ON THE GEOMETRY OF ORBITS OF PATH GROUP ACTIONS INDUCED BY HERMANN ACTIONS

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Introduction

Let $M = G/K$ be a Riemannian symmetric space of compact type and K' a closed subgroup of G. Then K' acts on M isometrically by

$$b \cdot (aK) := (ba)K, \qquad b \in K',\ aK \in M. \tag{1}$$

Then the subgroup $K' \times K$ of $G \times G$ acts on G isometrically by

$$(b, c) \cdot a := bac^{-1}, \qquad (b, c) \in K' \times K,\ a \in G.$$

Moreover this action induces a path group action on a path space. More precisely we consider the path group $\mathcal{G} := H^1([0, 1], G)$ of all Sobolev H^1-paths from $[0, 1]$ to G and the path space $V_{\mathfrak{g}} = L^2([0, 1], \mathfrak{g})$ of all L^2-paths from $[0, 1]$ to the Lie algebra $\mathfrak{g}$ of G. Then $\mathcal{G}$ is a Hilbert Lie group and $V_{\mathfrak{g}}$ a separable Hilbert space. $\mathcal{G}$ acts on $V_{\mathfrak{g}}$ isometrically via the gauge transformations:

$$g * u := gug^{-1} - \dot{g}g^{-1}, \qquad g \in \mathcal{G},\ u \in V_{\mathfrak{g}},$$

where $\dot{g}$ denotes the weak derivative of g with respect to the parameter $t \in [0, 1]$. Then the subgroup

$$P(G, K' \times K) := \{g \in \mathcal{G} \mid (g(0), g(1)) \in K' \times K\}$$

acts on $V_{\mathfrak{g}}$ by the same formula. It is known that the $P(G, K' \times K)$-action is a proper Fredholm (PF) action on a Hilbert space ([14]). Thus orbits of $P(G, K' \times K)$-actions are examples of homogeneous proper Fredholm (PF) submanifolds in Hilbert spaces ([12], [13]). Moreover the $P(G, K' \times K)$-action is also a tool to study the K'-action on M; a problem of K'-actions can be reduced to a problem of $P(G, K' \times K)$-actions due to linearity of the Hilbert space $V_{\mathfrak{g}}$ (e.g. [1]).

In this report[1] supposing that K' is a symmetric subgroup of G, that is, the K'-action is a *Hermann action* ([5]), we introduce our recent results on the submanifold geometry of orbits of $P(G, K' \times K)$-actions ([10]).

1. Submanifold geometry of orbits of Hermann actions

In this section we review the submanifold geometry of orbits of Hermann actions. For details, see Ohno [11] (see also Goertsches-Thorbergsson [2]).

Let G be a connected compact semisimple Lie group and K a closed subgroup of G. Suppose that K is a symmetric subgroup of G, that is, there exists an involution θ of G satisfying $G_0^\theta \subset K \subset G^\theta$, where G^θ denotes the fixed point subgroup of G and G_0^θ the identity component. We write $\mathfrak{g}$ and $\mathfrak{k}$ for the Lie algebras of G and K

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respectively. The differential of θ induces an involution of $\mathfrak{g}$, which is also denoted by θ. The direct sum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ into the ± 1-eigenspaces of θ is called the *canonical decomposition* associated to the pair $(\mathfrak{g}, \mathfrak{k})$. We fix an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ which is a negative multiple of the Killing form of $\mathfrak{g}$. We equip the corresponding bi-invariant Riemannian metric with G and the G-invariant Riemannian metric with the homogeneous space G/K. Then $M := G/K$ is a Riemannian symmetric space of compact type and the projection $\pi : G \to M$ is a Riemannian submersion with totally geodesic fiber.

Let K' be a symmetric subgroup of G with involution θ'. Denote by $\mathfrak{k}'$ the Lie algebra of K' and $\mathfrak{g} = \mathfrak{k}' + \mathfrak{m}'$ the canonical decomposition associated to the pair $(\mathfrak{g}, \mathfrak{k}')$. Then K' acts on M isometrically by the formula (1). This action is called a *Hermann action* ([5]). We know that a Hermann action is *hyperpolar* ([4]), that is, there exists a (totally geodesic) closed connected submanifold Σ of M satisfying the conditions that Σ meets every K'-orbit orthogonally and that Σ is flat with respect to the induced metric. Such a Σ is called a *section*. In fact, if we take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{m} \cap \mathfrak{m}'$ then $\exp \mathfrak{t}$ is a torus subgroup of G and thus $\Sigma := \pi(\exp \mathfrak{t})$ is a section of the Hermann action.

Take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{m} \cap \mathfrak{m}'$ and consider the root space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}(0) + \sum_{\alpha \in \Delta} \mathfrak{g}(\alpha) = \mathfrak{g}(0) + \sum_{\alpha \in \Delta^+} (\mathfrak{g}(\alpha) + \mathfrak{g}(-\alpha)),$$

$$\begin{aligned} \mathfrak{g}(0) &= \{ z \in \mathfrak{g}^{\mathbb{C}} \mid \forall \eta \in \mathfrak{t},\ \mathrm{ad}(\eta) z = 0 \}, \\ \mathfrak{g}(\alpha) &= \{ z \in \mathfrak{g}^{\mathbb{C}} \mid \forall \eta \in \mathfrak{t},\ \mathrm{ad}(\eta) z = \sqrt{-1} \langle \alpha, \eta \rangle z \}. \end{aligned}$$

The real form is

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha,$$

$$\begin{aligned} \mathfrak{g}_0 &= \{ x \in \mathfrak{g} \mid \forall \eta \in \mathfrak{t},\ \mathrm{ad}(\eta) x = 0 \}, \\ \mathfrak{g}_\alpha &= \{ x \in \mathfrak{g} \mid \forall \eta \in \mathfrak{t},\ \mathrm{ad}(\eta)^2 x = -\langle \alpha, \eta \rangle^2 x \}. \end{aligned}$$

Since θ commutes with $\mathrm{ad}(\eta)^2$ for all $\eta \in \mathfrak{t}$ we have

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \Delta^+} \mathfrak{k}_\alpha, \qquad \mathfrak{m} = \mathfrak{m}_0 + \sum_{\alpha \in \Delta^+} \mathfrak{m}_\alpha,$$

$$\begin{aligned} \mathfrak{k}_0 &= \mathfrak{g}_0 \cap \mathfrak{k}, \qquad & \mathfrak{k}_\alpha &= \mathfrak{g}_\alpha \cap \mathfrak{k}, \\ \mathfrak{m}_0 &= \mathfrak{g}_0 \cap \mathfrak{m}, \qquad & \mathfrak{m}_\alpha &= \mathfrak{g}_\alpha \cap \mathfrak{m}. \end{aligned}$$

Consider the composition

$$\theta \circ \theta' : \mathfrak{g} \to \mathfrak{g}$$

and the eigenspace decomposition

$$\mathfrak{g}^{\mathbb{C}} = \sum_{\epsilon \in U(1)} \mathfrak{g}(\epsilon),$$

$$\mathfrak{g}(\epsilon) = \{ z \in \mathfrak{g}^{\mathbb{C}} \mid (\theta \circ \theta')(z) = \epsilon z \}.$$

For each $\epsilon \in U(1)$ we denote by $\arg\epsilon$ its argument satisfying $-\pi \leq \arg\epsilon \leq \pi$. Since $\theta \circ \theta'$ commutes with $\mathrm{ad}(\eta)$ for all $\eta \in \mathfrak{t}$ we have

$$\mathfrak{g}^{\mathbb{C}} = \sum_{\epsilon \in U(1)} \mathfrak{g}(0,\epsilon) + \sum_{\alpha \in \Delta} \sum_{\epsilon \in U(1)} \mathfrak{g}(\alpha,\epsilon),$$

$$\mathfrak{g}(0,\epsilon) = \mathfrak{g}(0) \cap \mathfrak{g}(\epsilon), \qquad \mathfrak{g}(\alpha,\epsilon) = \mathfrak{g}(\alpha) \cap \mathfrak{g}(\epsilon).$$

The real form is

$$\mathfrak{g} = \sum_{\epsilon \in U(1)_{\geq 0}} \mathfrak{g}_{0,\epsilon} + \sum_{\alpha \in \Delta^+} \sum_{\epsilon \in U(1)} \mathfrak{g}_{\alpha,\epsilon}, \tag{2}$$

$$U(1)_{\geq 0} = \{\epsilon \in U(1) \mid \mathrm{Im}(\epsilon) \geq 0\},$$

$$\mathfrak{g}_{0,\epsilon} = (\mathfrak{g}(0,\epsilon) + \mathfrak{g}(0,\epsilon^{-1}) \cap \mathfrak{g},$$

$$\mathfrak{g}_{\alpha,\epsilon} = (\mathfrak{g}(\alpha,\epsilon) + \mathfrak{g}(-\alpha,\epsilon^{-1})) \cap \mathfrak{g}.$$

Since $\mathfrak{g}_{0,\epsilon}$ and $\mathfrak{g}_{\alpha,\epsilon}$ are invariant under θ we have

$$\mathfrak{k} = \sum_{\epsilon \in U(1)_{\geq 0}} \mathfrak{k}_{0,\epsilon} + \sum_{\alpha \in \Delta^+} \sum_{\epsilon \in U(1)} \mathfrak{k}_{\alpha,\epsilon}, \tag{3}$$

$$\mathfrak{m} = \sum_{\epsilon \in U(1)_{\geq 0}} \mathfrak{m}_{0,\epsilon} + \sum_{\alpha \in \Delta^+} \sum_{\epsilon \in U(1)} \mathfrak{m}_{\alpha,\epsilon}, \tag{4}$$

$$\mathfrak{k}_{0,\epsilon} = \mathfrak{g}_{0,\epsilon} \cap \mathfrak{k}, \qquad \mathfrak{k}_{\alpha,\epsilon} = \mathfrak{g}_{\alpha,\epsilon} \cap \mathfrak{k},$$

$$\mathfrak{m}_{0,\epsilon} = \mathfrak{g}_{0,\epsilon} \cap \mathfrak{m}, \qquad \mathfrak{m}_{\alpha,\epsilon} = \mathfrak{g}_{\alpha,\epsilon} \cap \mathfrak{m}.$$

We now take $w \in \mathfrak{t}$ and set $a := \exp w$. Then the tangent space and the normal space of the orbit $N := K' \cdot aK$ through aK are described as follows:

Proposition 1.1 (Ohno [11])**.**

$$T_{aK}N = dL_a\Big(\sum_{\substack{\epsilon \in U(1)_{\geq 0} \\ \epsilon \neq 1}} \mathfrak{m}_{0,\epsilon} + \sum_{\alpha \in \Delta^+} \sum_{\substack{\epsilon \in U(1) \\ \langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon \notin \pi\mathbb{Z}}} \mathfrak{m}_{\alpha,\epsilon} \Big), \tag{5}$$

$$T^{\perp}_{aK}N = dL_a\Big(\mathfrak{t} + \sum_{\alpha \in \Delta^+} \sum_{\substack{\epsilon \in U(1) \\ \langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon \in \pi\mathbb{Z}}} \mathfrak{m}_{\alpha,\epsilon} \Big). \tag{6}$$

Moreover the decomposition (5) *is just the simultaneous eigenspace decomposition of the family shape operators* $\{A^N_\xi\}_{\xi \in \mathfrak{t}}$. *In fact*

$dL_a(\mathfrak{m}_{0,\epsilon})$: *the eigenspace of eigenvalue* 0,

$dL_a(\mathfrak{m}_{\alpha,\epsilon})$: *the eigenspace of eigenvalue* $-\langle \alpha, \xi\rangle \cot(\langle \alpha, w\rangle + \frac{1}{2}\arg\epsilon)$.

If involutions θ and θ' commute then $\epsilon = \pm 1$ and thus we have:

Corollary 1.2 (Goertsches-Thorbergsson [2])**.** *Suppose that* $\theta \circ \theta' = \theta' \circ \theta$. *Then*

$$T_{aK}N = dL_a\Big(\mathfrak{m}_0 \cap \mathfrak{k}' + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w\rangle \notin \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{m}' + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w\rangle + \pi/2 \notin \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{k}' \Big),$$

$$T^{\perp}_{aK}N = dL_a\Big(\mathfrak{t} + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w\rangle \in \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{m}' + \sum_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, w\rangle + \pi/2 \in \pi\mathbb{Z}}} \mathfrak{m}_\alpha \cap \mathfrak{k}' \Big),$$

$$
\begin{aligned}
dL_a(\mathfrak{m}_0 \cap \mathfrak{k}') &\quad : \quad \textit{the eigenspace of eigenvalue } 0,\\
dL_a(\mathfrak{m}_\alpha \cap \mathfrak{m}') &\quad : \quad \textit{the eigenspace of eigenvalue } -\langle \alpha, \xi \rangle \cot\langle \alpha, w \rangle,\\
dL_a(\mathfrak{m}_\alpha \cap \mathfrak{k}') &\quad : \quad \textit{the eigenspace of eigenvalue } \langle \alpha, \xi \rangle \tan\langle \alpha, w \rangle.
\end{aligned}
$$

2. Submanifold geometry of orbits of $P(G, K' \times K)$-actions

In this section we consider orbits of the $P(G, K' \times K)$-action, where $M = G/K$ is a Riemannian symmetric space of compact type and K' a symmetric subgroup of G. Note that the $P(G, K' \times K)$-action is hyperpolar because the Hermann action $K' \curvearrowright M$ is hyperpolar ([14], [4], [1]). In fact, if we take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{m} \cap \mathfrak{m}'$ then the space of constant paths $\hat{\mathfrak{t}} := \{\hat{x} \in V_{\mathfrak{g}} \mid x \in \mathfrak{t}\}$, where $\hat{x}$ denotes a constant path with value x, is a section of the $P(G, K' \times K)$-action. The second fundamental form and the shape operator of $P(G, K' \times K)$-orbits were computed in [8]. The principal curvatures of $P(G, K' \times K)$-orbits are described as follows.

Theorem 2.1 ([10]). *Let $K' \curvearrowright M$ be a Hermann action. Fix a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{m} \cap \mathfrak{m}'$. Take $w \in \mathfrak{t}$ and consider the orbit $P(G, K' \times K) * \hat{w}$ through $\hat{w} \in \hat{\mathfrak{t}}$. Then for each $\xi \in \mathfrak{t}$ the principal curvatures of $P(G, K' \times K) * \hat{w}$ in the direction of $\hat{\xi} \in \hat{\mathfrak{t}}$ are given by*

$$
\begin{aligned}
\{0\} \cup &\left\{ \frac{-\langle \alpha, \xi \rangle}{\langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon + m\pi} \;\middle|\; \alpha \in \Delta^+,\ \epsilon \in U(1),\ \langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \notin \pi\mathbb{Z},\ m \in \mathbb{Z} \right\}\\
&\cup \left\{ \frac{\langle \alpha, \xi \rangle}{n\pi} \;\middle|\; \alpha \in \Delta^+,\ \epsilon \in U(1),\ \langle \alpha, w \rangle + \frac{1}{2} \arg \epsilon \in \pi\mathbb{Z},\ n \in \mathbb{Z}\backslash\{0\} \right\}.
\end{aligned}
$$

The multiplicities are respectively given by

$$
\infty, \qquad \dim \mathfrak{m}_{\alpha,\epsilon}, \qquad \sum_{\epsilon} \dim \mathfrak{m}_{\alpha,\epsilon}.
$$

In particular, if $w \in \mathfrak{t}$ is a regular point of the Hermann action then $\frac{\langle \alpha, \xi \rangle}{n\pi}$ vanishes.

In the proof of Theorem 2.1 we essentially use a formula for principal curvatures of PF submanifolds obtained from curvature adapted submanifolds in compact symmetric spaces through the parallel transport map ([7], [9]).

If $\theta \circ \theta' = \theta' \circ \theta$ then $\epsilon = \pm 1$ and thus we have the following corollary:

Corollary 2.2 ([10]). *Suppose that involutions θ and θ' commute. Then the principal curvatures of the orbit $P(G, K' \times K) * \hat{w}$ in the direction of $\hat{\xi} \in \hat{\mathfrak{t}}$ are given by*

$$
\begin{aligned}
\{0\} \cup &\left\{ \frac{-\langle \alpha, \xi \rangle}{\langle \alpha, w \rangle + m\pi} \;\middle|\; \alpha \in \Delta^+,\ \langle \alpha, w \rangle \notin \pi\mathbb{Z},\ m \in \mathbb{Z} \right\}\\
\cup &\left\{ \frac{-\langle \alpha, \xi \rangle}{\langle \alpha, w \rangle + (m + \frac{1}{2})\pi} \;\middle|\; \alpha \in \Delta^+,\ \langle \alpha, w \rangle + \frac{\pi}{2} \notin \pi\mathbb{Z},\ m \in \mathbb{Z} \right\}\\
\cup &\left\{ \frac{\langle \alpha, \xi \rangle}{n\pi} \;\middle|\; \alpha \in \Delta^+,\ \langle \alpha, w \rangle \in \pi\mathbb{Z} \ \text{ or } \ \langle \alpha, w \rangle + \frac{\pi}{2} \in \pi\mathbb{Z},\ n \in \mathbb{Z}\backslash\{0\} \right\}.
\end{aligned}
$$

The multiplicities are respectively given by

$$
\infty, \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{m}'), \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{k}'), \qquad \dim(\mathfrak{m}_\alpha \cap \mathfrak{m}') + \dim(\mathfrak{m}_\alpha \cap \mathfrak{k}').
$$

In particular, if $w \in \mathfrak{t}$ is a regular point of the Hermann action then $\frac{\langle \alpha, \xi \rangle}{n\pi}$ vanishes.

Using these results we finally consider the austere property of $P(G, K' \times K)$-orbits. Recall that a submanifold N of a Riemannian manifold M is called *austere* if for each normal vector ξ at each point of N the set of eigenvalues with multiplicities of the shape operator A^N_ξ is invariant under the multiplication by (-1). This notion was originally introduced in the study of calibrated geometry ([3]). Similarly we can also define austere PF submanifolds in Hilbert spaces ([8], [9]).

The following theorem shows that under some (technical) conditions we can obtain austere $P(G, K' \times K)$-orbits from austere K'-orbits.

Theorem 2.3 ([10]). *Let $K' \curvearrowright M$ be a Hermann action. Suppose that involutions θ and θ' commute and that G is simple. If an orbit $K' \cdot (\exp w)K$ through $\exp w \in G/K$ for $w \in \mathfrak{t}$ is an austere submanifold of G/K then the orbit $P(G, K' \times K) * \hat{w}$ through $\hat{w} \in \hat{w}$ is an austere PF submanifold of the Hilbert space $V_{\mathfrak{g}}$.*

Ikawa [6] classified austere orbits of Hermann actions under such conditions. Thus applying above theorem to his result we can obtain examples of homogeneous austere PF submanifolds in Hilbert spaces.

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