Flat fronts in H^3

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In this talk, I introduce some results obtained with coauthors M. Kokubu, M. Umehara and K. Yamada. Primary references are:

[GMM] J. A. Gálvez, A. Martínez and F. Milán, *Flat surfaces in hyperbolic* 3*-space*, Math. Ann., **316** (2000), 419–435.

[KRSUY] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic space*, Pacific J. Math. **221** (2005), 303–351.

[KRUY1] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Flat fronts in hyperbolic 3-space and their caustics*, to appear in J. Math. Soc. Japan.

[KRUY2] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Asymptotic behavior of regular ends of flat fronts in hyperbolic* 3*-space*, in preparation.

[KUY1] M. Kokubu, M. Umehara and K. Yamada, *An elementary proof of Small's formula for null curves in* PSL$(2,\boldsymbol{C})$ *and an analogue for Legendrian curves in* PSL$(2,\boldsymbol{C})$, Osaka J. Math., **40(3)** (2003), 697–715.

[KUY2] M. Kokubu, M. Umehara and K. Yamada, *Flat fronts in hyperbolic* 3*-space*, Pacific J. Math., **216** (2004), no.1, 149–175.

1. Preliminaries

L^4 is the Minkowski 4-space with the inner product $\langle\cdot,\cdot\rangle$ of signature $(-,+,+,+)$. The hyperbolic 3-space H^3 is the upper half component of the two-sheeted hyperboloid in L^4 with the metric induced by the inner product $\langle\cdot,\cdot\rangle$. Identifying L^4 with the set of 2×2-hermitian matrices Herm(2) via

$$L^4 \ni (x_0,x_1,x_2,x_3) \leftrightarrow \begin{pmatrix} x_0+x_3 & x_1+\sqrt{-1}x_2 \\ x_1-\sqrt{-1}x_2 & x_0-x_3 \end{pmatrix} \in \mathrm{Herm}(2),$$

H^3 is represented as

$$\begin{aligned} H^3 &= \{x=(x_0,x_1,x_2,x_3)\in L^4\,;\,\langle x,x\rangle=-1, x_0>0\} \\ &= \{X\in \mathrm{Herm}(2)\,;\, \det X=1,\ \mathrm{trace} X>0\} \\ &= \{aa^*\,;\, a\in SL(2,C)\} = SL(2,C)/SU(2). \end{aligned}$$

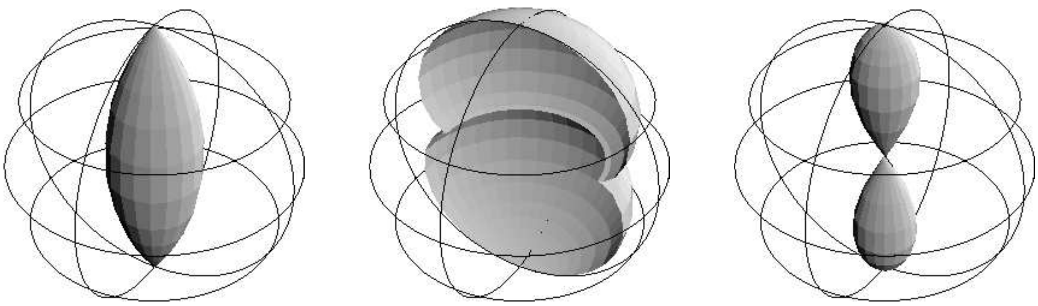
A flat front f from a Riemann surface with local coordinate z to H^3 can be constructed from two complex analytic one-forms $\omega=\hat{\omega}dz$ and $\theta=\hat{\theta}dz$ on the Riemann surface as follows:

$$f=\mathcal{E}\bar{\mathcal{E}}^t=\mathcal{E}\mathcal{E}^*\ , \quad \text{where} \quad \mathcal{E}=\begin{pmatrix} A & B \\ C & D\end{pmatrix} \quad \text{solves} \quad d\mathcal{E}=\mathcal{E}\begin{pmatrix} 0 & \theta \\ \omega & 0\end{pmatrix}\ .$$

Here we are identifying Hermitian matrices with determinant 1 with points (x_0,x_1,x_2,x_3) in the Minkowski model for H^3 via

$$\mathcal{E}\mathcal{E}^*=\begin{pmatrix} x_0+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0-x_3\end{pmatrix}\ .$$

FIGURE 1.1. Flat fronts of revolution in H^3



(a) cylinder (b) snowman (c) hourglass

We define $g = \int \omega$ and $g_* = \int \theta$ and the Hopf differential $Q = \hat{Q}dz^2 = \omega\theta$. Then the hyperbolic Gauss maps are

$$G = \tfrac{A}{C}\ , \quad G_* = \tfrac{B}{D}\ .$$

Because $\det \mathcal{E} = 1$, we have

$$dG = \frac{-\omega}{C^2}\ , \quad dG_* = \frac{\theta}{D^2}\ , \quad G - G_* = (CD)^{-1}\ ,$$

so

$$Q = -(CD)^2 dG dG_* = \frac{-dGdG_*}{(G-G_*)^2}\ .$$

Another computation gives

$$\frac{G''}{G'} - \frac{2G'}{G-G_*} = \frac{\hat{\omega}'}{\hat{\omega}}\ , \quad \frac{G_*''}{G_*'} - \frac{2G_*'}{G_*-G} = \frac{\hat{\theta}'}{\hat{\theta}}\ .$$

Defining

$$S(G) = \{G, z\} = \left(\frac{G''}{G'}\right)' - \frac{1}{2}\left(\frac{G''}{G'}\right)^2$$

as the Schwarzian derivative, and defining $s(\hat{\omega}) = (\hat{\omega}'/\hat{\omega})' - (1/2)(\hat{\omega}'/\hat{\omega})^2$, we have

$$S(g) - S(G) = 2Q \ \text{ iff } \ s(\hat{\omega}) - \{G, z\} = 2\hat{Q}\ ,$$

$$S(g_*) - S(G_*) = 2Q \ \text{ iff } \ s(\hat{\theta}) - \{G_*, z\} = 2\hat{Q}\ .$$

We know that $S(g) - S(G) = 2Q$ and $S(g_*) - S(G_*) = 2Q$ hold, by using

$$\frac{G''}{G'} = \frac{\hat{\omega}'}{\hat{\omega}} - \frac{2D\hat{\omega}}{C}\ , \quad \frac{G_*''}{G_*'} = \frac{\hat{\theta}'}{\hat{\theta}} - \frac{2C\hat{\theta}}{D}\ .$$

Changing $\mathcal{E}$ to

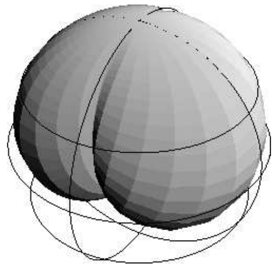
$$\mathcal{E} \cdot \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix}$$

does not change the surface if $\gamma \in R$, but does change θ and ω to $e^{i\gamma}\theta$ and $e^{-i\gamma}\omega$. So θ and ω have a $U(1)$-ambiguity.

General isometric motions of f are described by the transformation

$$\mathcal{E} \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \mathcal{E}\ ,$$

Figure 1.2. Peach front in H^3



where (a_{ij}) is a general matrix in $SL_2(\mathbb{C})$. Under this transformation, G changes as follows:

$$G \to \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}} .$$

A point $(x_0, x_1, x_2, x_3) \in H^3$ in the Minkowski model becomes

$$\frac{(x_1, x_2, x_3)}{1 + x_0}$$

in the Poincare model.

Remark 1.1. It was shown in [KUY1] that one has the Small-type formula

$$\mathcal{E} = \begin{pmatrix} A & dA/\omega \\ C & dC/\omega \end{pmatrix} ,$$

where $A = CG$ and $C = i\sqrt{\frac{\omega}{dG}}$.

2. Results

The first result gives criteria for cuspidal edge and swallowtail singularities.

Theorem 2.1. *Let $f\colon M^2 \to H^3$ be a flat front with canonical forms $(\omega = \hat{\omega}\, dz, \theta = \hat{\theta}\, dz)$, where z is a local complex coordinate.*

1. *A point $p \in M^2$ is a singular point if and only if $|\hat{\omega}(p)| = |\hat{\theta}(p)|$ holds.*
2. *The image of f around a singular point p is locally diffeomorphic to a cuspidal edge if and only if*

$$\hat{\omega}'\hat{\theta} - \hat{\theta}'\hat{\omega} \neq 0 \qquad \text{and} \qquad \operatorname{Im}\left(\frac{(\hat{\theta}'/\hat{\theta}) - (\hat{\omega}'/\hat{\omega})}{\sqrt{\hat{\omega}\hat{\theta}}}\right) \neq 0$$

hold at p, where $' = d/dz$.

3. *The image of f around a singular point p is locally diffeomorphic to a swallowtail if and only if*

$$\hat{\omega}'\hat{\theta} - \hat{\theta}\hat{\omega}' \neq 0, \qquad \operatorname{Im}\left(\frac{(\hat{\theta}'/\hat{\theta}) - (\hat{\omega}'/\hat{\omega})}{\sqrt{\hat{\omega}\hat{\theta}}}\right) = 0$$

$$\text{and} \qquad \operatorname{Re}\left(\frac{s(\hat{\theta}) - s(\hat{\omega})}{\hat{\omega}\hat{\theta}}\right) \neq 0$$

FIGURE 2.1. 3-noids.

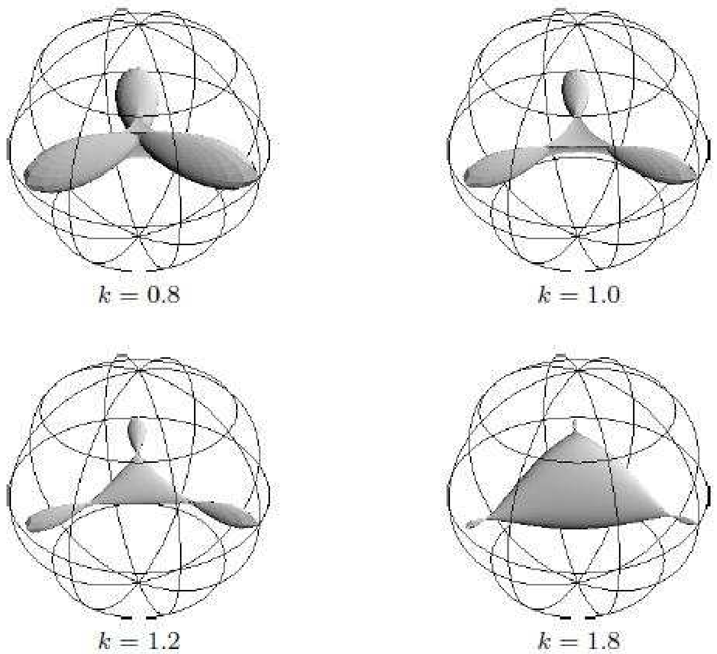
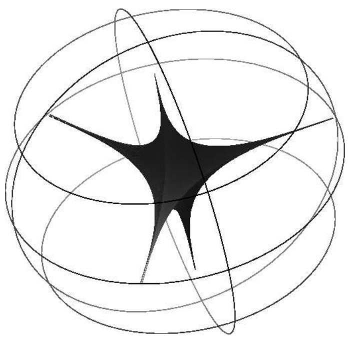
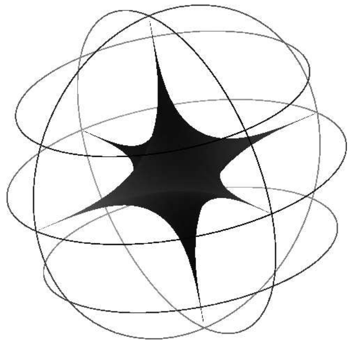


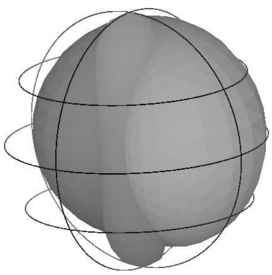
FIGURE 2.2. Caustics of 3-noids and 4-noids, and of the front with $G = z$, $G_* = z^2$.



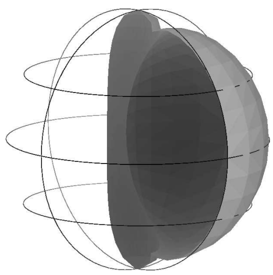
The caustic of a 3-noid



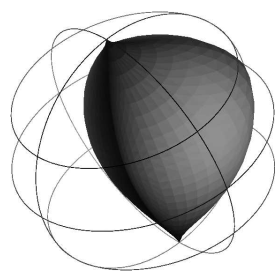
The caustic of a 4-noid



$(G, G_*) = (z, z^2)$



$(G, G_*) = (z, z^2)$ (half cut)



$(G, G_*) = (z, z^2)$ (caustic)

FIGURE 2.3. A genus 1 complete front with 5 embedded ends (left), with its caustic (center). Also, a genus 2 flat front with 10 embedded ends (right), where the Riemann surface is $\overline{M}^2 = \{(z, w) \in (\boldsymbol{C} \cup \{\infty\})^2\,;\, w^2 = z(z^2-1)(z^2-9/4)\}$ with 10 points removed, and $G = w$, $G_* = (5w - 2z(dw/dz))/5$.

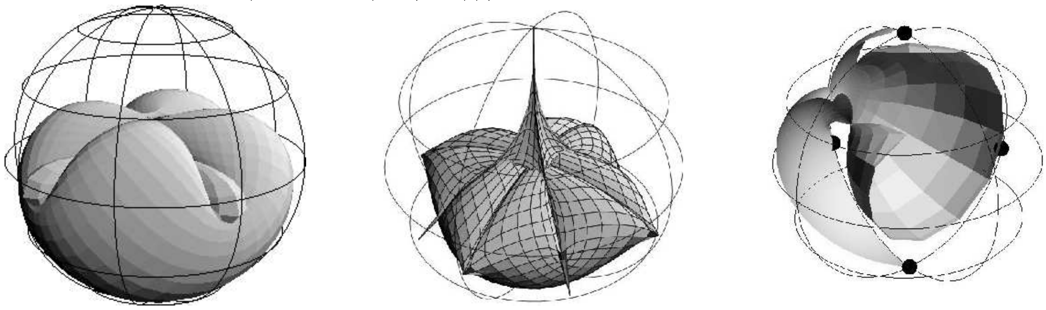
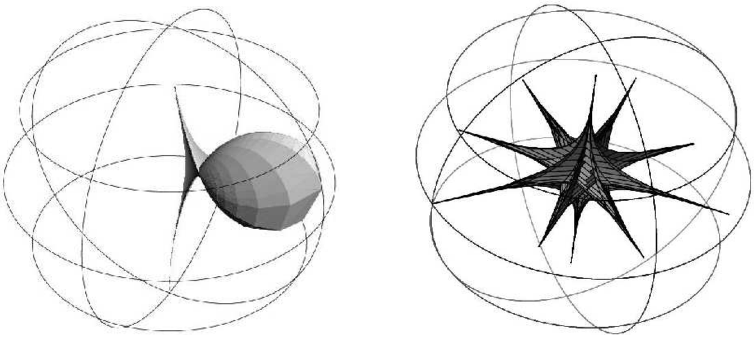


FIGURE 2.4. A p-front that is not globally a caustic, and the caustic with dihedral cross $\boldsymbol{Z}^2$ symmetry for $G = z^3$ and $G_* = z^{-5}$ and Riemann surface $\boldsymbol{C} \setminus \{z\,;\, z^8 = 1\}$.



hold at p*, where* $s(\hat{\omega})$ *is the Schwarzian derivative* $\{h, z\}$ *of the function* $h(z) := \int_{z_0}^{z} \omega$ *with respect to* z*, that is,*

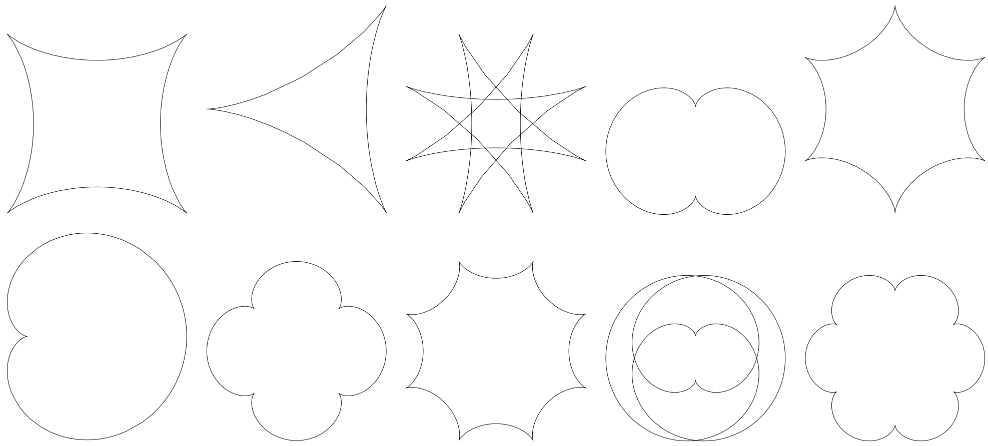
$$s(\hat{\omega}) = \{h, z\} = \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2 = \left(\frac{\hat{\omega}'}{\hat{\omega}}\right)' - \frac{1}{2}\left(\frac{\hat{\omega}'}{\hat{\omega}}\right)^2. \tag{2.1}$$

The next result is about the rarity of singularities other than cuspidal edges and swallowtails.

Definition 2.2. *A front* $f\colon M^2 \to H^3$ *is called complete if there exist a compact set* $C \subset M^2$ *and a symmetric* 2*-tensor* T *on* M^2 *such that* T *is identically* 0 *outside* C *and* $ds^2 + T$ *is a complete Riemannian metric of* M^2*, where* ds^2 *is the first fundamental form of* f*.*

Theorem 2.3. *Let* $f\colon M^2 \to H^3$ *be a complete flat front which is not a covering of an hourglass (hourglasses are rotationally symmetric), and let* $\{f_t\}$ *be the family of parallel fronts of* f*. Then, except for only finitely many values of* t*, all the singular points of* f_t *are locally diffeomorphic to cuspidal edges or swallowtails.*

FIGURE 2.5. Various cycloid curves



The next result relates completeness, weak completeness, the finite type property and the finite topology property.

Definition 2.4. *We say that f is weakly complete if $ds^2_{1,1} = |\omega|^2 + |\theta|^2$ is complete and Riemannian on M^2.*

Definition 2.5. *We say that a flat front f is of* finite type *if $ds^2_{1,1}$ has finite total curvature.*

Theorem 2.6. *A complete flat front is weakly complete and of finite type. Conversely, if $f\colon M^2 \to H^3$ is a weakly complete flat front of finite type, then there exists a finite set of real numbers $t_1, \dots, t_n$ such that $f_t\colon M^2 \to H^3$ is a complete flat front for all $t \in \boldsymbol{R} \setminus \{t_1, \dots, t_n\}$.*

The following theorem is an important property of flat surfaces in H^3, because there do in fact exist flat Möbius bands in $\boldsymbol{R}^3$ and S^3. For S^3 this is a deep fact, since such a front in S^3 can be of class C^∞, but is never C^ω, see a work of Gálvez and Mira.

Theorem 2.7. *Any flat p-front is orientable.*

The following result relates finite topology and weak completeness to properties of the corresponding caustic.

Theorem 2.8. *For a flat front $f\colon M^2 \to H^3$, the following assertions are equivalent*:

(1) *M^2 is biholomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$ for some compact Riemann surface $\overline{M}^2$ containing the points p_j, and f is a weakly complete flat front, all of whose ends are regular.*
(2) *The caustic C_f is a weakly complete p-front of finite type, all of whose ends are regular.*

Finally, we note that caustics can have ends with cross sections that asymptotically are cycloids, as will be shown in [KRUY2].