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1 Introduction

There are many closed surfaces in the unit sphere S^3 which admit nontrivial isometric deformations. For example, every flat torus in S^3 with nonconstant mean curvature admits nontrivial isometric deformations ([3]). In this note we show that each flat torus in S^3 conserves its total mean curvature during any isometric deformation.

Let M be an oriented flat torus, and let $I(M, S^3)$ denote the set of all isometric immersions $f : M \to S^3$. For each $f \in I(M, S^3)$, the total mean curvature of f is given by

$$\tau(f) = \int_M H \, dA,$$

where H denotes the mean curvature of f. Furthermore, we set

$$\tau(M) = \{\tau(f) : f \in I(M, S^3)\}.$$

By using a method for constructing the flat tori in S^3, we investigate the set $\tau(M)$ and obtain the following result.

Theorem 1 *Let G be a lattice of $\mathbb{R}^2$ such that M is isometric to $\mathbb{R}^2/G$. Then $\tau(M) \subset X(G)$, where $X(G)$ is a countable set defined by*

$$X(G) = \left\{ \frac{1}{2n} \langle \vec{\alpha}, \vec{\beta} \rangle : n \in \mathbb{N}, \ \vec{\alpha}, \vec{\beta} \in G \right\}.$$

In particular, the set $\tau(M)$ is at most countable.

As an immediate consequence of Theorem 1, we obtain

Theorem 2 *Let M be an oriented flat torus, and let $f_t : M \to S^3$, $t \in \mathbb{R}$, be a smooth one-parameter family of isometric immersions. Then $\tau(f_t) = \tau(f_0)$ for all $t \in \mathbb{R}$.*

2 Periodic admissible pairs and flat tori in S^3

In this section we give an ontline of a method for constructing all the flat tori isometrically immersed in S^3. For details, see [1, 2, 3]

Definition *A periodic admissible pair (p.a.p.) is a pair of periodic regular curves* $\gamma_1 : \mathbb{R} \to S^2$ *and* $\gamma_2 : \mathbb{R} \to S^2$ *such that*
(a) $k_1(s_1) > k_2(s_2)$ *for all* $s_1, s_2 \in \mathbb{R}$,
(b) $|\gamma_i'(s)|\sqrt{1 + k_i(s)^2} = 2 \quad (i = 1, 2)$,
where $k_i(s)$ *denotes the geodesic curvature of* $\gamma_i(s)$.

We first explain that each p.a.p. $\Gamma = (\gamma_1, \gamma_2)$ induces a flat torus M_Γ isometrically immersed in S^3. Let $\mathbb{H}$ denote the set of all quaternions, and let $\mathbb{R}^4$ be the 4-dimensional Euclidean space identified with $\mathbb{H}$ as follows:

$$(x_1, x_2, x_3, x_4) \longleftrightarrow x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}.$$

The unit spheres S^2 and S^3 are given by

$$S^2 = \{x \in \operatorname{Im}\mathbb{H} : |x| = 1\}, \quad S^3 = \{x \in \mathbb{H} : |x| = 1\}.$$

Note that the unit sphere S^3 has a group structure induced by the multiplicative structure of $\mathbb{H}$. The unit tangent bundle of S^2, denoted by US^2, is identified with a subset of $S^2 \times S^2$ as follows:

$$US^2 = \{(x, v) \in S^2 \times S^2 : \langle x, v\rangle = 0\},$$

where the canonical projection $p_1 : US^2 \to S^2$ is given by $p_1(x, v) = x$. Furthermore, we define a double covering map $p_2 : S^3 \to US^2$ by

$$p_2(a) = (a\mathbf{i}a^{-1}, a\mathbf{j}a^{-1}).$$

Consider a curve $\hat{\gamma}_i : \mathbb{R} \to US^2$ given by

$$\hat{\gamma}_i(s) = (\gamma_i(s), \gamma_i'(s)/|\gamma_i'(s)|),$$

and denote by $I(\gamma_i)$ the element of the homology group $H_1(US^2)$ represented by the closed curve $\hat{\gamma}_i : [0, l_i] \to US^2$, where l_i denotes the period of γ_i. . Let $c_i : \mathbb{R} \to S^3$ be a lift of the curve $\hat{\gamma}_i : \mathbb{R} \to US^2$ with respect to the covering p_2. Since $H_1(US^2) \cong \mathbb{Z}_2$ and the double covering p_2 satisfies the relation $p_2(-a) = p_2(a)$, we see that

$$c_i(s + l_i) = \begin{cases} c_i(s) & I(\gamma_i) = 0, \\ -c_i(s) & I(\gamma_i) = 1. \end{cases}$$

Using the group structure of S^3, we define a map $F_\Gamma : \mathbb{R}^2 \to S^3$ by

$$F_\Gamma(s_1, s_2) = c_1(s_1)c_2(s_2)^{-1}.$$

Then it follows that the map F_Γ is a doubly periodic immersion and induces a flat Riemannian metric g_Γ on $\mathbb{R}^2$. We now consider the group

$$G(\Gamma) = \{\varphi \in \mathrm{Diff}(\mathbb{R}^2) : F_\Gamma \circ \varphi = F_\Gamma\}.$$

Since each element of $G(\Gamma)$ is a parallel translation of $\mathbb{R}^2$, we obtain a flat torus $M_\Gamma = (\mathbb{R}^2, g_\Gamma)/G(\Gamma)$, and an isometric immersion

$$f_\Gamma : M_\Gamma \to S^3$$

satisfying the relation $f_\Gamma \circ \pi_\Gamma = F_\Gamma$, where $\pi_\Gamma : \mathbb{R}^2 \to M_\Gamma$ denotes the canonical projection. Furthermore, it follows that all the flat tori isometrically immersed in S^3 are constructed by using this procedure. In fact, modifying the proof of [2, Theorem 3.1], we obtain

Proposition 1 *Let $f : M \to S^3$ be an isometric immersion of a flat torus M into the unit sphere S^3. Then there exist a p.a.p. Γ and a covering map $\rho : M \to M_\Gamma$ such that $A \circ f = f_\Gamma \circ \rho$ for some isometry $A : S^3 \to S^3$.*

We now explain that the Riemannian structure of M_Γ and the total mean curvature of f_Γ can be written in terms of geometric data of $\Gamma = (\gamma_1, \gamma_2)$. Let

$$\vec{v}_i = \frac{1}{2}(K_i,\ L_i), \quad L_i = \int_0^{l_i} |\gamma_i'(s)| ds, \quad K_i = \int_0^{l_i} k_i(s)|\gamma_i'(s)| ds,$$

and define $W(\Gamma)$ to be a lattice of $\mathbb{R}^2$ generated by the following vectors

$$\begin{cases} \vec{v}_1,\ \vec{v}_2 & \text{if} \quad I(\Gamma) = (0,\ 0), \\ 2\vec{v}_1,\ \vec{v}_2 & \text{if} \quad I(\Gamma) = (1,\ 0), \\ \vec{v}_1,\ 2\vec{v}_2 & \text{if} \quad I(\Gamma) = (0,\ 1), \\ \vec{v}_1 + \vec{v}_2,\ \vec{v}_1 - \vec{v}_2 & \text{if} \quad I(\Gamma) = (1,\ 1), \end{cases}$$

where $I(\Gamma) = (I(\gamma_1),\ I(\gamma_2))$. Then we obtain

Proposition 2 ([3], Theorem 4.1) *For each p.a.p. $\Gamma = (\gamma_1, \gamma_2)$, the flat torus M_Γ is isometric to $\mathbb{R}^2/W(\Gamma)$.*

Proposition 3 ([3], Theorem 5.3) *For each p.a.p. $\Gamma = (\gamma_1, \gamma_2)$,*

$$\tau(f_\Gamma) = \begin{cases} \langle \vec{v}_1, \vec{v}_2 \rangle & if \quad I(\Gamma) = (0,0), \\ 2\langle \vec{v}_1, \vec{v}_2 \rangle & if \quad I(\Gamma) \neq (0,0). \end{cases}$$

3 Proof of Theorem 1

Let G be a lattice of $\mathbb{R}^2$ such that the flat torus M is isometric to $\mathbb{R}^2/G$, and let $f \in I(M, S^3)$. Then it follows from Proposition 1 that there exist a p.a.p. $\Gamma = (\gamma_1,\ \gamma_2)$ and a Riemannian covering $\rho : M \to M_\Gamma$ such that

$$A \circ f = f_\Gamma \circ \rho,$$

where A denotes an isometry of S^3. Furthermore, Proposition 2 implies that the flat torus M_Γ is isometric to $\mathbb{R}^2/W(\Gamma)$. So, there exists a linear isometry $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$T(G) \subset W(\Gamma).$$

Let $\{\vec{\xi}_1, \vec{\xi}_2\}$ be a generator of the lattice G, and let $\{\vec{\eta}_1, \vec{\eta}_2\}$ be a generator of the lattice $W(\Gamma)$. Then there exist integers c_{ij} such that

$$\begin{cases} T(\vec{\xi}_1) = c_{11}\vec{\eta}_1 + c_{12}\vec{\eta}_2, \\ T(\vec{\xi}_2) = c_{21}\vec{\eta}_1 + c_{22}\vec{\eta}_2. \end{cases}$$

This implies that

$$\tau(f) = n\tau(f_\Gamma), \quad n = |c_{11}c_{22} - c_{12}c_{21}|.$$

On the other hand, it follows from Proposition 3 that

$$\tau(f_\Gamma) = \frac{1}{2}\langle \vec{a}, \vec{b} \rangle, \quad \vec{a}, \vec{b} \in W(\Gamma).$$

Since $n\vec{\eta}_i \in T(G)$, there exist $\vec{\alpha},\ \vec{\beta} \in G$ such that

$$n\vec{a} = T(\vec{\alpha}), \quad n\vec{b} = T(\vec{\beta}).$$

Hence

$$\tau(f) = \frac{1}{2n}\langle T(\vec{\alpha}),\ T(\vec{\beta})\rangle = \frac{1}{2n}\langle \vec{\alpha},\ \vec{\beta} \rangle \in X(G).$$

This completes the proof.

References

[1] Y. Kitagawa, *Periodicity of the asymptotic curves on flat tori in S^3*, J. Math. Soc. Japan, **40** (1988), 457-476.

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[3] Y. Kitagawa, *Isometric deformations of flat tori in the 3-sphere with nonconstant mean curvature*, Tohoku Math. J., **52** (2000), 283 - 298.