DEGENERATE CENTER MAP FOR A CENTROAFFINE SURFACES

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1. Basic theory of affine hypersurfaces

Let $f : M \to \mathbb{R}^{n+1}$ be an immersion of an n-dimensional oriented manifold M into $\mathbb{R}^{n+1}$. We denote by $\Gamma(TM)$ the space of sections of the tangent bundle TM, by D the standard flat affine connection in $\mathbb{R}^{n+1}$, and by Det the standard parallel volume form of $\mathbb{R}^{n+1}$. Throughout this paper, we assume f is nondegenerate. Let ξ be the *Blaschke normal vector field* of f. By definition, the following hold (see [5], [6]):

(1) At each point x of M, the tangent space $T_{f(x)}\mathbb{R}^{n+1}$ is decomposed as

$$T_{f(x)}\mathbb{R}^{n+1} = f_* T_x M \oplus \mathbb{R}\xi_x.$$

(2) The $(0, 2)$-tensor field h^e, defined by

$$D_X f_* Y = f_* \nabla^e_X Y + h^e(X, Y)\xi$$

for X, $Y \in \Gamma(TM)$, is a semi-Riemannian metric on M.

(3) The 1-form τ, defined by

$$D_X \xi = -f_* SX + \tau(X)\xi$$

for $X \in \Gamma(TM)$, vanishes identically.

(4) The volume form θ defined by

$$\theta(X_1, \ldots, X_n) := Det(f_* X_1, \ldots, f_* X_n, \xi)$$

for $X_1, \cdots, X_n \in \Gamma(TM)$, is compatible with the orientation of M.

(5) θ coincides with the volume form

$$\mathrm{Vol}_{h^e}(X_1, \ldots, X_n) := |\det(h^e(X_i, X_j))|^{1/2}$$

determined by h^e, where $X_1, \ldots, X_n$ are vector fields on M.

It is well known that such a ξ is uniquely determined. We then call ∇^e and h^e the *equiaffinely-induced connection* and the *equiaffine metric* of f, respectively.

The line through each point of f in the direction of ξ is called the *equiaffine normal line*. As we know, the affine hypersurface f is a proper affine hypersphere if and only if the equiaffine normals meet at one point, *the center*, which can be generalized to a map for affine hypersurfaces as follows. For more details, we refer to [2].

Let r be the equiaffine support function of f with respect to the origin $o \in \mathbb{R}^{n+1}$. By definition, it is a function on M written as

$$f = Z_x + r(x)\xi_x,$$

where Z is an $\mathbb{R}^{n+1}$-valued function tangent to f.

Definition 1.1. For an immersion $f : M \to \mathbb{R}^{n+1}$, we set $c : M \to \mathbb{R}^{n+1}$ by

$$c(x) := c_f(x) := f(x) - r(x)\xi_x, \text{ for } x \in M,$$

and call it the *center map* of f.

Proposition 1.2. *An immersion $f : M \to \mathbb{R}^{n+1}$ is a proper affine hypersphere if and only if the center map c of f is constant.*

We assume that $f : M \to \mathbb{R}^{n+1}$ is a *centroaffine* immersion as well. By definition it means for each point the position vector is transversal to the tangent space, and the symmetric $(0, 2)$-tensor field h defined by

$$D_X f_* Y = f_* \nabla_X Y + h(X, Y) f$$

is nondegenerate. We call ∇ and h the *centroaffinely-induced connection* and the *centroaffine metric* of f, respectively. We denote the difference tensor of the centroaffinely-induced connection ∇ and the Levi-Civita connection $\widetilde{\nabla}$ of the centroaffine metric h by

$$K := \nabla - \widetilde{\nabla} \in \Gamma(TM^{(1,2)}),$$

and define the *centroaffine Tchebychev vector field* T and the *centroaffine Tchebychev operator* $\Im$ by

$$T := tr_h K \in \Gamma(TM),$$

$$\Im := \widetilde{\nabla} T \in \Gamma(TM^{(1,1)}).$$

We formulize the center map for centroaffine immersion.

Proposition 1.3. *Let c be the center map of an immersion $f : M \to \mathbb{R}^{n+1}$. Then the following formula holds:*

$$c = -\frac{2}{n+2} f_* T,$$

where T is the centroaffine Tchebychev vector field of f.

In the following, we consider when the center map c is degenerate. By definition, we have

$$c_* X = -\frac{2}{n+2} D_X f_* T = -\frac{2}{n+2} (f_* \nabla_X T + h(X, T) f)$$

for $X \in \Gamma(TM)$, so that c is degenerate if and only if there is a nonvanishing vector $X \in \Gamma(TM)$ such that

$$\begin{cases} h(X,T) = 0, \\ \nabla_X T \quad = 0. \end{cases} \tag{1.1}$$

2. Classification of centroaffine surfaces with degenerate center map

2.1. **Definite case.** Let $f : M \to \mathbb{R}^3$ be a centroaffine surface with definite centroaffine metric h and nonvanishing centroaffine Tchebychev vector field T. Then for any point $x \in M$, we can take local coordinates (u, v) such that

$$h(\partial_u, \partial_u) = \lambda,\ h(\partial_v, \partial_v) = 1,\ h(\partial_u, \partial_v) = 0,$$

and the centroaffine Tchebychev vector field

$$T = \alpha\partial_u \tag{2.1}$$

for some nonvanishing functions $\alpha(u, v)$ and $\lambda(u, v)$. If we assume the surface is not a proper affine sphere, using the Gauss equation and comparing the coefficients of f in the integrability conditions, we can easily get a contradiction. Then the following Theorem holds.

Theorem 2.1. *Let $f : M \to \mathbb{R}^3$ be a centroaffine surface with definite centroaffine metric. The center map c of f is degenerate if and only if f is a proper affine sphere.*

2.2. **Indefinite case.** In this section, we assume that f is a centroaffine surfaces with indefinite centroaffine metric. We take local asymptotic coordinates (u, v) such that the centroaffine metric h satisfies

$$h_{11} = h_{22} = 0,\ h_{12} = h_{21} = -\varphi$$

for some non-zero function φ.

The Gauss formulas for the centroaffine surface f are given by

$$\begin{cases} f_{uu} = (\dfrac{\varphi_u}{\varphi} + \rho_u)f_u + \dfrac{\alpha}{\varphi}f_v, \\ f_{uv} = -\varphi f + \rho_v f_u + \rho_u f_v, \\ f_{vv} = (\dfrac{\varphi_v}{\varphi} + \rho_v)f_v + \dfrac{\beta}{\varphi}f_u \end{cases} \tag{2.2}$$

with the integrability conditions

$$\begin{cases} (\log|\varphi|)_{uv} = -\varphi - \dfrac{\alpha\beta}{\varphi^2} + \rho_u\rho_v, \\ \alpha_v + \rho_u\varphi_u = \rho_{uu}\varphi, \\ \beta_u + \rho_v\varphi_v = \rho_{vv}\varphi, \end{cases} \tag{2.3}$$

where $\rho = -\frac{1}{4}\log(-\frac{K}{d^4})$, K is the Euclidean Gauss curvature, $d = \langle f, n\rangle$ is the Euclidean support function, and

$$\begin{cases} \alpha = \varphi \dfrac{\det[f, f_u, f_{uu}]}{\det[f, f_u, f_v]}, \\ \beta = \varphi \dfrac{\det[f, f_v, f_{vv}]}{\det[f, f_v, f_u]}. \end{cases} \tag{2.4}$$

Then the centroaffine Tchebychev vector field T is computed as

$$T = 2\frac{\rho_v}{\varphi}\partial_u + 2\frac{\rho_u}{\varphi}\partial_v. \tag{2.5}$$

Considering T is a null vector or not, we can get the following Theorem.

Theorem 2.2. *Let $f : M \to \mathbb{R}^3$ be an indefinite centroaffine surface. Its center map is degenerate if and only if one of the following holds.*

- *(i) f is a proper affine sphere, if T vanishes,*
- *(ii) f is a centroaffine minimal surfaces with scalar curvature 1 expressed as $f = b_u + vb$, if nonvanishing T is a null vector for h,*
- *(iii) ρ, α, β, φ for f satisfy $\rho_v^2(\alpha_v + \rho_v\alpha) = \rho_u^2(\beta_u + \rho_u\beta) = \rho_u\rho_v(\rho_{uv} + \rho_u\rho_v)\varphi$, otherwise.*

Moreover, a centroaffine minimal surface does not belong to the class (iii).

3. Degenerate center map for centroaffine ruled surfaces

Let $a(u)$, $b(u)$ be linearly independent $\mathbb{R}^3$-valued functions in one variable. Suppose that $b(u)$ is a nondegenerate centroaffine space curve and u is the centroaffine arc-length parameter of $b(u)$, which is defined by

$$\varepsilon(u) := \frac{\det[b'(u), b''(u), b'''(u)]}{\det[b(u), b'(u), b''(u)]} = \pm 1.$$

Denote the *centroaffine first curvature* by

$$\kappa_1(u) := -\frac{\det[b(u), b''(u), b'''(u)]}{\det[b(u), b'(u), b''(u)]},$$

and the *centroaffine second curvature* by

$$\kappa_2(u) := \frac{\det[b(u), b'(u), b'''(u)]}{\det[b(u), b'(u), b''(u)]}$$

(see[4]).

Let $f : M \to \mathbb{R}^3$ be a non-degenerate centroaffine ruled surface given by $f(u, v) = a(u) + vb(u)$. Because

$$\begin{cases} f_u = a' + vb', \\ f_v = b, \end{cases}$$

we have $\det[f_u, f_v, f] = \det[a', b, a] + v\det[b', b, a] \neq 0$.

Theorem 3.1. *Let $f : M \to \mathbb{R}^3$ be a non-degenerate centroaffine ruled surface given by $f(u,v) = a(u)+vb(u)$. The center map c of f is degenerate to a curve if and only if $a(u)$ is written as $a(u) = \phi(u)b'(u)$, where $\phi(u)$ is a nonzero function such that $3\phi'(u) + \phi(u)\kappa_2(u) \neq 0$, $\kappa_2(u)$ is the centroaffine second curvature of $b(u)$. Moreover, the center map of f is given by*

$$c(u,v) = -\frac{1}{2}(3\phi'(u) + \phi(u)\kappa_2(u))b(u).$$

Example. For a cubic curve $b(u) = (u, u^2, u^3)$, we can change the parameter to be the centroaffine arc-length parameter s and rewrite the curve as

$$b(s) = (e^{6^{-1/3}s}, e^{2\cdot 6^{-1/3}s}, e^{3\cdot 6^{-1/3}s}).$$

The second centroaffine curvature is given by

$$\kappa_2 = 6^{2/3}.$$

Then we can get the center map of $f(s,v) = b'(s) + vb(s)$ as follows,

$$c(s,v) = -\frac{1}{2}6^{2/3}(e^{6^{-1/3}s}, e^{2\cdot 6^{-1/3}s}, e^{3\cdot 6^{-1/3}s}) = -\frac{1}{2}\kappa_2 b(s).$$

Remark 3.2. Centroaffine ruled surfaces with $3\phi'(u) + \phi(u)\kappa_2(u) = 0$ are proper affine spheres.

Corollary 3.3. *The center map of the centroaffine ruled surface $f(u,v) = b'(u) + vb(u)$ is centroaffinely equivalent to $b(u)$ if and only if the second centroaffine curvature of $b(u)$ is constant.*

Corollary 3.4. *The center map of the centroaffine ruled surface $f(u,v) = b'(u) + vb(u)$ is projectively equivalent to $b(u)$.*

Corollary 3.5. *Given a nondegenerate centroaffine space curve $b(u)$ with centroaffine arc-length parameter u and centroaffine second curvature $\kappa_2(u)$, we can construct a centroaffine ruled surface f whose center map is $b(u)$. In fact, the center map of $f(u,v) = \phi(u)b'(u) + vb(u)$ is $b(u)$, for $\phi(u)$ satisfying $\phi(u) = -\frac{2}{3}\mu^{-1}(u)\int \mu(u)du$, where $\mu(u) = e^{\frac{1}{3}\int \kappa_2(u)du}$.*

References

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