NEW CHARACTERIZATIONS OF THE CATENOID AND HELICOID

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1. The catenoid

The catenoid is the first nontrivial minimal surface discovered. It was Euler who found it in 1744 in the process of proving that when the catenary is rotated about an axis it generates a surface of smallest area. In 1860 Bonnet showed that the catenoid is the only nonplanar minimal surface of revolution.

Recently Bernstein and Breiner proved that all embedded minimal annuli in a slab have area bigger than or equal to the minimum area of the catenoids in the same slab [1]. That minimum is attained by the catenoidal waist along the boundary of which the rays from the center of the slab are tangent to the waist. This waist is said to be maximally stable because its proper subset is stable and any subset of the catenoid properly containing the waist is unstable.

It is tempting to conjecture that Bernstein-Breiner's theorem should also hold for a minimal surface with genus in a slab as they conjectured, and for a multiply connected minimal surface in a slab as well. Indeed, the author and Benoît Daniel prove this conjecture provided the intersections of the minimal surface with horizontal planes have the same orientation [2]. The orientation of the horizontal section is induced by the surface. If the minimal surface in a slab has more than two boundary components or nonzero genus, it may happen that the horizontal section of the surface consists of two or more closed curves which have both clockwise and counterclockwise orientations. If this is not the case, we can prove the conjecture.

The part for the catenoid is a preliminary report. The paper with full proofs will appear later.

2. Theorem

Given the catenoid $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 : \cosh z = \sqrt{x^2 + y^2}\}$, $\mathcal{C}_a^b := \mathcal{C} \cap \{a \leq z \leq b\}$ is called a catenoidal waist. If there exists a point $p = (0, 0, c)$ such that the rays emanating from p are tangent to $\mathcal{C}$ along the waist circles $\mathcal{C} \cap \{z = a, b\}$, then $\mathcal{C}_a^b$ is called a *maximally stable* waist. This is because the homotheties centered at p give a foliation of a tubular neighborhood of $\mathcal{C}_a^b$, which generates a Jacobi fields J on $\mathcal{C}_a^b$ with $|J| > 0$ and vanishing only on $\partial\mathcal{C}_a^b$. Let $\beta > 0$ be the unique solution to the equation

$$\tanh x = 1/x. \tag{2.1}$$

Then it is easy to see that the tangent to the graph of $y = \cosh x$ at $x = \beta$ passes through the origin. It follows that $\mathcal{C}_{-\beta}^{\beta}$ is maximally stable.

Theorem 2.1. *Given a horizontal slab $\mathbb{H}_{-a}^a := \{-a \leq z \leq a\}$ in $\mathbb{R}^3$, let Σ be a compact minimal surface in $\mathbb{H}_{-a}^a$ with $\partial\Sigma \subset \partial\mathbb{H}_{-a}^a$ such that all the components of its intersection with any horizontal plane have the same orientation. Then there exists a catenoidal waist $\mathcal{W} \subset \mathbb{H}_{-a}^a$ whose flux has the same vertical component as Σ such that*

$$\text{Area}(\Sigma) \geq \text{Area}(\mathcal{W}). \tag{2.2}$$

Also

$$\text{Area}(\mathcal{W}) \geq \text{Area}\left(\frac{a}{\beta}\,\mathcal{C}_{-\beta}^{\beta}\right), \tag{2.3}$$

where β satisfies $\tanh\beta = \frac{1}{\beta}$ and $\frac{a}{\beta}\mathcal{C}_{-\beta}^{\beta}$ is the homothetic expansion of the catenoid $\mathcal{C}_{-\beta}^{\beta}$ by the factor of $\frac{a}{\beta}$. The boundary circles of $\frac{a}{\beta}\,\mathcal{C}_{-\beta}^{\beta}$ lie on the boundary of $\mathbb{H}_{-a}^a$ and $\frac{a}{\beta}\,\mathcal{C}_{-\beta}^{\beta}$ is maximally stable. Moreover,

$$\text{Area}(\Sigma) = \text{Area}\left(\frac{a}{\beta}\,\mathcal{C}_{-\beta}^{\beta}\right)$$

if and only if $\Sigma = \frac{a}{\beta}\,\mathcal{C}_{-\beta}^{\beta}$ up to translation.

Proof. The minimality of Σ in $\mathbb{R}^3$ implies that the Euclidean coordinates x, y, z of $\mathbb{R}^3$ are harmonic on Σ. The critical points of x, y, z are isolated on Σ. Let $\tilde{\Sigma}$ be the set of regular points of z. Let $u = z|_\Sigma$ and define $v = u^*$, the harmonic conjugate of u on $\tilde{\Sigma}$. Note that v is multi-valued on $\tilde{\Sigma}$ but dv is well defined there. Then $w = u + iv$ is a local complex parameter on $\tilde{\Sigma}$. The Gauss map $g : \Sigma \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function which is used to express the metric of Σ:

$$ds^2 = \cosh^2\kappa|dw|^2, \;\; \kappa = \ln|g| = \text{Re}(\log g), \;\; \cosh\kappa = \frac{1}{2}\left(|g| + \frac{1}{|g|}\right).$$

The key idea of the proof is to take the average of the harmonic function $\kappa(u, v)$ along the level curves of u, $\gamma_c : \{u = \text{c}\}$. To do so, we need to find the total variation of v along γ_c:

$$\int_{\gamma_c} dv = \int_{\gamma_c} dz^* := f(c). \tag{2.4}$$

$f(c)$ equals the vertical component of the flux of Σ along γ_c, which is in fact constant for $-a \leq c \leq a$.

The average $h(u)$ of $\kappa(u, v)$ along γ_u is defined by

$$h(u) = \frac{1}{f}\int_{\gamma(u)} \kappa(u, v)dv.$$

Since $\kappa(u,v)$ is harmonic, so is $h(u)$ and

$$h''(u) = \Delta h = 0.$$

Hence $h(u)$ is linear in an open interval where u is regular. Let's compute the slope of $h(u)$. By the Cauchy-Riemann equations,

$$h'(u) = \frac{1}{f}\int_{\gamma(u)} \kappa_u(u,v)dv = \frac{1}{f}\int_{\gamma(u)} \kappa^*_v(u,v)dv = \frac{1}{f}(\kappa^*(u,f) - \kappa^*(u,0)).$$

Since

$$\log g = \ln|g| + i\arg g = \kappa(u,v) + i\,\kappa^*(u,v),$$

$\kappa^*(u,f) - \kappa^*(u,0)$ equals 2π times the total rotation number $r(\gamma_u)$ of the set γ_u which is the union of a finite number of closed curves.

If all the components of γ_c have the same orientation, obviously

$$r(\gamma_c) \geq 1. \tag{2.5}$$

$h(u)$ is piecewise linear and continuous at the height of the horizontal points.

Now let's compute the area of Σ.

$$\text{Area}(\Sigma) = \int_{-a}^{a}\int_0^f \cosh^2\kappa(u,v)dv\,du \geq \int_{-a}^{a} f\cosh^2 h(u)du,$$

where we have the inequality due to the convexity of $\cosh^2 x$. If $h(u)$ vanishes at some height d, define

$$k(u) = \frac{2\pi}{f}(u-d).$$

If $h(u)$ has no zero and $\min h(u) = h(-a) > 0$, define

$$k(u) = \frac{2\pi}{f}(u+a) + h(-a),$$

and if $\max h(u) = h(a) < 0$, define

$$k(u) = \frac{2\pi}{f}(u-a) + h(a).$$

Then we have for every u

$$h(u) \leq k(u) \leq 0 \quad \text{or} \quad 0 \leq k(u) \leq h(u).$$

It follows that

$$\cosh h(u) \geq \cosh k(u).$$

Therefore

$$\text{Area}(\Sigma) \geq \int_{-a}^{a} f\cosh^2 h(u)du \geq \int_{-a}^{a} f\cosh^2 k(u)du. \tag{2.6}$$

By the way, for some $d_0 \in \mathbb{R}$

$$k(u) = \frac{2\pi}{f}u + d_0 = \text{Re}\left(\frac{2\pi}{f}w + d_0\right).$$

So

$$\text{Area}(\Sigma) \geq \text{Area}(\mathcal{C}(2\pi w/f + d_0) \cap \mathbb{H}^a_{-a}),$$

which proves (2.2) with $\mathcal{W} = \mathcal{C}(2\pi w/f + d_0) \cap \mathbb{H}^a_{-a}$. Note here that

$$\text{Area}(\mathcal{C}(2\pi w/f + d_0) \cap \mathbb{H}^a_{-a}) \geq \text{Area}(\mathcal{C}(2\pi w/f) \cap \mathbb{H}^a_{-a}). \tag{2.7}$$

Now we need to find the catenoidal waist in $\mathbb{H}^a_{-a}$ with smallest area. (2.1) implies that $\beta = ea$ and hence

$$\text{Area}(\mathcal{C}(2\pi w/f) \cap \mathbb{H}^a_{-a}) \geq \text{Area}(\mathcal{C}(\beta w/a) \cap \mathbb{H}^a_{-a}). \tag{2.8}$$

Since the central waist circle of $\mathcal{C}(\beta w/a)$ has radius $r = a/\beta$, we have

$$\mathcal{C}(\beta w/a) = \frac{a}{\beta}\,\mathcal{C}.$$

Therefore

$$\mathcal{C}(\beta w/a) \cap \mathbb{H}^a_{-a} = \frac{a}{\beta}\,\mathcal{C}^{\beta}_{-\beta},$$

which, together with (2.7) and (2.8), gives (2.3). Clearly $\frac{a}{\beta}\,\mathcal{C}^{\beta}_{-\beta}$ is maximally stable. □

3. The helicoid

Catalan proved that the helicoid and the plane are the only ruled minimal surfaces in $\mathbb{R}^3$. Recently Eunjoo Lee, a former student of the author, has given some new characterizations of the helicoid as follows.

Definition:
Let $Cyl = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq 1\}$,
$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 < z < 1\}$,
d_1, d_2 : *diameters on the top and bottom disks of* ∂Cyl,
ℓ: *the central axis of* Cyl,
H_C: *the helicoid* $\subset Cyl$ *containing* $d_1 \cup d_2 \cup \ell$,
h_1, h_2: *the helices* $\subset \partial H_C \cap S$.

Theorem 3.1. *Let* $\Sigma \subset Cyl$ *be a minimal surface spanning* $d_1 \cup d_2 \cup h_1 \cup h_2$. *If the slope of* h_i *as a curve on the flat surface* S *is* ≥ 1, *then* $\Sigma = H_C$.

Theorem 3.2. *Let* $\Sigma \subset Cyl$ *be a minimal surface spanning* $d_1 \cup d_2 \cup h_1 \cup h_2$. *If the angle swept from* d_1 *to* d_2 *is* $\leq \pi$, *then* $\Sigma = H_C$.

Theorem 3.3. *Let* $\Sigma \subset Cyl$ *be a minimal disk and* $\gamma_1, \gamma_2 \subset S$ *simple curves connecting* ∂d_1 *to* ∂d_2 *such that* $\partial\Sigma = d_1 \cup d_2 \cup \gamma_1 \cup \gamma_2$. *If* γ_1 *is axially symmetric to* γ_2 *and* Σ *is perpendicular to* S *along* $\gamma_1 \cup \gamma_2$, *then* $\Sigma = H_C$.

Theorem 3.4. *Let* Σ *be a disk type surface in* Cyl *which is not necessarily minimal such that* $\partial\Sigma \subset \partial Cyl$ *and* $\Sigma \supset d_1 \cup d_2 \cup \ell$. *Then* $\text{Area}(\Sigma) \geq \text{Area}(H_C)$ *and equality holds if and only if* $\Sigma = H_C$.

Theorem 3.5. *Let* Σ *be a disk type surface in* Cyl *which is not necessarily minimal. And let* $\gamma_1, \gamma_2 \subset S$ *be simple curves connecting* ∂d_1 *to* ∂d_2 *such that* $\partial\Sigma = d_1 \cup d_2 \cup \gamma_1 \cup \gamma_2$. *If* γ_1 *is axially symmetric to* γ_2 *and the total* *curvature of* γ_1 *is* $\leq \pi$*, then* $\text{Area}(\Sigma) \geq \text{Area}(H_C)$ *and equality holds if and only if* $\Sigma = H_C$*.*

References

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