + e_3 - e_4$, similarly we have $l = 4$, and then $\mathrm{Ad}(K)H$ is austere. □

Proposition 5.6. *In the case where R is one of types B_l, C_l and BC_l, an austere orbit except orbits through restricted root vectors is the following:*

When $R = B_2$ where the multiplicities of the restricted roots are constant, the orbit through

$$H = e_1 + \frac{e_1 + e_2}{\sqrt{2}}$$

is austere. This orbit is a principal orbit.

Remark 5.7. In the case of $R = B_2$, there exist two singular orbits and these are not isometric. Hence from Proposition 2.9, a principal austere orbit in Proposition 5.6 is not a weakly reflective submanifold.

Proof. First we consider the case of $R = B_l$, where $R_+ = \{e_i,\ e_i \pm e_j \mid i < j\}$. From Lemma 5.3, we can assume

$$H = \frac{\alpha}{\|\alpha\|} \pm \frac{\beta}{\|\beta\|} \quad (\alpha, \beta \in R_+).$$

i) When α and β are both short roots, we can put $H = \alpha \pm \beta$. Furthermore, since any short root α can be translated to e_1 by the action of the Weyl group, we can assume $H = e_1 \pm \beta$. If $\beta = e_1$, then $H = 2e_1$ and this is equivalent to the orbit through a root vector. If $\beta = e_j$ $(j \geq 2)$, then $H = e_1 \pm e_j$ is a root vector.

ii) When α and β are both long roots, we can put $H = \alpha \pm \beta$. Since any long root α can be translated to $e_1 + e_2$ by the action of the Weyl group, we can assume $H = (e_1 + e_2) \pm \beta$. Furthermore β can be translated to one of

$$\beta = e_1 \pm e_2,\ e_1 + e_3,\ e_2 + e_3,\ e_3 + e_4$$

by the action of elements of the Weyl group which fix e_1 and e_2.

In the case of $\beta = e_1 \pm e_2$, H is equivalent to a root or zero vector.

In the case of $\beta = e_1 + e_3$,

$$H = (e_1 + e_2) \pm (e_1 + e_3) = \begin{cases} 2e_1 + e_2 + e_3, \\ e_2 - e_3 \quad (\text{root}). \end{cases}$$

When $H = 2e_1 + e_2 + e_3$, the only possibility to be the form $H = (\text{a positive root}) \pm (\text{a positive root})$ is $H = (e_1 + e_2) + (e_1 + e_3)$. Thus $R_+ - R_+^\Delta$ must be $R_+ - R_+^\Delta = \{e_1 + e_2,\ e_1 + e_3\}$. On the other hand, since $\langle e_1, H\rangle \neq 0$, we have $e_1 \in R_+ - R_+^\Delta$. This is a contradiction. Hence this orbit is not austere.

In the case of $\beta = e_2 + e_3$,

$$H = (e_1 + e_2) \pm (e_2 + e_3) = \begin{cases} e_1 + 2e_2 + e_3 \sim 2e_1 + e_2 + e_3, \\ e_1 - e_3 \quad (\text{root}). \end{cases}$$

In the case of $\beta = e_3 + e_4$,

$$H = (e_1 + e_2) \pm (e_3 + e_4) = \begin{cases} e_1 + e_2 + e_3 + e_4, \\ e_1 + e_2 - e_3 - e_4 \sim e_1 + e_2 + e_3 + e_4. \end{cases}$$

In this case,

$$H = (e_1 + e_2) + (e_3 + e_4) = (e_1 + e_3) + (e_2 + e_4) = (e_1 + e_4) + (e_2 + e_3)$$

are possibilities to be the form $H = \alpha \pm \beta$. Thus $R_+ - R_+^\Delta$ must satisfy

$$R_+ - R_+^\Delta \subset \{e_1 + e_2,\ e_3 + e_4,\ e_1 + e_3,\ e_2 + e_4,\ e_1 + e_4,\ e_2 + e_3\}.$$

On the other hand, since $\langle e_1, H\rangle \neq 0$, we have $e_1 \in R_+ - R_+^\Delta$. This is a contradiction. Hence this orbit is not austere.

iii) When α is a short root and β is a long root, we can assume $\alpha = e_1$ and

$$H = e_1 \pm \frac{\beta}{\sqrt{2}} \quad \text{where} \quad \beta = e_1 + e_2,\ e_2 + e_3.$$

In the case of $H = e_1 + \frac{e_1+e_2}{\sqrt{2}}$, if $l \geq 3$, then $e_3 \in R_+ - R_+^\Delta$. On the other hand, there is no $\mu \in R_+$ such that

$$H = n\left(e_3 \pm \frac{\mu}{\|\mu\|}\right).$$

Thus we have $l = 2$. In this case $R_+ - R_+^\Delta = \{e_1,\ e_2,\ e_1 + e_2,\ e_1 - e_2\}$. If we define $f : R_+ - R_+^\Delta \to R_+ - R_+^\Delta$ by

$$f(e_1) = e_1 + e_2, \qquad f(e_2) = e_1 - e_2,$$

then H satisfies the condition (5.3) of Lemma 5.3. Hence this orbit is austere if the multiplicities of the restricted roots are constant.

In the case of $H = e_1 - \frac{e_1+e_2}{\sqrt{2}}$, we can express H as

$$H = -\frac{1}{\sqrt{2}+1}\left\{\left(1 + \frac{1}{\sqrt{2}}\right)e_2 - \frac{1}{\sqrt{2}}e_1\right\}.$$

Permuting e_1 and e_2 by the action of the Weyl group and replacing $e_2 \mapsto -e_2$, we have that this orbit is equivalent to the orbit through

$$H = e_1 + \frac{e_1 + e_2}{\sqrt{2}}.$$

In the case of $\beta = e_2 + e_3$,

$$H = e_1 \pm \frac{e_2 + e_3}{\sqrt{2}} \sim e_1 + \frac{e_2 + e_3}{\sqrt{2}}.$$

In this case $e_3 \in R_+ - R_+^\Delta$. On the other hand, there is no $\mu \in R_+$ such that

$$H = n\left(e_3 \pm \frac{\mu}{\|\mu\|}\right).$$

Hence this orbit is not austere.

Second we consider the case of $R = C_l$, where $R_+ = \{2e_i,\ e_i \pm e_j \mid i < j\}$. For this purpose we shall use the dual mapping and transfer the result of the case $R = B_l$ by the dual mapping. A mapping

$$\mathfrak{a} - \{0\} \to \mathfrak{a} - \{0\}; H \mapsto H^* = \frac{2H}{\langle H, H\rangle}$$

is called a dual mapping. This maps a root system to a root system, more precisely, a long root is moved to a short root and a short root is moved to a long root. The root systems of type B_l and C_l are dual by this mapping, and other irreducible root systems are self-dual. If there exists f which satisfies (5.3) for $H \in \mathfrak{a}$, then there exists f^* which satisfies (5.3) for H^*.

In the above discussion, in the case of $R = B_l$ $(l \geq 3)$, we showed that there are no austere orbits except orbits through a restricted root vector. Thus we also have that there are no austere orbits except orbits through a restricted root vector in the case of $R = C_l$ $(l \geq 3)$. When $l = 2$, $C_2 = B_2$.

Finally we shall consider the case of $R = BC_l$, where $R_+ = \{e_i,\ 2e_i,\ e_i \pm e_j \mid i < j\}$. From Lemma 5.3, we can put

$$H = \frac{\alpha}{\|\alpha\|} \pm \frac{\beta}{\|\beta\|} \quad (\alpha, \beta \in R_+).$$

If $\|\alpha\| = \|\beta\|$, then we can put $H = \alpha \pm \beta$. When α and β are both short roots or both long roots, H is a scalar multiple of a root vector. When α and β are both middle roots, we can assume $\alpha = e_1 + e_2$ and

$$H = (e_1 + e_2) \pm \beta \quad (\beta = e_1 + e_3,\ e_2 + e_3,\ e_3 + e_4).$$

By the action of the Weyl group, these are equivalent to

$$H = 2e_1 + e_2 + e_3, \quad H = e_1 + e_2 + e_3 + e_4$$

or a restricted root vector. In the case of $H = 2e_1 + e_2 + e_3$, we have $l = 3$ and

$$R_+ - R_+^\Delta = \{e_1,\ e_2,\ e_3,\ 2e_1,\ 2e_2,\ 2e_3,\ e_1 \pm e_2,\ e_1 \pm e_3,\ e_2 + e_3\}.$$

Since there is no f which satisfies (5.3), this orbit is not austere. In the case of $H = e_1 + e_2 + e_3 + e_4$, we have $l = 4$ and

$$\begin{aligned} R_+ - R_+^\Delta &= \{e_i, 2e_i \mid 1 \leq i \leq 4\} \\ &\quad \cup \{e_1 + e_2,\ e_1 + e_3,\ e_1 + e_4,\ e_2 + e_3,\ e_2 + e_4,\ e_3 + e_4\}. \end{aligned}$$

Since there is no f which satisfies (5.3), this orbit is not austere.

It remains the case where $\|\alpha\| < \|\beta\|$. When α is a short root and β is a long root, H is a scalar multiple of a root vector. By the dual mapping, we can identify two cases, where α is a short root and β is a middle root, and where α is a middle root and β is a long root. Therefore we shall discuss the former. In this case, we can assume $\alpha = e_1$ and

$$H = e_1 \pm \frac{\beta}{\sqrt{2}} \quad \text{where} \quad \beta = e_1 + e_2,\ e_2 + e_3.$$

Similarly with the case of the restricted root system of type B,

$$H = e_1 + \frac{e_1 + e_2}{\sqrt{2}}, \qquad H = e_1 + \frac{e_2 + e_3}{\sqrt{2}}$$

have a possibility to be austere. When $H = e_1 + \frac{e_2+e_3}{\sqrt{2}}$, we have $l = 3$ and

$$R_+ - R_+^\Delta = \{e_1,\ e_2,\ e_3,\ 2e_1,\ 2e_2,\ 2e_3,\ e_1 \pm e_2,\ e_1 \pm e_3,\ e_2 + e_3\}.$$

Since there is no f which satisfies (5.3), this orbit is not austere. When $H = e_1 + \frac{e_1+e_2}{\sqrt{2}}$, the orbit has a possibility to be austere if $l = 2$. In this case, the orbit is a principal orbit. This orbit is austere if the sum of the multiplicities of long roots and short roots coincides with the multiplicity of middle roots. From the classification of symmetric pairs, there does not exist such a symmetric pair. $\square$

Proposition 5.8. *In the case where R is of type G_2, the orbit through*

$$H = \alpha_1 + \frac{\alpha_2}{\sqrt{3}}$$

is the only austere orbit except orbits through a restricted root vector. This orbit is a principal orbit.

Remark 5.9. By the same discussion in Remark 5.7, this principal austere orbit is not a weakly reflective submanifold from Proposition 2.9.

Proof. The fundamental system F of the restricted root system of type G_2 is given by $F = \{\alpha_1 = e_1 - e_2,\ \alpha_2 = -2e_1 + e_2 + e_3\}$ and the set R_+ of positive roots is

$$R_+ = F \cup \left\{ \begin{array}{l} \alpha_1 + \alpha_2 = -e_1 + e_3,\ 2\alpha_1 + \alpha_2 = -e_2 + e_3, \\ 3\alpha_1 + \alpha_2 = e_1 - 2e_2 + e_3,\ 3\alpha_1 + 2\alpha_2 = -e_1 - e_2 + 2e_3 \end{array} \right\}.$$

In the case of G_2, since all restricted roots have constant multiplicities, the condition (5.4) of Lemma 5.3 is always satisfied. From Lemma 5.3, we can put

$$H = \frac{\alpha}{\|\alpha\|} \pm \frac{\beta}{\|\beta\|} \quad (\alpha, \beta \in R_+).$$

When α and β are both short roots, we can put $H = \alpha \pm \beta$ $(\alpha \neq \beta)$. Furthermore, since any short root α can be translated to α_1 by the action of the Weyl group, we can assume

$$H = \alpha_1 \pm \beta \quad \text{where} \quad \beta = \alpha_1 + \alpha_2,\ 2\alpha_1 + \alpha_2.$$

In the case of $\beta = \alpha_1 + \alpha_2$,

$$H = \alpha_1 \pm (\alpha_1 + \alpha_2) = 2\alpha_1 + \alpha_2,\ -\alpha_2.$$

Then H is a root vector. In the case of $\beta = 2\alpha_1 + \alpha_2$,

$$H = \alpha_1 \pm (2\alpha_1 + \alpha_2) = 3\alpha_1 + \alpha_2,\ -\alpha_1 - \alpha_2.$$

Then H is a root vector.

When α and β are both long roots, we can put $H = \alpha \pm \beta$ $(\alpha \neq \beta)$. Since any long root α can be translated to α_2 by the action of the Weyl group, we can assume

$$H = \alpha_2 \pm \beta \quad \text{where} \quad \beta = 3\alpha_1 + \alpha_2,\ 3\alpha_1 + 2\alpha_2.$$

In the case of $\beta = 3\alpha_1 + \alpha_2$,

$$H = \alpha_2 \pm (3\alpha_1 + \alpha_2) = -3\alpha_1,\ 3\alpha_1 + 2\alpha_2.$$

Then H is a scalar multiple of a root vector. In the case of $\beta = 3\alpha_1 + 2\alpha_2$,

$$H = \alpha_2 \pm (3\alpha_1 + 2\alpha_2) = -3\alpha_1 - \alpha_2,\ 3(\alpha_1 + \alpha_2).$$

Then H is a scalar multiple of a root vector.

When α is a short root and β is a long root, we can assume $\alpha = \alpha_1$ and

$$H = \alpha_1 \pm \frac{\beta}{\sqrt{3}} \quad \text{where} \quad \beta = \alpha_2,\ 3\alpha_1 + \alpha_2,\ 3\alpha_1 + 2\alpha_2.$$

We note that the orbit though H is a principal orbit. In these cases, H is equivalent to a scalar multiple of $\alpha_1 + \frac{\alpha_2}{\sqrt{3}}$ by the action of the Weyl group. In the case of $H = \alpha_1 + \frac{\alpha_2}{\sqrt{3}}$,

$$H = \frac{1}{\sqrt{3}+1}\left(\alpha_1 + \alpha_2 + \frac{3\alpha_1 + \alpha_2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}+2}\left(2\alpha_1 + \alpha_2 + \frac{3\alpha_1 + 2\alpha_2}{\sqrt{3}}\right).$$

Thus from Lemma 5.3 this orbit is austere. This completes the proof. □

Proposition 5.10. *In the case of $R = F_4$, there are no austere orbits except orbits through a restricted root vector.*

Proof. In this case R_+ is given by

$$R_+ = \{e_i\}_{1\le i\le 4} \cup \{e_i \pm e_j\}_{1\le i<j\le 4} \cup \left\{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\right\}$$

From Lemma 5.3, we can assume

$$H = \frac{\alpha}{\|\alpha\|} \pm \frac{\beta}{\|\beta\|} \quad (\alpha, \beta \in R_+).$$

When α and β are both short roots, we can put $H = \alpha \pm \beta$. In this case, since any short root can be translated to e_1 by the action of the Weyl group, we can put

$$H = e_1 \pm \beta \quad \text{where} \quad \beta = e_i \ (i \ge 2),\ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4).$$

In the case of $\beta = e_i$, H is a root vector. In the case of $\beta = \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$.

$$H = e_1 \pm \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) = \begin{cases} e_1 + \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4), \\ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \quad \text{(a root)}. \end{cases}$$

Thus we consider the case of $H = e_1 + \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$. In this case, $\langle e_4, H\rangle \ne 0$, however, there does not exist $\beta \in R_+$ such that

$$H = n\left(e_4 \pm \frac{\beta}{\|\beta\|}\right).$$

Hence from Lemma 5.3 this orbit is not austere.

When α and β are both long root, we can assume $H = e_1 + e_2 \pm e_i \pm e_j$ $(i < j)$. Moreover, we exclude H which is a scalar multiple of a root vector. Then

$$H = 2e_1 + e_2 \pm e_i,\ e_1 + 2e_2 \pm e_i \ (i = 3, 4).$$

The reflection $s_{e_3 - e_4}$ permutes e_3 and e_4, and fixes e_1, e_2. Therefore we can put

$$H = 2e_1 + e_2 \pm e_3.$$

In this case, $\langle e_3, H\rangle \ne 0$, however, there does not exist $\beta \in R_+$ such that

$$H = n\left(e_3 \pm \frac{\beta}{\|\beta\|}\right).$$

Hence this orbit is not austere.

When α is a short root and β is a long root, we can put

$$H = e_1 + \frac{\pm e_i \pm e_j}{\sqrt{2}} \quad (i < j).$$

Moreover, by the action of the Weyl group we can assume

$$H = e_1 + \frac{\pm e_1 + e_2}{\sqrt{2}} \quad \text{or} \quad H = e_1 + \frac{e_2 + e_3}{\sqrt{2}}.$$

In the case of $H = e_1 + \frac{e_2 + e_3}{\sqrt{2}}$, we have $\langle e_3, H\rangle \ne 0$. However, there does not exist $\beta \in R_+$ such that

$$H = n\left(e_3 \pm \frac{\beta}{\|\beta\|}\right).$$

Hence this orbit is not austere. In the case of $H = e_1 + \frac{\pm e_1 + e_2}{\sqrt{2}}$, we have $\langle e_2 + e_3, H\rangle \ne 0$. However, there does not exist $\beta \in R_+$ such that

$$H = n\left(\frac{e_2 + e_3}{\sqrt{2}} + \frac{\beta}{\|\beta\|}\right).$$

Hence this orbit is not austere. $\square$

Proposition 5.11. *In the case of $R = E_8$, there are no austere orbits except the orbits through a restricted root vector.*

Proof. In the case of $R = E_8$, R_+ is given by

$$R_+ = \{\pm e_i + e_j \mid 1 \le i < j \le 8\} \cup \left\{ \frac{1}{2}(e_8 + \sum_{i=1}^{7}(-1)^{\nu(i)} e_i) \;\middle|\; \sum_{i=1}^{7} \nu(i) \in 2\boldsymbol{Z} \right\}$$

Since all restricted roots have constant multiplicities, the condition (5.4) of Lemma 5.3 is always satisfied. From Lemma 5.3, we can put $H = e_1 + e_2 + \beta$ where

$$\beta = \begin{cases} \pm e_1 \pm e_2, & \\ \pm e_1 \pm e_i & (3 \le i \le 8), \\ \pm e_2 \pm e_i & (3 \le i \le 8), \\ \pm e_i \pm e_j & (3 \le i < j \le 8), \\ \frac{1}{2}\sum_{i=1}^{\nu(i)}(-1)^{\nu(i)} e_i & (\sum_{i=1}^{8} \nu(i) \in 2\boldsymbol{Z}). \end{cases}$$

i) In the case of $\beta = \pm e_1 \pm e_2$,

$$H = \begin{cases} e_1 + e_2 + e_1 + e_2 = 2(e_1 + e_2) & \text{(twice of a root)}, \\ e_1 + e_2 + e_1 - e_2 = 2e_1, & \\ e_1 + e_2 - e_1 + e_2 = 2e_2, & \\ e_1 + e_2 - e_1 - e_2 = 0. & \end{cases}$$

When $H = 2e_1$,

$$R - R^{\Delta} = \{\pm e_1 \pm e_j \mid 2 \le j \le 8\} \cup \left\{ \pm\frac{1}{2}(e_1 + \sum_{i=2}^{8}(-1)^{\nu(i)} e_i) \;\middle|\; \sum_{i=2}^{8} \nu(i) \in 2\boldsymbol{Z} \right\}$$

Then there does not exist $\beta \in R_+$ such that $H = n(\frac{1}{2}\sum_{i=1}^{8} e_i \pm \beta)$. Hence this orbit is not austere. When $H = 2e_2$, since the reflection $s_{e_1 - e_2}$ permutes e_1 and e_2, this orbit is equivalent to the orbit through $H = 2e_1$. Hence this orbit is not austere.

ii) In the case of $\beta = \pm e_1 \pm e_i$ $(3 \le i \le 8)$,

$$H = \begin{cases} 2e_1 + e_2 \pm e_i, \\ e_2 \pm e_i \quad \text{(a root)}. \end{cases}$$

The reflection $s_{e_3 - e_i}$ $(i \ge 4)$ fixes e_1, e_2 and permutes e_3 and e_i. Thus the orbit through $H = 2e_1 + e_2 \pm e_i$ is equivalent to the orbit through $H = 2e_1 + e_2 \pm e_3$. Then $\langle e_1 + e_4, H\rangle \ne 0$, however, there does not exist $\beta \in R_+$ such that $H = n(e_1 + e_4 \pm \beta)$. Hence this orbit is not austere.

iii) In the case of $\beta = \pm e_2 \pm e_i$ $(3 \le i \le 8)$,

$$H = \begin{cases} e_1 + 2e_2 \pm e_i, \\ e_1 \pm e_i \quad \text{(a root)}. \end{cases}$$

By the action of the Weyl group, the orbit through $H = e_1 + 2e_2 \pm e_i$ is equivalent to the orbit through $H = 2e_1 + e_2 \pm e_i$. Hence this orbit is not austere.

iv) In the case of $\beta = \pm e_i \pm e_j$ $(3 \le i < j \le 8)$, we can assume

$$H = e_1 + e_2 \pm e_3 \pm e_4.$$

Then $\langle e_1 + e_5, H\rangle \ne 0$, however, there does not exist $\beta \in R_+$ such that $H = n(e_1 + e_5 \pm \beta)$. Thus this orbit is not austere.

v) v-1) In the case of

$$\beta = \frac{1}{2}(-e_1 - e_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}e_i), \quad \sum_{i=3}^{8}\nu(i) \in 2\boldsymbol{Z},$$

then

$$H = \frac{1}{2}(e_1 + e_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}e_i) \quad \text{(a root)}.$$

v-2) In the case of

$$\beta = \frac{1}{2}(-e_1 + e_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}e_i), \quad \sum_{i=3}^{8}\nu(i) \in 2\boldsymbol{Z} + 1,$$

then

$$H = \frac{1}{2}(e_1 + 3e_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}e_i) \sim \frac{1}{2}(3e_1 + e_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}e_i).$$

v-3) In the case of

$$\beta = \frac{1}{2}(e_1 - e_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}e_i), \quad \sum_{i=3}^{8}\nu(i) \in 2\boldsymbol{Z} + 1,$$

then

$$H = \frac{1}{2}(3e_1 + e_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}e_i).$$

In this case, there exists an i such that $\nu(i) = 1$. Permuting e_i and e_3 by the action of the Weyl group, we have

$$H = \frac{1}{2}(3e_1 + e_2 - e_3 + \sum_{i=4}^{8}(-1)^{\nu(i)}e_i), \quad \sum_{i=4}^{8}\nu(i) \in 2\boldsymbol{Z}.$$

Then $\langle e_2 - e_3, H\rangle \neq 0$, however, there does not exist $\beta \in R_+$ such that $H = n(e_2 - e_3 \pm \beta)$. Thus this orbit is not austere.

vi) In the case of

$$\beta = \frac{1}{2}(e_1 + e_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}e_i), \quad \sum_{i=3}^{8}\nu(i) \in 2\boldsymbol{Z},$$

then

$$H = \frac{1}{2}(3e_1 + 3e_2 + \sum_{i=3}^{8}(-1)^{\nu(i)}e_i).$$

In this case $\langle e_1 + e_3, H\rangle \neq 0$, however, there does not exist $\beta \in R_+$ such that $H = n(e_1 + e_3 \pm \beta)$. Thus this orbit is not austere. □

Proposition 5.12. *In the case of $R = E_7$, there are no austere orbits except the orbits through a restricted root vector.*

Proof. In the case of $R = E_7$, all restricted roots have constant multiplicities. Thus the condition (5.4) of Lemma 5.3 is always satisfied. $\mathfrak{a} = \{\sum_{i=1}^{8} \xi_i e_i \mid \xi_8 = -\xi_7\}$ and

$$\begin{aligned} R_+ &= \{\pm e_i + e_j \mid 1 \le i < j \le 6\} \cup \{e_8 - e_7\} \\ &\quad \cup \left\{ \frac{1}{2}(e_8 - e_7 + \sum_{i=1}^{6} (-1)^{\nu(i)} e_i) \;\middle|\; \sum_{i=1}^{6} \nu(i) \in 2\boldsymbol{Z} + 1 \right\} \end{aligned}$$

From Lemma 5.3, we can assume $H = e_7 - e_8 + \beta$ where

$$\beta = \begin{cases} \pm e_i \pm e_j \quad (1 \le i < j \le 6), \\ \pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu(i)} e_i). \end{cases}$$

In the case of $\beta = \pm e_i \pm e_j$ $(1 \le i < j \le 6)$, we take k such that $1 \le k \le 6$ and $k \ne i, j$. Then $e_j + e_k \in R_+$ and $\langle H, e_j + e_k \rangle \ne 0$, however, there does not exist $\alpha \in R$ such that $H = n(e_j + e_k + \alpha)$. Thus this orbit is not austere.

In the case of $\beta = \pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu(i)} e_i)$ then

$$H = \begin{cases} \frac{1}{2}(3e_7 - 3e_8 + \sum_{i=1}^{6} (-1)^{\nu(i)} e_i), \\ \frac{1}{2}(e_7 - e_8 - \sum_{i=1}^{6} (-1)^{\nu(i)} e_i) \quad \text{(a root)}. \end{cases}$$

Therefore it suffices to consider the case of $H = \frac{1}{2}(3e_7 - 3e_8 + \sum_{i=1}^{6} (-1)^{\nu(i)} e_i)$. In this case, either $e_1 + e_2$ or $e_1 - e_2$ is an element of $R_+ - R_+^{\Delta}$. Denote this element by α. Then there does not exist $\beta \in R_+$ such that $H = n(\alpha \pm \beta)$. Hence this orbit is not austere. This completes the proof. □

Proposition 5.13. *In the case of $R = E_6$, there are no austere orbits except orbits through a restricted root vector.*

Proof. In the case of $R = E_6$, all restricted roots have constant multiplicities. Thus the condition (5.4) of Lemma 5.3 is always satisfied. $\mathfrak{a} = \{\sum_{i=1}^{8} \xi_i e_i \mid \xi_6 = \xi_7 = -\xi_8\}$ and

$$\begin{aligned} R_+ &= \{\pm e_i + e_j \mid 1 \le i < j \le 5\} \\ &\quad \cup \left\{ \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e_i) \;\middle|\; \sum_{i=1}^{5} \nu(i) \in 2\boldsymbol{Z} \right\} \end{aligned}$$

From Lemma 5.3, we can assume $H = e_1 + e_2 + \beta$ where

$$\beta = \begin{cases} \pm(e_1 - e_2), \\ \pm e_2 \pm e_i \quad (3 \le i \le 5), \\ \pm e_i \pm e_j \quad (3 \le i < j \le 5), \\ \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e_i). \end{cases}$$

i) In the case of $\beta = \pm(e_1 - e_2)$, then $H = 2e_1,\ 2e_2$. For

$$\alpha = \frac{1}{2}(e_8 - e_7 - e_6 + e_1 + e_2 + e_3 - e_4 - e_5) \in R_+ - R_+^{\Delta}$$

there does not exist $\beta \in R_+$ such that $H = n(\alpha \pm \beta)$. Thus this orbit is not austere.

ii) In the case of $\beta = \pm e_2 \pm e_i$ $(3 \le i \le 5)$, then

$$H = e_1 + e_2 \pm e_2 \pm e_i = \begin{cases} e_1 + 2e_2 \pm e_i, \\ e_1 \pm e_i \quad \text{(root)}. \end{cases}$$

Therefore it suffices to consider the case of $H = e_1 + 2e_2 \pm e_i$. In this case, for j with $3 \leq j \leq 5$ and $j \neq i$, we have $e_1 + e_j \in R_+ - R_+^{\Delta}$. However, there does not exist $\beta \in R_+$ such that $H = n(e_1 + e_j \pm \beta)$. Thus this orbit is not austere.

iii) In the case of $\beta = \pm e_i \pm e_j$ $(3 \leq i < j \leq 5)$, then $H = e_1 + e_2 \pm e_i \pm e_j$. For k with $3 \leq k \leq 5$ and $k \neq i, j$, we have $e_1 + e_k \in R_+ - R_+^{\Delta}$. However, there does not exist $\beta \in R_+$ such that $H = n(e_1 + e_k \pm \beta)$. Thus this orbit is not austere.

iv) In the case of

$$\beta = \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)} e_i) \quad \text{where} \quad \sum_{i=1}^{5} \nu(i) \in 2\boldsymbol{Z},$$

then

$$H = e_1 + e_2 \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)} e_i).$$

Here

$$H = \begin{cases} e_1 + e_2 + \dfrac{1}{2}(e_8 - e_7 - e_6 - e_1 - e_2 + \displaystyle\sum_{i=3}^{5}(-1)^{\nu(i)} e_i), \\ e_1 + e_2 - \dfrac{1}{2}(e_8 - e_7 - e_6 + e_1 + e_2 + \displaystyle\sum_{i=3}^{5}(-1)^{\nu(i)} e_i) \end{cases}$$

are root vectors.

In the case of

$$H = \begin{cases} e_1 + e_2 + \dfrac{1}{2}(e_8 - e_7 - e_6 + e_1 - e_2 + \displaystyle\sum_{i=3}^{5}(-1)^{\nu(i)} e_i), \\ e_1 + e_2 + \dfrac{1}{2}(e_8 - e_7 - e_6 - e_1 + e_2 + \displaystyle\sum_{i=3}^{5}(-1)^{\nu(i)} e_i), \\ e_1 + e_2 - \dfrac{1}{2}(e_8 - e_7 - e_6 + e_1 - e_2 + \displaystyle\sum_{i=3}^{5}(-1)^{\nu(i)} e_i), \\ e_1 + e_2 - \dfrac{1}{2}(e_8 - e_7 - e_6 - e_1 + e_2 + \displaystyle\sum_{i=3}^{5}(-1)^{\nu(i)} e_i), \end{cases}$$

we have $\langle e_1 - e_2, H \rangle \neq 0$. However there does not exist $\beta \in R_+$ such that $H = n(e_1 - e_2 \pm \beta)$. Thus this orbit is not austere.

Finally, in the case of

$$H = \begin{cases} e_1 + e_2 + \dfrac{1}{2}(e_8 - e_7 - e_6 + e_1 + e_2 + \displaystyle\sum_{i=3}^{5}(-1)^{\nu(i)} e_i), \\ e_1 + e_2 - \dfrac{1}{2}(e_8 - e_7 - e_6 - e_1 - e_2 + \displaystyle\sum_{i=3}^{5}(-1)^{\nu(i)} e_i) \end{cases}$$

we have $\langle e_1 + e_3, H \rangle \neq 0$. However there does not exist $\beta \in R_+$ such that $H = n(e_1 + e_3 \pm \beta)$. Thus this orbit is not austere. This completes the proof. $\square$

By discussions above, we completed the proof of Theorem 5.1 and Theorem 4.1.

6. Miscellaneous results

In this section, we shall concern with some results on weakly reflective submanifolds besides orbits of s-representations.

Proposition 6.1. *Let M_1 and M_2 be weakly reflective submanifolds in Riemannian manifolds $\tilde{M}_1$ and $\tilde{M}_2$, respectively. Then $M_1 \times M_2$ is a weakly reflective submanifold in $\tilde{M}_1 \times \tilde{M}_2$.*

Proof. We take $(x_1, x_2) \in M_1 \times M_2$ and a normal vector $(\xi_1, \xi_2) \in T^{\perp}_{(x_1,x_2)}(M_1 \times M_2)$. Let σ_{ξ_1} and σ_{ξ_2} be reflections of M_1 and M_2 in $\tilde{M}_1$ and $\tilde{M}_2$ with respect to ξ_1 and ξ_2, respectively. Then $\sigma_{\xi_1} \times \sigma_{\xi_2}$ is an isometry of $\tilde{M}_1 \times \tilde{M}_2$ and satisfies

$$\begin{aligned}
&(\sigma_{\xi_1} \times \sigma_{\xi_2})(x_1, x_2) = (x_1, x_2), \\
&d(\sigma_{\xi_1} \times \sigma_{\xi_2})_{(x_1,x_2)}(\xi_1, \xi_2) = -(\xi_1, \xi_2), \\
&(\sigma_{\xi_1} \times \sigma_{\xi_2})(M_1 \times M_2) = M_1 \times M_2.
\end{aligned}$$

Thus $M_1 \times M_2$ is a weakly reflective submanifold in $\tilde{M}_1 \times \tilde{M}_2$. □

The following proposition states that the cone over a weakly reflective submanifold in a sphere is also a weakly reflective submanifold in a Euclidean space.

Proposition 6.2. *Let M be a weakly reflective submanifold in a unit sphere $S^{n-1}(1)$. Then the cone*

$$C(M) = \{tx \mid t \in \mathbf{R},\ t > 0,\ x \in M\}$$

over M is a weakly reflective submanifold in a Euclidean space $\mathbf{R}^n$.

Proof. Fix $x \in M$. We note that for arbitrary $t \in \mathbf{R}$ $(t > 0)$, we have

$$T^{\perp}_{tx}(C(M)) = T^{\perp}_x M \subset T_x S^{n-1}(1).$$

For $\xi \in T^{\perp}_x M$, a reflection σ_ξ of M with respect to ξ satisfies

$$\sigma_\xi(x) = x, \qquad (d\sigma_\xi)_x \xi = -\xi, \qquad \sigma_\xi(M) = M.$$

Since σ_ξ is an isometry of $S^{n-1}(1)$, it can be expressed as an orthogonal matrix. Thus σ_ξ acts on $\mathbf{R}^n$ and satisfies

$$\sigma_\xi(tx) = t\sigma_\xi(x) = tx, \qquad (d\sigma_\xi)_x \xi = -\xi.$$

In addition, for $x' \in M,\ t' \in \mathbf{R},\ t' > 0$, we have

$$\sigma_\xi(t'x') = t'\sigma_\xi(x') \in C(M).$$

Therefore $\sigma_\xi(C(M)) = C(M)$. Hence $C(M)$ is a weakly reflective submanifold in $\mathbf{R}^n$. □

Next we shall describe the relationship between weakly reflective submanifolds in an odd dimensional sphere and in a complex projective space.

Proposition 6.3. *Let M be a weakly reflective submanifold in the hypersphere $S \subset \mathbf{C}^{n+1}$. If M is invariant under the $U(1)$-action, that is the multiplication of $e^{i\theta}$ on $\mathbf{C}^{n+1}$, and if a reflection of M with respect to each normal vector is a unitary transformation, then the image $P(M)$ of M is a weakly reflective submanifold in $\mathbf{C}P^n$, where P is the natural projection $P : S \to \mathbf{C}P^n$.*

Proof. By the definition of P, $T_x(U(1)x) = \ker dP_x$ at each point $x \in S$. Thus $dP_x : T_x^\perp(U(1)x) \to T_x(\mathbf{C}P^n)$ is an isometric linear isomorphism. Since M is invariant under the $U(1)$-action, $P(M)$ is a submanifold in $\mathbf{C}P^n$. Moreover, since $T_x(U(1)x) \subset T_xM$ for $x \in M$, we have $T_x^\perp(U(1)x) \supset T_x^\perp M$. Thus $dP_x : T_x^\perp M \to T_{p(x)}^\perp(P(M))$ also gives an isometric linear isomorphism. Let σ_ξ denote a reflection of M with respect to $\xi \in T_x^\perp M$. From the assumption, σ_ξ is a unitary transformation of $\mathbf{C}^{n+1}$. Hence σ_ξ induces an isometry of $\mathbf{C}P^n$. Since $\sigma_\xi(x) = x$ and $\sigma_\xi(M) = M$, we have $\sigma_\xi(P(x)) = P(x)$ and $\sigma_\xi(P(M)) = P(M)$. In addition we have

$$\begin{aligned} d\sigma_\xi(dP_x(\xi)) &= d(\sigma_\xi \circ P)_x(\xi) = d(P \circ \sigma_\xi)_x(\xi) \\ &= dP_x\sigma_\xi(\xi) = dP_x(-\xi) = -dP_x(\xi). \end{aligned}$$

Thus σ_ξ is a reflection of $P(M)$ with respect to a normal vector $dP_x(\xi)$ at $P(x)$. Hence $P(M)$ is a weakly reflective submanifold in $\mathbf{C}P^n$. □

Corollary 6.4. *An orbit of the s-representation of an irreducible compact Hermitian symmetric pair through a restricted root vector induces a weakly reflective submanifold in a complex projective space.*

Proof. The center of the linear isotropy subgroup of an irreducible compact Hermitian symmetric pair is $U(1)$ ([5]). Thus all orbits are invariant under $U(1)$. Furthermore an orbit in the hypersphere S through a restricted root vector is a weakly reflective submanifold. From the proof of Proposition 4.4, a reflection of an orbit through a restricted root vector with respect to each normal vector is a unitary transformation. Hence, from Proposition 6.3, we have the conclusion. □

References

[1] L. Bérard Bergery, Sur de nouvelles variétés riemanniennes d'Einstein, Publ. Inst. E. Cartan **6** (1982), 1–60.

[2] V. Borrelli and C. Gorodski, Minimal Legendrian submanifolds of S^{2n+1} and absolutely area-minimizing cones. Differential Geom. Appl., vol.21 no.3 (2004), 337–347.

[3] N. Bourbaki, Groupes et algebres de Lie, Hermann, Paris, 1975.

[4] R. Harvey and H. B. Lawson, Jr., Calibrated geometries, Acta Math., **148** (1982), 47–157.

[5] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Academic Press, New York, 1978.

[6] D. Hirohashi, T. Kanno and H. Tasaki, Area-minimizing of the cone over symmetric R-spaces, Tsukuba J. Math., vol.24 no.1 (2000) pp.171–188.

[7] D. Hirohashi, H. Song, R. Takagi and H. Tasaki, Minimal orbits of the isotropy groups of symmetric spaces of compact type, Differential Geom. Appl., vol.13 no.2 (2000), 167–177.

[8] Dominic S. P. Leung, The reflection principle for minimal submanifolds of Riemannian symmetric spaces, J. Differential Geometry, 8 (1973), 153–160.

[9] A. Kollross, A classification of hyperpolar and cohomogeneity one actions, Trans. Amer. Math. Soc. 354 (2002), no. 2, 571–612.

[10] P. S. Mostert, On a compact Lie group acting on a manifold, Ann. Math. 65 (1957), 447–455.

[11] R. Palais and C.-L. Terng, A general theory of canonical forms, Trans. Amer. Math. Soc. 300 (1987), no. 2, 771–789.

[12] R. Palais and C.-L. Terng, Critical point theory and submanifold geometry, Lecture Notes in Mathematics 1353, Springer (1988).

[13] F. Podestà, Some remarks on austere submanifolds, Boll. Un. Mat. Ital. B(7) **11** (1997), no.2, suppl., 157–160.

[14] M. Takeuchi, Modern spherical functions, Translations of Mathematical Monographs, 135. American Mathematical Society, Providence, RI, 1994.

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