Horocyclic surfaces in hyperbolic 3-space

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This talk is based on the joint work [3] with C.Takizawa.

§1 Introduction

Horocyclic surfaces in hyperbolic 3-space are defined analogously to ruled surfaces in the Euclidean space $\mathbb{R}^3$:

		foliated by
ruled surfaces in $\mathbb{R}^3$	$\cdots$	lines
horocyclic surfaces in H^3	$\cdots$	horocycles

The notion of horocyclic surfaces was introduced by S.Izumiya, K.Saji and M.Takahashi ([1]) as one of important objects in their **horospherical geometry**.

In this talk I explain

- to construct horocyclic surfaces associated with spacelike curves in the lightcone
- to describe geometric properties of horocyclic surfaces in terms of invariants (curvatures and torsions) of corresponding spacelike curves in the lightcone
- in particular to classify singularities of horocyclic surfaces constructed above

§2 Curves and surfaces in hyperbolic 3-space

$\langle$ **The standard Lorentzian model of hyperbolic 3-space** $\rangle$ Let $\mathbb{R}^4_1$ be a 4-dimensional vector space $\mathbb{R}^4$ with the inner product $\langle , \rangle$ defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i \quad \text{for} \quad \boldsymbol{x} = (x_0, x_1, x_2, x_3), \boldsymbol{y} = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4.$$

We say that a non-zero vector $\boldsymbol{v} \in \mathbb{R}^4_1$ is *spacelike, lightlike*, or *timelike* if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$, $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ or $\langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0$, respectively.

We define

$$\begin{aligned} H^3_+ &= \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, \text{ and } x_0 \geq 0\} &&: \text{ the hyperbolic 3-space} \\ LC_+ &= \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \text{ and } x_0 > 0\} &&: \text{ the future light cone} \\ S^3_1 &= \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\} &&: \text{ the de Sitter space} \end{aligned}$$

Then H^3_+ is a complete Riemannian manifold of constant sectional curvature -1.

$\langle$ **Curves in** H^3_+ $\rangle$ Let $\gamma : I \to H^3_+$ be a unit speed curve defined on an open interval I of $\mathbb{R}$. We use $\boldsymbol{t}(s)$ for the tangent vector $\gamma'(s)$ with $\|\boldsymbol{t}(s)\| = 1$. The vector $\boldsymbol{t}'(s) - \gamma(s)$ is orthogonal to $\gamma(s)$ and $\boldsymbol{t}(s)$. We assume that $\boldsymbol{t}'(s) - \gamma(s)$ is not zero. Then we have a unit vector $\boldsymbol{n}(s) = (\boldsymbol{t}'(s) - \gamma(s))/\|\boldsymbol{t}'(s) - \gamma(s)\|$. Moreover we define $\boldsymbol{e}(t)$ such that $\{\gamma(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ is a positive orthonormal basis. Then the following formula (Frene-Seret formula) holds:

$$\begin{pmatrix} \gamma'(s) \\ \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \\ \boldsymbol{e}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \kappa_h(s) & 0 \\ 0 & -\kappa_h(s) & 0 & \tau_h(s) \\ 0 & 0 & -\tau_h(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{e}(s) \end{pmatrix}.$$

A curve γ in H^3_+ is called *a horocycle* if $\kappa_h(s) \equiv 1$ and $\tau_h(s) \equiv 0$.

Proposition 2.1 (1) Let $\gamma : I \to H^3_+$ be a horocycle . Then $\gamma(s) + \boldsymbol{n}(s)$ is a constant lightlike vector which is denoted by l_0 and called to be *the polar vector* of a horocycle γ. Moreover we have

$$\gamma(s) = \gamma(s_0) + (s - s_0)\boldsymbol{t}(s_0) + \frac{1}{2}(s - s_0)^2 l_0 \quad \text{for some fixed } s_0 \in I.$$

(2) For vectors $l_0 \in LC_+$ and $\omega_0 \in S^3_1$ with $\langle l_0, \omega_0 \rangle = 0$, $\{\boldsymbol{x} \in H^3_+ \mid \langle \boldsymbol{x}, l_0 \rangle + 1 = 0,\ \langle \boldsymbol{x}, \omega_0 \rangle = 0\}$ is a horocycle of H^3_+ whose polar vector is given by l_0.

$\langle$ **Surfaces in** H^3_+ $\rangle$ Let M be a 2-dimensional connected oriented manifold and $f : M \to H^3_+$ an immersion.
We define two kinds of " Gauss maps":

Gauss map	$\mathbb{E} : M \to S^3_1$ a unit normal vector field in H^3_+
Lightcone Gauss map	$\mathbb{L}^\pm = f \pm \mathbb{E} : M \to LC_+$

and define associated shape operators at each point $p \in M$:

Shape operator	$A_p : T_pM \to T_pM$	defined by $d\mathbb{E} = -df(A_p)$
Hyperbolic shape operator	$S^\pm_p = -1_{T_pM} \pm A_p$	where $d\mathbb{L}^\pm = df_p \pm d\mathbb{E}_p = -df_p(S^\pm_p)$

We denote the eigenvalues of A_p by $\kappa_i(p)$ and those of $S^\pm_p$ by $\bar{\kappa}^\pm_i(p)$ $(i = 1, 2)$, respectively. Evidently, A_p and $S^\pm_p$ have same eigenvectors and the relation $\bar{\kappa}^\pm_i(p) = -1 \pm \kappa_i(p)$. We call $\kappa_i(p)$ *principal curvatures* and $\bar{\kappa}^\pm_i(p)$ *hyperbolic principal curvatures* of (M, f) at $p \in M$, respectively. The Gauss curvature of M at p is given by $K(p) = -1 + \det A_p = -1 + \kappa_1(p)\kappa_2(p)$ and similarly *the hyperbolic Gauss curvature* $K^\pm_h$ is defined by $K^\pm_h(p) = \det S^\pm_p = \bar{\kappa}^\pm_1(p)\bar{\kappa}^\pm_2(p)$.

Now we recall the geometry of horospheres. For a non-zero vector $\boldsymbol{v} \in \mathbb{R}^4_1$ and a real number $c \in \mathbb{R}$, we define the hyperplane by $HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}$. Then the intersections $M = H^3_+ \cap HP(\boldsymbol{v}, c)$ (if not empty) are totally umbilical surfaces of H^3_+ with $\kappa^2 = \frac{c^2}{\langle \boldsymbol{v}, \boldsymbol{v} \rangle + c^2}$, where κ denotes the principal curvature . In particular for $\boldsymbol{v} \in LC_+$ and $c < 0$, $M = H^3_+ \cap HP(\boldsymbol{v}, c)$ has the constant principal curvature 1 or -1 and hence its Gauss curvature vanishes. It is called *a horosphere*. Let $M = H^3_+ \cap HP(\boldsymbol{v}, -1)$ with $\boldsymbol{v} \in LC_+$ be a horosphere. We consider an inclusion map $\iota : M \to H^3_+$. For $\boldsymbol{x} \in M$,

$\mathbb{E}(\boldsymbol{x}) = \boldsymbol{v} - \boldsymbol{x}$ is the unit normal vector of M at $\boldsymbol{x} \in M$. Therefore we have the lightcone Gauss map $\mathbb{L}^+(\boldsymbol{x}) = \boldsymbol{x} + \mathbb{E}(\boldsymbol{x}) = \boldsymbol{v}$ at $\boldsymbol{x} \in M$. It is a constant map and hence the hyperbolic Gauss curvature K_h^+ vanishes.

Definition 2.2 ([1]) An immersed surface (M, f) in H^3_+ is *a horospherical flat surface* (briefly *a horo-flat surface*) if $K_h^+(p) = 0$ or $K_h^-(p) = 0$ at any point $p \in M$.

Horospheres are examples of horo-flat surfaces and they are also totally umbilical. Now we assume that an immersed surface (M, f) is a horo-flat surface without umbilical points. Then the shape operator A of (M, f) has two different eigenvalues 1 and $\kappa(\neq 1)$. We denote by $\mathcal{F}_1$ the line foliation corresponding to the principal curvature 1. Then the lines of $\mathcal{F}_1$ are horocycles in H^3_+ whose polar vectors coincide with the lightcone Gauss map $\mathbb{L}$ (Theorem 4.4 in [1]).

Definition 2.3 ([1]) A surface (M, f) in H^3_+ is called *a horocyclic surface* if it is foliated by horocycles in H^3_+.

§3 Spacelike curves in the lightcone

We introduce the Frenet-Serret formula for spacelike curves in the lightcone LC_+. Let $l : I \to LC_+$ be a unit speed spacelike curve defined on an open interval I of $\mathbb{R}$. We set $\boldsymbol{t}(t) = l'(t)$ and define a function κ on I by $\kappa(t) = \langle l''(t), l''(t)\rangle$. We put $\boldsymbol{n}(t) = -\frac{\kappa(t)}{2} l(t) - l''(t)$. Then we have $\langle \boldsymbol{n}(t), \boldsymbol{n}(t)\rangle = 0$ and $\langle \boldsymbol{n}(t), l(t)\rangle = 1$. We denote by $\boldsymbol{e}(t)$ the unit spacelike vector which is orthogonal to $l(t), \boldsymbol{t}(t)$ and $\boldsymbol{n}(t)$ such that the basis $\{l(t), \boldsymbol{t}(t), \boldsymbol{e}(t), \boldsymbol{n}(t)\}$ has a positive orientation. Then we have an pseudo orthonormal frame field $\{l(t), \boldsymbol{t}(t), \boldsymbol{e}(t), \boldsymbol{n}(t)\}$ along l . We define a function τ on I by $\tau(t) = \langle \boldsymbol{n}'(t), \boldsymbol{e}(t)\rangle$. Then we have the following formula of Frenet-Serret type :

$$\begin{pmatrix} l'(t) \\ \boldsymbol{t}'(t) \\ \boldsymbol{e}'(t) \\ \boldsymbol{n}'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\kappa(t)}{2} & 0 & 0 & -1 \\ -\tau(t) & 0 & 0 & 0 \\ 0 & \frac{\kappa(t)}{2} & \tau(t) & 0 \end{pmatrix} \begin{pmatrix} l(t) \\ \boldsymbol{t}(t) \\ \boldsymbol{e}(t) \\ \boldsymbol{n}(t) \end{pmatrix}. \tag{3.1}$$

§4 Construction of horocyclic surfaces

We construct horocyclic surfaces in H^3_+ by a slightly different method to [1] and study their properties. Let $l : I \to LC_+$ and $\omega : I \to S^3_1$ be smooth maps from an open interval I of $\mathbb{R}$ with $\langle l(t), \omega(t)\rangle = 0$ for any $t \in I$. Now we set subsets $\tilde{M}$ of $H^3_+ \times I$ and M of H^3_+ as follows:

$$\begin{aligned} \tilde{M} &= \{(\boldsymbol{x}, t) \in H^3_+ \times I \mid \langle \boldsymbol{x}, l(t)\rangle + 1 = 0, \ \langle \boldsymbol{x}, \omega(t)\rangle = 0\}, \\ M &= \{\boldsymbol{x} \in H^3_+ \mid \text{there exists } t \in I \text{ with } (\boldsymbol{x}, t) \in \tilde{M}\}. \end{aligned}$$

Then the set $\tilde{M}$ is a 2-dimensional submanifold of $H^3_+ \times I$. We restrict the projections of $H^3_+ \times I$ onto H^3_+ and of $H^3_+ \times I$ onto I to $\tilde{M}$, which are denoted by π_1 and π_2, respectively. Then $M = \pi_1(\tilde{M})$ and M is a horocyclic surface in H^3_+ possibly with singularites. In fact, for each $t \in I$ we put

$$C_t = \pi_1 \circ \pi_2^{-1}(t) = \{\boldsymbol{x} \in H^3_+ \mid \langle \boldsymbol{x}, l(t)\rangle + 1 = 0, \ \langle \boldsymbol{x}, \omega(t)\rangle = 0\}.$$

By Proposition 2.1 (2), it follows that C_t is a horocycle in H^3_+ whose polar vector is $l(t)$.

Now we will construct a parametrization $f : \mathbb{R}\times I \to H^3_+$ with $f(\mathbb{R}\times I) = M$. We choose a smooth curve $\tilde{\gamma}(t) = (\gamma(t), t)$ in $\tilde{M}$. Then γ is a smooth curve in H^3_+ which is contained in M. For each $t \in I$, we put $a_2(t) = l(t) - \gamma(t)$ and $a_3(t) = \omega(t)$. Then $\gamma(t), a_2(t)$, and $a_3(t)$ are orthonormal. We define a unit spacelike vector $a_1(t)$ such that $a_1(t)$ is orthogonal to $\gamma(t), a_2(t), a_3(t)$ and the basis $\{\gamma(t), a_1(t), a_2(t), a_3(t)\}$ has a positive orientation. Then we have $a_1(t) \in T_{\gamma(t)}C_t$ and the horocycle C_t is parametrized as in Proposition 2.1 (1). We now define a map f of $\mathbb{R} \times I$ to H^3_+ by

$$f(s, t) = \gamma(t) + sa_1(t) + \frac{1}{2}s^2 l(t).$$

Then $f(\mathbb{R} \times I) = M$.

In order to study the detailed geometric properties of f , we consider a special class of horocyclic surfaces which satisfy the following condition:

(**HC**) we assume that $l : I \to LC_+$ has a unit speed and that $l'(t) = \omega(t)$.

Under the assumption above, the structure equation for the orthonormal frame fields $\{\gamma(t), a_1(t), a_2(t), a_3(t)\}$ is of the following form:

$$\begin{pmatrix} \gamma'(t) \\ a_1'(t) \\ a_2'(t) \\ a_3'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & c_3(t) \\ 0 & 0 & 0 & c_5(t) \\ 0 & 0 & 0 & c_6(t) \\ c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ a_1(t) \\ a_2(t) \\ a_3(t) \end{pmatrix}, \quad c_3(t) + c_6(t) = 1.$$

The induced metric g on $\mathbb{R} \times I$ by f is given by

$$g = ds^2 + \frac{1}{4}(s^2 + 2c_5(t)s + 2c_3(t))^2 dt^2.$$

This implies the following:

Lemma 4.1 Under the assumption (HC), a point $(s_0, t_0) \in \mathbb{R} \times I$ is a singular point of f if and only if

$$s_0^2 + 2c_5(t_0)s_0 + 2c_3(t_0) = 0.$$

At a non-singular point $(s, t) \in \mathbb{R} \times I$, the shape operator A is given by

$$A\left(\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial s}, \quad A\left(\frac{\partial}{\partial t}\right) = \left(1 - \frac{2}{s^2 + 2c_5(t)s + 2c_3(t)}\right)\frac{\partial}{\partial t}.$$

Consequently we obtain the following.

Proposition 4.2 A horocyclic surface M constructed above is horo-flat.

Now we describe the geometric properties of a horocyclic surface M in terms of invariants κ, τ defined in (3.1) of the corresponding spacelike curve $l : I \to LC_+$. First we show the criteria for the existence of singularities on the horocycle C_t for $t \in I$.

Proposition 4.3 (1) If $\kappa(t) > 1$, there exist two singular points of f on the horocycle C_t.
(2) If $\kappa(t) = 1$, there exists only one singular point of f on the horocycle C_t.
(3) If $\kappa(t) < 1$, there exists no singularity of f on the horocycle C_t.

Applying the criteria above, we can construct complete non-singular horo-flat horocyclic surfaces.Let $l : \mathbb{R} \to LC_+$ be a unit speed spacelike curve defined on the whole $\mathbb{R}$. Suppose that there exists a positive number $\varepsilon > 0$ such that $\kappa(t) \leq 1 - \varepsilon$ for any $t \in \mathbb{R}$.

Then by Proposition 4.3 (3), f is an immersion of $\mathbb{R} \times \mathbb{R}$ into H^3_+ . Moreover we see that the induced Riemannian metric g on $\mathbb{R} \times \mathbb{R}$ by f is geodesically complete.

§5 Classification of singularities

Now we study the cases of Proposition 4.3 (1) and (2) in the previous section. For a fixed $t_0 \in I$ with $\kappa(t_0) \geq 1$, we may assume that a point $(0, t_0) \in \mathbb{R} \times I$ is a singular point of f. By Lemma 4.1, it follows that $c_3(t_0) = 0$ and the other singular point on C_{t_0} is given by $(-2c_5(t_0), t_0)$. Moreover we assume that $c_5(t_0) \geq 0$. Then we have the following theorem:

Theorem 5.1 (A) Suppose that $\kappa(t_0) > 1$. Then we have the following:
(1) The point $(0, t_0)$ is the cuspidal edge if $\frac{1}{2}\kappa'(t_0) - \sqrt{\kappa(t_0) - 1}\tau(t_0) \neq 0$.
(2) The point $(0, t_0)$ is the swallowtail if $\frac{1}{2}\kappa'(t_0) - \sqrt{\kappa(t_0) - 1}\tau(t_0) = 0$ and
$\frac{1}{2}\kappa''(t_0) - \sqrt{\kappa(t_0) - 1}\tau'(t_0) - \tau(t_0)^2 \neq 0$.
(3) The point $(-2c_5(t_0), t_0)$ is the cuspidal edge if $\frac{1}{2}\kappa'(t_0) + \sqrt{\kappa(t_0) - 1}\tau(t_0) \neq 0$.
(4) The point $(-2c_5(t_0), t_0)$ is the swallowtail if $\frac{1}{2}\kappa'(t_0) + \sqrt{\kappa(t_0) - 1}\tau(t_0) = 0$ and
$\frac{1}{2}\kappa''(t_0) + \sqrt{\kappa(t_0) - 1}\tau'(t_0) - \tau(t_0)^2 \neq 0$.
(B) Suppose that $\kappa(t_0) = 1$. Then we have $c_5(t_0) = 0$ and the point $(0, t_0)$ is only one singular point on C_{t_0}. Moreover we have the following:
(1) The point $(0, t_0)$ is the cuspidal edge if $\kappa'(t_0) \neq 0$.
(2) The point $(0, t_0)$ is the cuspidal beaks if $\kappa'(t_0) = 0$, $\kappa''(t_0) > 0$, and $\kappa''(t_0) \neq 2\tau(t_0)^2$.
(3) The point $(0, t_0)$ is the cuspidal lips if if $\kappa'(t_0) = 0$, $\kappa''(t_0) < 0$.

Here the cuspidal edge is a germ of surface diffeomorphic to $\{(x_1, x_2, x_3) | x_1 = u^2, x_2 = u^3, x_3 = v\}$, the swallowtail is a germ of surface diffeomorphic to $\{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$, the cuspidal beaks is a germ of surface diffeomorphic to $\{(x_1, x_2, x_3) | x_1 = 3u^4 - 2u^2v^2, x_2 = u^3 - uv^2, x_3 = v\}$ and the cuspidal lips is a germ of surface diffeomorphic to $\{(x_1, x_2, x_3) | x_1 = 3u^4 + 2u^2v^2, x_2 = u^3 + uv^2, x_3 = v\}$.

Remark This theorem is an another formulation of the classification theorem (Theorem 6.2 in [1]) proved by S.Izumiya, K.Saji and M.Takahashi. Compared with their theorem, our classification is to use invariants (the curvatures κ and the torsions τ) of spacelike curves in the lightcone.

Recently useful criteria in order to recognize the singularities of surfaces have been developed (cf. [2], [1]). We apply their recognition lemmas and prove the Theorem above.

References

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