REGULAR TRIPLETS IN COMPACT SYMMETRIC SPACES

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1. Introduction

This article is based on the collaboration with Tadashi Nagano.

In the first part of this article we briefly review basic notions of our geometric theory of symmetric spaces developping in the series of papers [7], [8], [9], [10] and [11].

And in the next part we introduce a regular triplet in a symmetric space M. A triplet $\{o, p, q\}$ of three points o, p and q in M is called a regular triplet if the composite of the point-symmetries at each point is the identity map. If M has no pole, M has a regular triplet $\{o, p, q\}$ if and only if M has a polar $M^+_{(p:o)}$ which is isomorphic with the meridian to it (Theorem 3.8). And then o, p and q are vertices of"a right regular triangle" each of which sides is isomorphic with $M^+_{(o,p)}$. We also proved that if M without pole has a regular triplet, there is a subspace in M which is isomorphic with $\boldsymbol{R}P^2$ and contains the regular triplet (Theorem 3.10).

2. Preliminaries

We define the *category of symmetric spaces* as the following: (i) the category of symmetric spaces is a subcategory of that of smooth manifolds, (ii) for every point x in a *symmetric space* M there is the point-symmetry $s_x : M \to M$, that is, s_x is involutive and x is an isolated fixed point of s_x, (iii) a map $f : M \to N$ from a symmetric space M into another symmetric space N is a *morphism* if it satisfies $f \circ s_x = s_{f(x)} \circ f$ for every point x in M, (iv) every point-symmetry is a morphism.

This definition of a symmetric space and a morphism are essentially the same as that of O. Loos in [6]. By using this definition of symmetric spaces we can take disconnected symmetric spaces into consideration. For example, we can consider a finite set as a trivial symmetric space, that is, a symmetric space whose point-symmetry is the identity map at every point. We make use of it to define the 2-numbers of symmetric spaces in [4].

We can prove that there exists a unique affine connection which is invariant under every point-symmetry. Thus our definition of a symmetric space is the same as usual one (for example in [5]). Since it follows that every invariant tensor of odd degrees is identically zero, the invariant affine connection is torsion-free and its curvature tensor is parallel. Since a morphism is an affine map with respect to this invariant connection, a smooth map $f : M \to N$ is a morphism if and only if f is totally geodesic, if M is connected. Every morphism $f : M \to N$ is a composite of a epimorphism and a monomorphism. Especially the image $f(M)$ is also a symmetric space and we call it a *subspace* of N. For example, for an automorphism σ of a symmetric space M, the fixed point set of σ is a subspace of M.

From now on, we consider in the category of Riemannian symmetric spaces unless otherwise stated, that is, we assume that a symmetric space is equipped with a Riemannian metric and every morphism satisfies the isometry condition, more precisely, it is isometric on the orthogonal complement to the kernel of the differential at each point. In particular, every symmetry preserves the Riemannian metric. Hereafter we study compact symmetric spaces only.

Let $M = G/K$ be a connected symmetric space, here G is the identity component of the automorphism group of M and K is the isotropy subgroup of G at a point o in M. Each connected component of the fixed point set of s_o is called a *polar* of o in M and denoted by $M^+_{(p)}$ or $M^+_{(p:o)}$ if it contains a point p. The connected component of the fixed point set of $s_p \circ s_o$ which contains p is called the *meridian* to $M^+_{(p)}$ at p, denoted by $M^-_{(p)}$ or $M^-_{(o,p)}$. By the definition, we can easily see that $M^+_{(p)}$ and $M^-_{(p)}$ are *c-orthogonal* to each other, that is, the tangent spaces of $M^+_{(p)}$ and $M^-_{(p)}$ at p are the orthogonal complements to each other in that of M. Each polar is a K-orbit and the meridians to the same polar are all G-congruent with each other. A polar and a meridian are subspaces in M naturally.

For example, if M is a compact Lie group with bi-invariant metric, the symmetry at $x \in M$ is defined by $s_x(y) = xy^{-1}x$ for every element y in M. Thus the polars of the identity element in M coincide with the conjugate classes of involutive elements of M. In the case of $M = U(n)$, we have $M^+_{(p)} = G_k(\boldsymbol{C}^n)$, the complex Grassmannian manifold of the k-dimensional subspaces in $\boldsymbol{C}^n$, and $M^-_{(p)} = U(k) \times U(n-k)$ where $0 \leq k \leq n$.

Polars of M deeply relate to the topology of M, for instance, M is orientable if and only if every polar has even dimension: the Euler number χM of M is equal to the sum of those of all the polars of M unless $\chi M = 0$. On the other hand, each meridian $M^-_{(p)}$ has the same rank as that of M. So we can find that the natural homomorphism $\pi_1(M^-_{(p)}) \to \pi_1(M)$ is surjective.

When a polar $M^+_{(p)}$ consists of a single point p, p is called a *pole* of o in M if p is not o. A pole does not necessarily exist. And if it does, it is not unique. For example, a torus T^r has $2^r - 1$ poles and a simple Lie group $Spin(4n), n \geq 2$ has three poles.

When there is a pole p of o in M, we have a double covering morphism $\pi : M \to N$ onto another symmetric space N, satisfying $\pi(o) = \pi(p)$. Under this situation, let $C(o,p)$ denote a set of the midpoints of the geodesic arcs between o and p, which is a subspace of M. Each connected component of $C(o,p)$ is called a *centriole* for the pair (o,p) in M. The image of every centriole under π is a polar of $\pi(o)$ in N. Conversely, every polar of $\pi(o)$ in N is a image of a polar of o or a centriole for (o,p) in M under π. Moreover, M and N are locally isomorphic and there is a unique symmetric space $M^{\%}$ in a local isomorphism class of M which is not a covering space on any other symmetric spaces if M is semisimple. This $M^{\%}$ is called the *bottom space* of M. The bottom space has no pole but the converse is not true: $M = SU(n)$ with $n =$ odd have no pole and $M^{\%} = SU(n)/\boldsymbol{Z_n}$.

For example, n-dimensional sphere $M = S^n$ has a pole and the centriole is the equator S^{n-1}. The bottom space of S^n is a projective space $\boldsymbol{R}P^n$ and its polar $\boldsymbol{R}P^{n-1}$ is the image of the centriole S^{n-1} under the double covering morphism.

By the classification of polars and meridians ([3], [4], for example), we can find that every polar of M is a pole if and only if M is a Riemannian product of spheres and a torus.

The theoretical significance of the notion of polars and the meridians lies in the following fact.

Theorem 2.1 (1.15 in [8])**.** *Every compact connected symmetric space is globally determined by any one pair of a polar and the meridian to it.*

3. Regular triplets

Let M be a compact connected symmetric space, and o, p and q three points in M. A triplet $\{o, p, q\}$ is called *a regular triplet* in M if they satisfy $s_o \circ s_p \circ s_q = \mathrm{id}_M$. For example, the real, complex, quaternion, or Cayley projective plane has a regular triplet. In fact, the polar $M^+_{(p)}$ of a point o in M is a projective line (hence a sphere or a circle) and by taking q, the pole of p in $M^+_{(p)}$, we have a regular triplet $\{o, p, q\}$. Then o, p and q are vertices of “a right regular triangle” each of which sides is a projective line.

Remark As we will see later, every symmetric space which admits a regular triplet has such “a right regular triangle”.

Since every point-symmetry is involutive, the next lemma follows immediately.

Lemma 3.1. *If $\{o, p, q\}$ is a regular triplet, we have $s_o \circ s_p = s_q$, $s_p \circ s_q = s_o$ and $s_o \circ s_q = s_o$.*

Since the composite of two point-symmetries is an inner involution, we have the following consequence.

Corollary 3.2. *If M has a regular triplet, M is of inner type.*

Remark. A simple symmetric space $M = G/K$ is of inner type if and only if one of the following conditions holds: (i) the rank of K is the same as that of G and (ii) the Euler number χM of M is positive.

The following lemmas which we can prove easily describe some of fundamental properties of regular triplets and we use them to prove the main theorem (Theorem 3.8).

Lemma 3.3. *Let $\{o, p, q\}$ be a regular triplet. Then the group generated by $\{s_o, s_p, s_q\}$ is isomorphic with that generated by $\left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$ and its automorphism group is isomorphic with the symmetric group of degree 3.*

Lemma 3.4. *If $\{o, p, q\}$ is a regular triplet, s_o, s_p and s_q are commutative with each other.*

Lemma 3.5. *Assume that M has no pole and let $\{o, p, q\}$ be a regular triplet in M, then $\{s_o, s_p, s_q\}$ acts trivially on $\{o, p, q\}$.*

Lemma 3.6. *Assume that M has no pole and let $\{o, p, q\}$ be a regular triplet in M and m the midpoint of a minimal geodesic arc between p and q, then we have $s_m(o) = o$.*

Lemma 3.7. *Assume that M has no pole and let $\{o, p, q\}$ be a regular triplet in M, then $M^+_{(p:o)}$ coincides with $M^+_{(q:o)}$.*

Theorem 3.8. *If M has no pole, then the following three conditions are equivalent:* (i) *M has a regular triplet;* (ii) *there is a polar in M which is congruent with the meridian to it; and* (iii) *there is a polar in M which has a pole in it.*

Remark. A simple M has no pole if and only if M is (i)the bottom space $M^{\%}$; or (ii) the root system of M is of type A_{2r} or type E_6. This condition is relating with the fundamental group of $M^{\%}$.

Now we consider the case when M admits a pole. Let p be a pole of o in M and π the double covering morphism from M onto N which maps o and p onto the same point.

Theorem 3.9. *M has a regular triplet if and only if $N = \pi(M)$ has a regular triplet.*

Since the complete list of (M^+, M^-) is known ([3], [4], for example), we can find all M with no pole which has a regular triplet by Theorem 3.8.

In [10] we investigated the relation between self-intersections and regular triplets on the sequence of symmetric spaces: $\boldsymbol{R}P^2 \subset \boldsymbol{C}P^2 \subset \boldsymbol{H}P^2 \subset FII \subset EIII \subset EVI \subset EVIII$. Here we use the notation of symmetric spaces in [5]. There is another sequence of quaternion Kähler symmetric spaces of exceptional type: $GI \subset FI \subset EII \subset EVI \subset EIX$, where each of these spaces has a regular triplet. Thus we know that every inner symmetric space of exceptional type has a regular triplet. And every Kähler symmetric space with a pole (that is, whose root system is of type C_r) has a regular triplet. In fact, each of them contains a symmetric R-space as a centriole, which is isomorphic with its c-orthogonal space.

Remark. K. Atsuyama studied the bottom spaces which have a regular triplet from the view point of "projective geometry in a wider sense" in [1]. B. Y. Chen studied regular triplets in [2], where he call them *antipodal* subset and he proved in it the equivalence between the conditions (i) and (ii) in Theorem 3.8 and Theorem 3.9.

Assume that $M = G/K$ has a regular triplet $\{o, p, q\}$ but no pole. The Lie algebra $\mathfrak{g}$ of G has an orthogonal directsum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, where $\mathfrak{k}$ and $\mathfrak{m}$ are eigenspaces of ds_o with respect to the eigenvalues 1 and -1 respectively. And furthermore, we have the following decomposition since s_o and s_p are commutative: $\mathfrak{g} = \mathfrak{k}_+ + \mathfrak{k}_- + \mathfrak{m}_+ + \mathfrak{m}_-$, where $ds_p = \mathrm{id}$ on $\mathfrak{k}_+$ and $\mathfrak{m}_+$ and $ds_p = -\mathrm{id}$ on $\mathfrak{k}_-$ and $\mathfrak{m}_-$. $\mathfrak{m}_-$ can be ientified with the tangent space $T_o M^-_{(p:o)}$ of the meridian $M^-_{(p:o)}$, $\mathfrak{k}_-$ with $T_p M^+_{(p:o)}$ and $\mathfrak{m}_+$ with $T_o M^-_{(q:o)}$. By the assumption we have $M^+_{(p:o)} \cong M^-_{(p:o)} \cong M^-_{(q:o)}$, hence $\mathfrak{k}_- \cong \mathfrak{m}_- \cong \mathfrak{m}_+$ as $\mathfrak{k}_+$-modules. Moreover, we have $[\mathfrak{k}_-, \mathfrak{m}_\pm] = \mathfrak{m}_\mp$ and $[\mathfrak{m}_+, \mathfrak{m}_-] = \mathfrak{k}_-$. And there is an automorphism of $\mathfrak{g}$ which maps these three spaces cyclicly. This property is something related with the triality of $Spin(8)$. In fact, if M is one of the following spaces: $FII \subset EIII \subset EVI \subset EVIII$, the isotropy subgroup contains a spin group as a normal subgroup.

By the above-mentioned facts we obtain the following.

Theorem 3.10. *Let M have a regular triplet but no pole. Then there is a subspace in M which is isomorphic with $\boldsymbol{R}P^2$ and contains the regular triplet.*

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