The Total Absolute Torsion of Open Curves in E^3

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0. Introduction

Let Σ be a C^3 curve in the 3-dimensional Euclidean space E^3. For curves in E^3, two geometric quantities called curvature and torsion are defined. The total integral of curvature is called the total absolute curvature. The study of the total absolute curvature has a long history since the work by Fenchel ([4]). Seeing that most works had been done for closed curves, the authors studied the total absolute curvature of open (i.e. not closed) curves in [2] and [3]. The total integral of the absolute value of torsion is called the total absolute torsion. The total absolute torsion has been studied for closed curves in E^3 (See, for example, [5], [6], [7]).

In this work, we study the total absolute torsion of open curves and determine the minimal possible value of the total absolute torsion in a certain family of open curves. The family of curves is described as follows. Let p, q be points in E^3, Π_p be an oriented plane through p, Π_q be an oriented plane through q and L be a positive constant not smaller than $|pq|$. We define $\mathcal{C}(p, \Pi_p, q, \Pi_q, L)$ as the set of all C^3 curves whose endpoints are p and q, osculating plane at p is Π_p, osculating plane at q is Π_q and length is L. Our main theorem asserts that the infimum of the total absolute torsion in $\mathcal{C}(p, \Pi_p, q, \Pi_q, L)$ can be calculated using a piecewise linear curve with only two edges.

To prove the theorem, we first extend the notion of the total absolute torsion to curves which are C^3 only piecewise. This makes it possible to study the total absolute torsion of piecewise linear curves. The notion of the total absolute torsion for piecewise linear curves is generalized to what we call the total rotation of unit normal vector fields along piecewise linear curves. We will show that the minimization of the total rotation leads to the minimization of the total absolute torsion (Lemma 1). Then, we will show that, for any piecewise linear curve with three edges, it is always possible to find a piecewise linear curve with two edges which has a unit normal vector field of smaller total rotation, preserving the boundary condition (Lemma 2). By an induction argument, we see that minimal value of the total rotation under the given boundary condition is attained by a unit normal vector field along a piecewise linear curve with 2 edges (Proposition 1). Finally, making use of an approximation of smooth curves by piecewise linear curves, we prove our main theorem.

1. Total absolute torsion

Let Σ be a C^3 curve in the 3-dimensional Euclidean space E^3. Let L be the length of Σ and $x(s)$ $(0 \leq s \leq L)$ be a parameterization of Σ by its arclength.

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Let $T(s) = dx/ds$ and $\kappa(s) = |dT/ds| = \left|d^2x/ds^2\right|$. $\kappa(s)$ is called the curvature of Σ. If $\kappa(s) \neq 0$ on Σ, a unit vector field $N(s) = \frac{dT/ds}{\kappa(s)}$ is defined. Note that $\langle N(s), T(s)\rangle = 0$, where $\langle\ ,\ \rangle$ is the inner product of E^3. Let $B(s) = T(s) \times N(s)$. Then $\{T(s), N(s), B(s)\}$ forms a positively oriented orthonormal frame field of E^3 defined along Σ. The torsion of Σ is, by definition, $\tau(s) = \left\langle \frac{dN}{ds}, B(s)\right\rangle$. The *total absolute torsion* is defined by

$$TAT(\Sigma) = \int_0^L |\tau(s)|\, ds.$$

We regard $T(s)$ as a curve in S^2, which will be denoted by T_Σ and called the tangent indicatrix of Σ. $N(s)$ becomes a unit tangent vector of T_Σ and $B(s)$ becomes a unit normal vector of T_Σ. The total absolute torsion $TAT(\Sigma)$ is equal to the total absolute curvature (as a curve in S^2) of T_Σ. $TAT(\Sigma)$ is also equal to the length of $B(s)$ as a curve in S^2.

We next consider the case when Σ has a point with vanishing curvature. If $k(s) = 0$ for $a \leq s \leq b$ (a and b may coincide), $\{T(s) : a \leq s \leq b\}$ shrinks to a point and T_Σ becomes a piecewise C^2 curve. The definition of the total absolute curvature can be extended to piecewise C^2 curves in S^2 as the sum of the total integral of the curvature and the rotation angles of the tangent vector at nonsmooth points, which makes it possible to define the total absolute torsion for Σ with points with vanishing curvature.

We make further extension of the definition of the total absolute torsion for curves which are C^3 only piecewise. If Σ is such a curve, the tangent indicatrix may be discontinuous, but we can define the total absolute torsion of Σ as the total absolute curvature of the tangent indicatrix whose discontinuous points are connected with great circular arcs. This covers the case when the curve is a piecewise linear curve. For a piecewise linear curve $P = p_0p_1 \cup p_1p_2 \cup \cdots \cup p_{n-1}p_n$, let $T_i = \overrightarrow{p_{i-1}p_i}/|\overrightarrow{p_{i-1}p_i}|$. Then the tangent indicatrix T_P of P becomes $T_P = T_1T_2 \cup \cdots \cup T_{n-1}T_n$, where T_iT_{i+1} is the geodesic segment in S^2 which joins T_i and T_{i+1}. Let φ_i be the exterior angle of T_P at T_i. Then the total absolute torsion of P, or the total absolute curvature of T_P, is given by

$$TAT(P) = TAC(T_P) = \sum_{i=1}^{n-1} \varphi_i.$$

The angle φ_i is equal to the angle between two oriented planes, one spanned by $\{\overrightarrow{p_{i-2}p_{i-1}}, \overrightarrow{p_{i-1}p_i}\}$ and one spanned by $\{\overrightarrow{p_{i-1}p_i}, \overrightarrow{p_ip_{i+1}}\}$. Hence φ_i is the angle (valued in $[0, \pi]$) between two vectors $T_{i-1} \times T_i$ and $T_i \times T_{i+1}$. If we set $B_i = T_i \times T_{i+1}/|T_i \times T_{i+1}|$, then φ_i is the distance $d(B_{i-1}, B_i)$ between B_{i-1} and B_i as points in S^2. Thus we have another expression of $TAT(P)$ as

$$TAT(P) = \sum_{i=1}^{n-1} d(B_{i-1}, B_i).$$

Now we extend the notion of the total absolute torsion for piecewise linear curves to a little more general notion which we call the total rotation of unit normal fields along piecewise linear curves. Let ν_i be a unit vector perpendicular to $p_{i-1}p_i \in P$. We denote the set of the unit vectors $\{\nu_i : i = 1, \ldots, n\}$ by $\bar{\nu}$. We call $\bar{\nu}$ a *unit normal vector field* along the piecewise linear curve P. We define the *total rotation* $TR(P, \bar{\nu})$ by

$$TR(P, \bar{\nu}) = \sum_{i=1}^{n-1} d(\nu_i, \nu_{i+1}),$$

where d is the distance in S^2. If $\bar{\nu} = \bar{B} = \{B_1, \cdots, B_{n-2}, B_{n-1}, B_{n-1}\}$, we have

$$TAT(P) = TR(P, \bar{B}).$$

2. Curves with fixed endpoints, end-osculating-planes, length

Let p, q be points in E^3. Let Π_p and Π_q be oriented planes which pass through p and q, respectively. Let L be a positive constant not smaller than $|pq|$. We define several classes of curves;
$\mathcal{C}(p, q)$: The set of all piecewise C^3 curves which has endpoints at p and q.
$\mathcal{C}(p, q, L)$: The set of elements of $\mathcal{C}(p, q)$ whose length is L.
$\mathcal{C}(p, \Pi_p, q, \Pi_q)$: The set of elements of $\mathcal{C}(p, q)$ such that T_Σ at the initial point is tangent to the oriented great circle corresponding to Π_p, and T_Σ at the terminal point is tangent to the oriented great circle corresponding to Π_q.
$\mathcal{C}(p, \Pi_p, q, \Pi_q, L) = \mathcal{C}(p, q, L) \cap \mathcal{C}(p, \Pi_p, q, \Pi_q)$.

Let n be a positive integer and let $\mathcal{P}_n$ be the set of all piecewise linear curves with n edges. For all $m < n$, we regard $\mathcal{P}_m$ as a subset of $\mathcal{P}_n$ by allowing angles between two edges to be zero. Let $\mathcal{P}_n(p, q) = \mathcal{P}_n \cap \mathcal{C}(p, q)$ and $\mathcal{P}_n(p, q, L) = \mathcal{P}_n \cap \mathcal{C}(p, q, L)$. Any element P of $\mathcal{P}_n(p, q)$ may be written as $P = pp_1 \cup p_1p_2 \cup \cdots \cup p_{n-2}p_{n-1} \cup p_{n-1}q$. P becomes an element of $\mathcal{C}(p, \Pi_p, q, \Pi_q)$ if and only if $p_1, p_2 \in \Pi_p$ and $\{\overrightarrow{pp_1}, \overrightarrow{p_1p_2}\}$ is positively oriented in Π_p and $p_{n-2}, p_{n-1} \in \Pi_q$ and $\{\overrightarrow{p_{n-2}p_{n-1}}, \overrightarrow{p_{n-1}q}\}$ is positively oriented in Π_q. If ν_i $(i = 1, \ldots, n)$ is a unit vector perpendicular to $p_{i-1}p_i$ (setting $p_0 = p$ and $p_n = q$), then $\bar{\nu} = \{\nu_1, \cdots, \nu_n\}$ defines a unit normal vector field along P. Let $\mathcal{PN}(p, q, L)$ be the set of all $(P, \bar{\nu})$ such that $P \in \mathcal{P}(p, q, L)$ and $\bar{\nu}$ is a unit normal field along P. Let $\mathcal{PN}_n(p, q, L)$ be the set of all elements of $\mathcal{PN}(p, q, L)$ which have n edges. For our purpose, we define the *exended total rotation* $\widetilde{TR}(P, \bar{\nu})$ by

$$\widetilde{TR}(P, \bar{\nu}) = \angle(\nu_p, \nu_1) + \sum_{i=1}^{n-1} \angle(\nu_i, \nu_{i+1}) + \angle(\nu_n, \nu_q).$$

Note that if $\nu_1 = \nu_p$ and $\nu_n = \nu_q$, then $\widetilde{TR}(P, \bar{\nu}) = TR(P, \bar{\nu})$.

Lemma 1.

$$\inf\{TAT(P) \mid P \in \mathcal{P}(p, \Pi_p, q, \Pi_q, L)\} = \inf\{\widetilde{TR}(P, \bar{\nu}) \mid (P, \bar{\nu}) \in \mathcal{PN}(p, q, L)\}.$$

Lemma 1 shows that the problem of minimizing the total absolute torsion in $\mathcal{P}(p, \Pi_p, q, \Pi_q, L)$ is reduced to the problem of minimizing the extended total rotation in $\mathcal{PN}(p,q,L)$. The following lemma for curves with 3 edges is the key in this work.

Lemma 2. *Let $(P,\bar\nu)$ be an element of $\mathcal{PN}_3(p,q,L)$. Then there exists an element $(P',\bar\nu')$ of $\mathcal{PN}_2(p,q,L)$ such that $\widetilde{TR}(P',\bar\nu') \le \widetilde{TR}(P,\bar\nu)$.*

We use Lemma 2 to prove the following lemma for piecewise linear curves with arbitrary number of edges.

Lemma 3. *Let $(P,\bar\nu)$ be an element of $\mathcal{PN}_n(p,q,L)$ with $n \ge 3$. Then there exists an element $(P',\bar\nu')$ of $\mathcal{PN}_{n-1}(p,q,L)$ such that $\widetilde{TR}(P',\bar\nu') \le \widetilde{TR}(P,\bar\nu)$.*

By an inductive argument based on Lemma 3, we obtain the following.

Proposition 1. *Let $(P,\bar\nu)$ be an element of $\mathcal{PN}_n(p,q,L)$. Then there exists an element $(P',\bar\nu')$ of $\mathcal{PN}_2(p,q,L)$ such that $\widetilde{TR}(P',\bar\nu') \le \widetilde{TR}(P,\bar\nu)$.*

Proposition 1, together with Lemma 1, gives the following proposition, which shows that our main theorem holds for piecewise linear curves. In the statement, $\angle(\,\cdot\,,\,\cdot\,)$ denotes the angle between two oriented planes.

Proposition 2. *For any p, q, Π_p, Π_q and L, there exist a point r, an oriented plane Π_1 containing the line segment pr, and an oriented plane Π_2 containing rq which have the following properties;*

(1) The sum of the lengths of the line segments pr and rq is L.
(2) The sum of the angles $\angle(\Pi_p,\Pi_1)+\angle(\Pi_1,\Pi_2)+\angle(\Pi_2,\Pi_q)$ gives the infumum of the total absolute torsion in $\mathcal{P}(p,\Pi_p,q,\Pi_q,L)$.

Finally, we use an approximation of a smooth curve by a picewise linear curve to prove the followinfg theorem.

Theorem. *For any p, q, Π_p, Π_q and L, there exist a point r, an oriented plane Π_1 containing the line segment pr, and an oriented plane Π_2 containing rq which have the following properties;*

(1) The sum of the lengths of the line segments pr and rq is L.
(2) The sum of the angles $\angle(\Pi_p,\Pi_1)+\angle(\Pi_1,\Pi_2)+\angle(\Pi_2,\Pi_q)$ gives the infumum of the total absolute torsion in $\mathcal{C}(p,\Pi_p,q,\Pi_q,L)$.

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