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A Variational problem related to conformal maps between Riemannian manifolds

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§0. Introduction

There exists a fundamental question:

Question What maps are closest to conformal ones?

We give a variational approach to this question. We consider the functional

$$\Phi(f) \ = \ \int_M \|T_f\|^2 dv_g \, ,$$

where T_f is a covariant symmetric tensor such that

$$T_f = 0 \quad \Longleftrightarrow \quad f \text{ is a weakly conformal map.}$$

In this note we give a brief summary of results for this functional.

§1. A variational problem for the conformality of maps

We use the following notations throughout this note:

Notations

(M, g), (N, h)	:	compact Riemannian manifolds without boundary.
m	:	the dimension of M
f	:	a smooth map from M into N.
X, Y	:	vector fields on M.
e_i	:	a local orthonormal frame on M.
f^*h	:	the pullback of a metric h by a map f, i.e., $(f^*h)(X, Y) = h(df(X), df(Y))$

We first recall notions of the conformality of maps:

Conformal and Weakly conformal

(1) A smooth map f is **weakly conformal** if there exists a **non-negative** function φ on M such that

$$(*) \qquad f^*h = \varphi g\,.$$

(2) A smooth map f is **conformal** if there exists a **positive** function φ on M satisfying $(*)$.

Note that f is weakly conformal if and only if it is conformal at x or $(df)_x = 0$ for any $x \in M$ [1].

We give a tensor of the conformality. Let $\|df\|$ denote the energy density of f in the theory of harmonic maps, i.e.,

$$\|df\|^2 = \sum_{i=1}^{m} h\big(df(e_i),\, df(e_i)\big).$$

We consider the following covariant symmetric tensor:

Tensor T_f

$$T_f \overset{def}{=} f^*h - \frac{1}{m}\|df\|^2 g,$$

i.e.,

$$T_f(X,\, Y) \overset{def}{=} h\big(df(X),\, df(Y)\big) - \frac{1}{m}\|df\|^2 g(X,\, Y)\,.$$

Remark. In the case of $m = 2$, the tensor T_f is equal to the stress energy tensor

$$S_f = f^*h - \frac{1}{2}\|df\|^2 g$$

for harmonic maps. (See Eells and Lemaire [2], p.392.)

We can verify the following basic properties for the tensor T_f:

[1] A map f is called conformal *at* $x \in M$ if it satisfies $(*)$ at x for a *positive* function φ.

Properties of tensor T_f

Lemma T.

(1) T_f is symmetric, i.e., $T_f(X, Y) = T_f(Y, X)$.

(2) f is weakly conformal if and only if $T_f = 0$.

(3) $\|T_f\|^2 = \|f^*h\|^2 - \frac{1}{m}\|df\|^4$.

(4) T_f is trace-free, i.e.,

$$\mathrm{Tr}_g T_f \;=\; \sum_{i,\,j=1}^{m} g(e_i,\, e_j) T_f(e_i,\, e_j) \;=\; 0\,.$$

(5) The trace of T_f with respect to the pullback f^*h is equal to the norm squared of T_f, i.e.,

$$\mathrm{Tr}_{f^*h} T_f \;=\; \sum_{i,\,j=1}^{m} (f^*h)(e_i,\, e_j) T_f(e_i,\, e_j) \;=\; \|T_f\|^2\,.$$

We are concerned with the following functional:

Functional $\Phi(f)$

$$\Phi(f) \;=\; \int_M \|T_f\|^2 dv_g\,.$$

This functional $\Phi(f)$ gives a quantity of the conformality of maps f. Note that if f is a conformal map, then $\Phi(f)$ vanishes. In this note we give a brief summary of the following results ([5], [4]):

1. First variation formula
2. Second variation formula
3. Weak conformality for maps from or into spheres
4. Quasi-monotonicity formula
5. Bochner type formula
6. Existence of minimizers in 3-homotopy class
7. Other variational problem

§2. First variation formula

In this section we give the first variation formula for the functional $\Phi(f)$. We first define the following "$f^{-1}TN$-valued" 1-form ξ_f [2]. The 1-form ξ_f plays an important role in our arguments.

1-form ξ_f

$$
\begin{aligned}
\xi_f(X) &= \sum_j T_f(X, e_j) df(e_j) \\
&= \sum_j h\big(df(X), df(e_j)\big) df(e_j) \; - \; \frac{1}{m} \|df\|^2 df(X) \,.
\end{aligned}
$$

Take any smooth deformation F of f, i.e., any smooth map

$$
F \; : \; (-\varepsilon, \varepsilon) \times M \; \longrightarrow \; N \;\; \text{s.t.} \;\; F(0, x) = f(x).
$$

Let $f_t(x) = F(t, x)$, and we often say a deformation $f_t(x)$ instead of

[2] Though I want to use the notation τ_f instead of ξ_f, it is confused with the notation of the tension field in the theory of harmonic maps.

a deformation $F(t, x)$. Let

$$X = dF(\tfrac{\partial}{\partial t})\big|_{t=0}$$

denote the variation vector field of the deformation F. Then we have the following first variation formula.

First variation formula

$$\frac{d\Phi(f_t)}{dt}\bigg|_{t=0} = -4\int_M h\,(X,\, \mathrm{div}_g\, \xi_f)\; dv_g\,.$$

In the above formula, dv_g denotes the volume form on M, and $\mathrm{div}_g\, \xi_f$ denotes the divergence of ξ_f, i.e., $\mathrm{div}_g\, \xi_f = \sum_{i=1}^{m}(\nabla_{e_i}\xi_f)(e_i)$.

We give here the notion of *C-stationary maps*.

C-stationary map

We call a smooth map f **C-stationary** if

$$\frac{d\Phi(f_t)}{dt}\bigg|_{t=0} = 0$$

for any smooth deformation f_t of f.

By the first variation formula, a smooth map f is *C-stationary* if and only if it satisfies the following equation:

Euler-Lagrange equation

$$\mathrm{div}_g\, \xi_f = 0\,.$$

§3. Second variation formula

In this section we give the second variation formula for the functional $\Phi(f)$. Take any smooth deformation F of f with two parameters, i.e., any smooth map

$$F \;:\; (-\varepsilon,\, \varepsilon) \times (-\delta,\, \delta) \times M \;\longrightarrow\; N \;\; \text{s.t.} \;\; F(0,\, 0,\, x) = f(x)\,.$$

Let $f_{s,\,t}(x) = F(s,\, t,\, x)$, and we often say a deformation $f_{s,\,t}(x)$ instead of a deformation $F(s,\, t,\, x)$. Let

$$X = dF(\tfrac{\partial}{\partial s})\big|_{s,\,t=0},\;\; Y = dF(\tfrac{\partial}{\partial t})\big|_{s,\,t=0}$$

denote the variation vector fields of the deformation F. Then we have the following second variation formula.

Second variation formula

$$\begin{aligned}
\frac{1}{4}\,\frac{\partial^2\Phi(f_{s,\,t})}{\partial s\partial t}\bigg|_{s,\,t=0} \;&=\; \int_M h\big(\mathrm{Hess}_F\big(\tfrac{\partial}{\partial s},\, \tfrac{\partial}{\partial t}\big),\, \mathrm{div}_g\,\xi_f\big)dv_g \\
&+ \int_M \sum_{i,\,j} h\big(\nabla_{e_i}X,\, \nabla_{e_j}Y\big)\, T_f(e_i,\, e_j)\, dv_g \\
&+ \int_M \sum_{i,\,j} h\big(\nabla_{e_i}X,\, df(e_j)\big)\, h\big(\nabla_{e_i}Y,\, df(e_j)\big)dv_g \\
&+ \int_M \sum_{i,\,j} h\left(\nabla_{e_i}X,\, df(e_j)\right)\, h\left(df(e_i),\, \nabla_{e_j}Y\right) dv_g \\
&- \frac{2}{m}\int_M \sum_{i} h\left(\nabla_{e_i}X,\, df(e_i)\right) \sum_{j} h\left(\nabla_{e_j}Y,\, df(e_j)\right)\, dv_g \\
&- \int_M \sum_{i,\,j} h\big({}^N\!R\,(df(e_i),\, X)\,Y,\, df(e_j)\big)\, T_f(e_i,\, e_j)\, dv_g\,.
\end{aligned}$$

In this formula, Hess_f denotes the Hessian of f, i.e., $\mathrm{Hess}_f(Z, W) = (\nabla_Z df)(W) = (\nabla_W df)(Z)$.

Remark. Note that the first term in the right hand side vanishes if f is a C-stationary map.

Remark. The last term of the right hand side is equal to

$$- \int_M \sum_i h\big({}^N R\,(df(e_i),\, X)\, Y,\, \xi_f(e_i)\big)\, dv_g\,.$$

§4. Weak conformality for maps from or into spheres

A C-stationary map f is called to be **stable** if the second variation at f is non-negative. We give two results for the weak conformality of stable C-stationary maps. (See Kawai-Nakauchi [4].)

Weak conformailty — the case of source spheres

Let f be a stable C-stationary map from the standard sphere S^m into a Riemannian manifold N. If $m \geq 5$, then f is a weakly conformal map.

Weak conformailty — the case of target spheres

Let f be a stable C-stationary map from a Riemannian manifold M into the standard sphere S^n. If $n \geq 5$, then f is a weakly conformal map.

The above results can be regarded as a type of Liouville theorems since the trivial case for the functional Φ is that of not constant maps, but weakly conformal maps. On the other hand, stable C-stationary maps are not weakly conformal in general. We see the following fact.

Existence of non-conformal stable C-stationary maps

There exists a stable C-stationary maps which is not weakly conformal.

This fact follows from a simple example. Let us define a map

$$\begin{array}{cccc} f \ : & S^1 \times S^1 \times \cdots \times S^1 & \longrightarrow & S^1(r) \times S^1 \times \cdots \times S^1 \\ & \Cup & & \Cup \\ & (x^1, x^2, \ldots, x^m) & \longmapsto & (rx^1, x^2, \ldots, x^m) \end{array}$$

where S^1 (resp. $S^1(r)$) denotes the sphere of dimension 1 with radius 1 (resp. r) centered at the origin of $\mathbb{R}^2$. Obviously f is not weakly conformal if $r \neq 1$. By simple calculations, we can verify that f is a C-stationary map, and that f is stable if r is sufficiently close to 1.

§5. Quasi-monotonicity formula

In this section we prove a kind of the monotonicity formula for C-stationary maps. We give this formula under the following weak condition.

C-stationary w.r.t. diffeomorphisms

We call a map f **C-stationary with respect to diffeomorphisms** on M if

$$\frac{d}{dt}\Phi(f \circ \varphi_t)\bigg|_{t=0} = 0$$

for any 1-parameter family φ_t of diffeomorphisms on M.

Note that the above definition of C-stationary maps is weaker than the previous one of C-stationary maps, since $f_t(x) = f \circ \varphi_t(x)$ is a deformation in the latter definition.

Let $B_\rho(x_0)$ be the open ball of a radius ρ with a center $x_0 \in M$. Then we have the following formula.

Quasi-monotonicity formula

For any *C-stationary map f with respect to diffeomorphisms*, we have

$$\frac{d}{d\rho}\left\{ e^{C\rho}\rho^{4-m}\int_{B_\rho(x_0)} \|T_f\|^2 dv_g \right\} \;\geq\; 4e^{C\rho}\rho^{4-m}\left(\varphi'(\rho) + \frac{C}{4}\varphi(\rho)\right)$$

where

$$\varphi(\rho) \;=\; \int_{B_\rho(x_0)} h\big(df\left(\tfrac{\partial}{\partial r}\right), \xi_f\left(\tfrac{\partial}{\partial r}\right)\big)\, dv_g\, .$$

Remark. If $\varphi(\rho)$ satisfies the condition $\varphi'(\rho) + \frac{C}{4}\varphi(\rho) \geq 0$, then $e^{C\rho}\rho^{4-m}\int_{B_\rho(x_0)} \|T_f\|^2 dv_g$ is monotone non-decreasing. We cannot expect such a monotonicity in general, since T_f is indefinite.

§6. Bochner type formula

Bochner type formulas are basic tools for various arguments in geometry. For the norm of T_f, we have the following Bochner type formula.

Bochner type formula

$$
\begin{aligned}
\frac{1}{4} \triangle \|T_f\|^2 \;=\; & \operatorname{div}_g \alpha_f \;-\; h\big(\tau_f,\, \operatorname{div}_g \xi_f\big) \;+\; \frac{1}{2}\|\nabla T_f\|^2 \\
& + \sum_{i,j,k} h\big((\nabla_{e_k} df)(e_i),\, (\nabla_{e_k} df)(e_j)\big) T_f(e_i,\, e_j) \\
& + \sum_{i,j} \; h\big(df\big(\sum_k {}^M R(e_i,\, e_k)e_k\big),\, df(e_j)\big)\; T_f(e_i,\, e_j) \\
& - \sum_{i,j,k} h\big({}^N R\big(df(e_i),\, df(e_k)\big)df(e_k),\, df(e_j)\big) T_f(e_i,\, e_j)
\end{aligned}
$$

where

$$\alpha_f(X) \;=\; h\big(\xi_f(X),\, \tau_f\big)\,.$$

In the above formula, $\tau_f = \operatorname{tr}(\nabla df) = \sum_j (\nabla_{e_j} df)(e_j)$ is the tension field of f in the theory of harmonic maps. (See Eells and Lemaire [1], p.9.)

Remark. The first term in the right hand side is of divergence form, and hence the integral of it over M vanishes.

Remark. The second term in the right hand side vanishes if f is a C-stationary map.

Remark. The last two terms of the right hand side are equal to

$$+ \sum_{i,k} \; h\big(df(\sum_k {}^M R(e_i,\, e_k)e_k),\, \xi_f(e_i)\big)$$

and

$$- \sum_{i,k} h\big({}^N R\big(df(e_i),\, df(e_k)\big)df(e_k),\, \xi_f(e_i)\big)$$

respectively.

§7. Existence of local minimizers

In this section we utilize the notion of 3-homotopy in the Sobolev spaces, which is given by White, and consider a variational problem of minimizing the functional $\Phi(f)$ in each 3-homotopy class. For any two maps f_1 and f_2 from M into N, these maps are ***k*-homotopic** ($k \in \mathbb{N}$) if they are homotopic to each other on k-dimensional skeletons of a triangulation on M.

By Nash's isometric embedding, we may assume that N is a submanifold of a Euclidean space $\mathbb{R}^q$. Let

$$\mathrm{L}^{1,p}(M,\, N) \;=\; \{\, f \in \mathrm{L}^{1,p}(M,\, \mathbb{R}^q) \mid\, f(x) \in N \text{ a.e. } \},$$

where $\mathrm{L}^{1,p}(M,\, \mathbb{R}^q)$ denotes the Sobolev space of $\mathbb{R}^q$-valued L^p-functions on M such that their derivatives are in L^p . Then White proved that the notion of the $[p-1]$-homotopy is compatible with the Sobolev space $\mathrm{L}^{1,p}(M,\, N)$, where $[\ \ ]$ denotes the Gauss symbol, i.e., $[r]$ is the maximum integer less than or equal to r. We recall the following results of White [8]. (See also White [7].)

Known results

(1) The $[p-1]$-homotopy is well-defined for any map $f \in \mathrm{L}^{1,p}(M,\, N)$.
(2) If f_j converges weakly to f_∞ in $\mathrm{L}^{1,p}(M,\, N)$, then f_j and f_∞ are $[p-1]$-homotopic for sufficient large j .

The functional $\Phi(f)$ is defined on $\mathrm{L}^{1,4}(M,\, N)$, in which the 3-homotopy is well-defined. Then for any given continuous map f_0 from M into N, we want to minimize the functional $\Phi(f)$ in the following class:

$$\mathcal{F} = \left\{ \; f \in \mathrm{L}^{1,4}(M,\, N) \;\middle|\; f \text{ is 3-homotopic to } f_0 \;\text{ and }\; \|f\|_{\mathrm{L}^{1,4}(M,N)} \le C_0 \; \right\},$$

where C_0 is a given positive constant. We see that the space $\mathcal{F}$ is not empty for sufficiently large C_0.

Existence of minimizers

There exists a minimizer of the functional $\Phi(f)$ in $\mathcal{F}$.

If a 3-homotopy class contains a conformal map, then the conformal map is a minimizer. Minimizers are expected to be *closest* to conformal maps, even if its 3-homotopy class does not contain any conformal map.

Remark. When M is 4-dimensional and $\pi_4(N) = 0$, any *continuous* minimizer is (freely) homotopic to f_0 in the ordinary sense.

§8. Other variational problem

By Lemma T (3), we see

$$\|T_f\|^2 \quad = \quad \underbrace{\|f^*h\|^2}_{\text{the norm of the pullback}} \quad - \quad \frac{1}{m}\underbrace{\|df\|^4}_{\text{the energy density of 4-harmonic maps}}.$$

Then we consider the following functional for pullbacks of metrics.

Functional $F(f)$

$$F(f) = \int_M \|f^*h\|^2 dv_g\,.$$

We mention here harmonic maps. A map f is called a **harmonic map** if it is a critical point of the energy functional $E(f) = \int_M \|df\|^2 dv_g$. The theory of harmonic maps make a rapid progress during the last fifty years, and gave various applications to other branches in mathematics and physics. From the viewpoint of pullbacks of metrics, the norm squared $\|df\|^2$ of the energy density is the *trace* of the pullback f^*h. Thus we see the following correspondence between the energy functional $E(f)$ and our functional $F(f)$.

the energy functional in the theory of harmonic maps	our functional
$E(f) = \int_M \|df\|^2 dv_g = \int_M \mathrm{tr}_g(f^*h) dv_g$	$F(f) = \int_M \|f^*h\|^2 dv_g\,.$
the **trace** of the pullback f^*h	the **norm** of the pullback f^*h

We give the notion of *symphonic maps.*

symphonic map

We call a smooth map f **symphonic** if

$$\left.\frac{dF(f_t)}{dt}\right|_{t=0} = 0$$

for any smooth deformation f_t of f.

The *norm* contains informations of more components than the *trace* while *symphonies* have more parts than *harmonies*. Compared with *harmonic maps*, a critical point of the functional $F(f)$ is called a **symphonic map**[3]. For symphonic maps, see Nakauchi-Takenaka [6] and Kawai-Nakauchi [3]. (In these papers we use the term "stationary maps" instead of "symphonic maps".)

References

[1] Eells, J. and Lemaire, L., A report on harmonic maps, Bull. London Math. Soc. 10 (1978), 1-68.

[2] Eells, J. and Lemaire, L., Another report on harmonic maps, Bull. London Math. Soc. 20 (1988), 385-524.

[3] Kawai, S. and Nakauchi, N., Some results for stationary maps of a functional related to pullback metrics, Nonlinear Analysis 74 (2011), 2284-2295.

[3] This is one of my favorite jokes, and I adopted the term of *symphonic maps*.

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