INDICES OF HYPERSURFACES IN A SPHERE

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1. Index of minimal hypersurfaces

Let $\varphi: \ M \to S^{n+1}(1)$ be an n-dimensional compact hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. We consider a differentiable map X given by

$$X: \ (-\varepsilon, \varepsilon) \times M \to S^{n+1}(1).$$

X is called a variation of φ if and only if, for any $t \in (-\varepsilon, \varepsilon)$,

$$\varphi_t: \ M \to S^{n+1}(1)$$

defined by $\varphi_t(p) = X(t,p)$ is an immersion with $\varphi_0 = \varphi$. The area of φ_t is given by

$$A(t) = \int_M dA_t$$

and the volume of φ_t is defined by

$$V(t) = \int_{[0,t]\times M} X^* dS^{n+1}(1).$$

For any t, if $V(t) = V(0)$, then the variation X is called volume-preserving. If the variational vector $\dfrac{\partial X}{\partial t}|_{t=0} = fN$ for a smooth function f, then the variation is called a normal variation, where N is the unit normal of φ. The following fact is known.

Lemma 1.1. *For a smooth function f satisfying*

$$\int_M f dA = 0,$$

there exists a volume-preserved normal variation X of φ such that its variational vector is equal to fN.

Let H denote the mean curvature of φ. The first variation formula of the area functional $A(t)$ is given by

$$\frac{dA(t)}{dt}|_{t=0} = -\int_M nHu dA,$$

where $u = \langle \frac{\partial X}{\partial t}|_{t=0}, N\rangle$. Hence, we know, for a compact minimal hypersurface, that is, $H = 0$

$$\frac{dA(t)}{dt}|_{t=0} = 0,$$

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namely, compact minimal hypersurfaces are critical points of the area functional $A(t)$. The second variation formula of $A(t)$ is given by

$$\frac{d^2A(t)}{dt^2}|_{t=0} = \int_M uJ_m udA$$

and

$$J_m u = -\Delta u - (S+n)u,$$

where S denotes the squared norm of the second fundamental form of φ and Δ stands for the Laplace-Beltrami operator. The J_m is called a Jacobi operator or a stability operator of the minimal hypersurface φ.

Define

$$Q(u) = \int_M uJ_m udA.$$

The dimension of the maximal subspace of $C^\infty(M)$, in which Q is negative, is called index of the minimal hypersurface φ, denoted by $\mathrm{Ind}(M)$. Thus, we know the index $\mathrm{Ind}(M)$ is equal to the number of negative eigenvalues of the Jacobi operator J_m.

Let $\lambda_1^{J_m}$ denote the first eigenvalue of the Jacobi operator J_m. Then

$$J_m u = \lambda_1^{J_m} u$$

and the $\lambda_1^{J_m}$ is given by

$$\lambda_1^{J_m} = \inf_{u \not\equiv 0} \frac{\int_M uJ_m udA}{\int_M u^2 dA}$$

From the definition the Jacobi operator $J_m = -\Delta - (S+n)$, we know

$$\lambda_1^{J_m} \le -n$$

and $\lambda_1^{J_m} = -n$ if and only if M is totally geodesic (see [6]). Furthermore, Wu [7] proved that for an n-dimensional compact non-totally geodesic minimal hypersurface M in $S^{n+1}(1)$, then $\lambda_1^{J_m} \le -2n$ and $\lambda_1^{J_m} = -2n$ if and only if M is a Clifford torus

$$S^{n-k}(\sqrt{\frac{n-k}{n}}) \times S^k(\sqrt{\frac{k}{n}}),$$

for $k = 1, 2, \cdots, n-1$,

On the other hand, for any fixed constant vector $\mathbf{a} \in \mathbf{R}^{n+2}$, we consider a function

$$f_a = \langle N, \mathbf{a} \rangle.$$

We can obtain

$$J_m f_a = -n f_a.$$

If M is a compact non-totally geodesic minimal hypersurface in the unit sphere, then $-n$ is an eigenvalue of J_m. Thus, we have

Theorem 1.1. *If M is an n-dimensional non-totally geodesic compact minimal hypersurface in the unit sphere $S^{n+1}(1)$, then $Ind(M) \ge n+3$.*

For a Clifford torus, we can prove

$$\lambda_1^{J_m} = -2n, \quad \lambda_2^{J_m} = -n, \quad \mathrm{Ind}(M) = n+3.$$

From the above theorem and the examples of Clifford torus, one can propose the following:

Conjecture. For an n-dimensional compact minimal hypersurface in the unit sphere $S^{n+1}(1)$, $\mathrm{Ind}(M) = n+3$ if and only if M is a Clifford torus.

For $n = 2$, Urbano gave an affirmative answer by using the Gauss-Bonnet Theorem and conformal transformations. For $n \geq 3$, the above conjecture still remains open.

2. Index of hypersurfaces with constant mean curvature

In this section, we assume $\varphi:\ M \to S^{n+1}(1)$ be an n-dimensional compact hypersurface with constant mean curvature in the unit sphere $S^{n+1}(1)$. If the variation of φ is volume-preserved, then we have

$$\int_M u dA = 0.$$

From the first variation formula:

$$\frac{dA(t)}{dt}|_{t=0} = -\int_M nHu dA,$$

we know that compact hypersurfaces with constant mean curvature are critical points of the area functional $A(t)$ of the volume-preserved variation. From the second variation formula of $A(t)$:

$$\frac{d^2A(t)}{dt^2}|_{t=0} = \int_M uJ_m u dA,$$

we know that $J_m u = -\Delta u - (S+n)u$ is also the Jacobi operator of hypersurfaces with constant mean curvature in the unit sphere.

Alias, Barros and Brasil [2] studied the first eigenvalue of the Jacobi operator J_m. They proved the following:

Theorem 2.1. *If M is an n-dimensional compact hypersurface with constant mean curvature in the unit sphere $S^{n+1}(1)$, then $\lambda_1^{J_m} = -n(1+H^2)$ and M is totally umbilical or*

$$\lambda_1^{J_m} \leq -2n(1+H^2) + \frac{n(n-2)|H|}{\sqrt{n(n-1)}} \max \sqrt{S - nH^2}$$

and the equality holds if and only if M is $S^1(c) \times S^{n-1}(\sqrt{1-c^2})$.

3. Index of hypersurfaces with constant scalar curvature

In this section, we consider n-dimensional compact hypersurfaces $\varphi : M \to S^{n+1}(1)$ with constant scalar curvature in the unit sphere $S^{n+1}(1)$.

For any C^2-function f on M, we define its gradient and Hessian by

$$df = (f_i), \quad Hess(f) = (f_{ij})$$

One defines a differential operator $\Box$ by

$$\Box f = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij}) f_{ij},$$

where H and h_{ij} denote the mean curvature and components of the second fundamental form of M. The differential operator $\Box$ was introduced by S.Y. Cheng and Yau [5] and it plays an very important role for studying compact hypersurfaces with constant scalar curvature in $S^{n+1}(1)$.

Let $\varphi : M \to S^{n+1}(1)$ be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$ and

$$X : \ (-\varepsilon, \varepsilon) \times M \to S^{n+1}(1)$$

is a variation of φ. We consider the functional

$$F_H(t) = \int_M nH(t) dA_t,$$

where $H(t)$ is the mean curvature of φ_t. Alencar, do Carmo and Colares [1] gave the first variation formula of this functional $F_H(t)$:

$$\frac{dF_H(t)}{dt}|_{t=0} = \int_M (-n(n-1)(r-1) + n) u dA$$

where $\frac{\partial X}{\partial t}|_{t=0}$ is the variational vector and $u = \langle \frac{\partial X}{\partial t}|_{t=0}, N \rangle$. Therefore, for all volume-preserving variations, compact hypersurfaces with constant scalar curvature in the unit sphere $S^{n+1}(1)$ are critical points of the functional $F_H(t)$. For all volume-preserving variations of φ, the second variation formula of this functional $F_H(t)$ is

$$\frac{d^2 F_H(t)}{dt^2}|_{t=0} = \int_M u J_s u dA$$

and

$$J_s u = -\Box u - \{n(n-1)H + nHS - f_3\} u.$$

The differential operator J_s given by

$$J_s = -\Box - \{n(n-1)H + nHS - f_3\},$$

are called Jacobi operator for the functional $F_H(t)$ of the compact hupersurface φ in $S^{n+1}(1)$, where $f_3 = \sum_{j=1}^{n} k_j^3$ and k_j's are the principal curvatures of M.

Define

$$Q_s(u) = \int_M u J_s u dA.$$

The dimension of the maximal subspace of $C^\infty(M)$, in which Q is negative, is called index of compact hypersurface φ with constant scalar curvature in the unit sphere $S^{n+1}(1)$, denoted by $\mathrm{Ind}_s(M)$. Thus, we know the index $\mathrm{Ind}_s(M)$ is equal to the number of negative eigenvalues of the Jacobi operator J_s when J_s is an elliptic operator.

Since the Laplace-Beltrami operator is always elliptic, the Jacobi operator J_m is always elliptic. But, in general, the operator $\Box$, and hence the Jacobi operator J_s are not elliptic. Since, when $r > 1$, the differential operator $\square$ is elliptic, the differential operator J_s is elliptic if $r > 1$.

Let $\lambda_1^{J_s}$ denote the first eigenvalue of the Jacobi operator J_s. Then

$$J_s u = \lambda_1^{J_s} u$$

and the $\lambda_1^{J_s}$ is given by

$$\lambda_1^{J_s} = \inf_{u \neq 0} \frac{\int_M u J_s u dA}{\int_M u^2 dA}$$

First of all, we will consider the first eigenvalue of the Jacobi operator J_s of both the totally umbilical and non-totally geodesic hypersurface and the Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$.

For a totally umbilical and non-totally geodesic hypersurface M in $S^{n+1}(1)$, we have

$$\begin{aligned} J_s &= -\square - \{n(n-1)H + nHS - f_3\} \\ &= -\{(n-1)H\Delta + n(n-1)H(1+H^2)\}. \end{aligned}$$

and

$$\begin{aligned} \lambda_1^{J_s} &= -n(n-1)r\sqrt{r-1} \\ &= -\{2n(n-1) + n^2(n-1)(r-1)\}H \\ &+ n(n-1)(r-1)\{(n-1)(r-1)+1\}\frac{1}{H}. \end{aligned}$$

For compact hypersurfaces $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$, with $r > 1$ in $S^{n+1}(1)$, we know that the first eigenvalue of the Jacobi operator J_s is given by

$$\begin{aligned} \lambda_1^{J_s} = &- \{2n(n-1) + n^2(n-1)(r-1)\}H \\ &+ n(n-1)(r-1)\{(n-1)(r-1)+1\}\frac{1}{H}. \end{aligned}$$

Remark 3.1. *We must notice that the first eigenvalue of the Jacobi operator J_s of both the totally umbilical and non-totally geodesic hypersurface and the Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$ has the same representation formula. But, as we have seen, the first eigenvalue of the Jacobi operator J_m of the totally umbilical hypersurface is different from one of the Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$. It is a very interesting fact.*

For an n-dimensional hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, Cheng [3] has studied the first eigenvalue of the Jacobi operator J_s and has proved the following:

Theorem 3.1. *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. Then, the Jacobi operator J_s is elliptic, the mean curvature H does not vanish on M and the first eigenvalue $\lambda_1^{J_s}$ of the Jacobi operator J_s satisfies*

$$\begin{aligned} \lambda_1^{J_s} \leq &-\{2n(n-1) + n^2(n-1)(r-1)\} \min |H| \\ &+ n(n-1)(r-1)\{(n-1)(r-1)+1\}\frac{1}{\min |H|} \end{aligned}$$

and the equality holds if and only if either M is totally umbilical and non-totally geodesic, or M is a Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \le m \le n-1$, with $r > 1$.

Corollary 3.1. *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. Then, the Jacobi operator J_s is elliptic and the first eigenvalue $\lambda_1^{J_s}$ of the Jacobi operator J_s satisfies*

$$\lambda_1^{J_s} \le -n(n-1)r\sqrt{r-1}$$

and the equality holds if and only if M is totally umbilical and non-totally geodesic.

From now, we consider the differential operator J_s on n-dimensional compact hypersurfaces in the unit sphere $S^{n+1}(1)$. We do not assume that scalar curvature of hypersurfaces is constant. We study the second eigenvalue of J_s. Cheng and Wei [4] have proved

Theorem 3.2. *Let M be an $n(\ge 5)$-dimensional compact hypersurface with scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. Then the second eigenvalue $\lambda_2^{J_s}$ of J_s satisfies*

$$\lambda_2^{J_s} \le \frac{1}{A(0)} \int_M \frac{n(S - nH^2)(1-r)}{2|H|} dv \le 0,$$

and $\lambda_2^{J_s} = 0$ if and only if M is totally umbilical and non-totally geodesic, where $A(0)$ denotes the area of M.

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