B.-Y. Chen's conjecture on hypersurfaces of Euclidean spaces

部分多様体論・湯沢 2014
2014-11-20, 湯沢

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1. Introduction

Definition 1.1. (a) A map $f : (M, g) \to (N, g_N)$ is **harmonic** iff f is a solution of the variational problem defined by $\int_M |df|^2 \, v_g$. Its Euler-Lagrange equation is $\tau \equiv 0$, where τ is the tension field of f. Roughly speaking, it means that f is "close to a constant map".
(b) A map $f : (M, g) \to (N, g_N)$ is **bi-harmonic** iff f is a solution of the variational problem defined by $\int_M |\tau|^2 \, v_g$. Roughly speaking, it means that f is "close to a harmonic map".
(c) A submanifold $M \subset (\overline{M}, \overline{g})$ is a **bi-harmonic submanifold** iff the inclusion map ι is bi-harmonic map w.r.t. $g = \iota^* \overline{g}$. Its Euler-Lagrange equation becomes

$$
(1.1) \qquad \begin{cases} (\perp) \colon \Delta\tau + \alpha^2(\tau) - g^{ij}(\overline{R}(\tau, \partial_i)\partial_j)^\perp = 0, \\ (\top) \colon -2\overline{g}((\delta\alpha)(\partial_i), \tau) + 2\overline{g}(\alpha(\partial_j, \partial_i), \nabla^j \tau) - \dfrac{1}{2}\nabla_i |\tau|^2 = 0, \end{cases}
$$

where α is the second fundamental equation. The equation $(\perp)$ is a 4th order elliptic equation.

Note that the conditions of bi-harmonic submanifold: "the inclusion map ι is bi-harmonic map" and "the source metric g is the induced metric $\iota^*\overline{g}$" are independent, and their combination becomes an over-determined PDE. Therefore, its solutions rarely exist, in general.

However, every minimal submanifold is bi-harmonic, and we have a lot of bi-harmonic submanifolds. B.-Y. Chen conjectured as follows.

B.-Y. Chen's conjecture: There are no non-minimal bi-harmonic submanifolds in E^m.

We consider this conjecture for local submanifolds $M^n \subset E^m$. Note that bi-harmonic submanifolds in E^m automatically become C^ω submanifolds.

Known results: Chen's conjecture is true for
(1) Curves in E^m (I. Dimitric, 1992),
(2) Surfaces in E^3 (B.-Y. Chen, 1991; G. Y. Jiang, 1986),
(3) Hypersurfaces in E^4 (T. Hasanis & T. Vlachos, 1995),

(4) Hypersurfaces in E^{n+1} with the number of principal curvatures $\#\mathrm{pc} \leq 2$ (I. Dimitric, 1989).

Our main results are as follows.

Theorem 1.2 (N). *There are no non-minimal hypersurfaces in E^{n+1} with $\#pc \leq 3$.*

Theorem 1.3 (I). *There are no non-minimal bi-harmonic hypersurfaces M^n in E^{n+1} with following properties:*

(1) Each principal curvature λ_i of M is simple at some point in M.

(2) $g(\nabla_{v_i} v_j, v_k) \neq 0$ for all distinct unit principal curvature vector fields $v_i, v_j, v_k \in$ $\mathrm{Ker}\, d\tau$ at some point in M.

2. Proof of the theorem

Let M be a non-minimal bi-harmonic hypersurface of $\mathbf{R}^{n+1}$. Since principal curvatures are simple, unit principal curvature vector fields $\{v_i\}$ forms a orthonormal frame field on M, $\alpha(v_i, v_j) = \delta_{ij}\lambda_i$. (1.1) becomes

$$(2.1)\qquad \begin{array}{ll} \text{(A)} & \Delta\tau + |\alpha|^2\tau = 0, \\ \text{(B)} & (\tau + 2\lambda_i)v_i[\tau] = 0 \quad (1 \leq \forall i \leq n), \end{array}$$

where $v_i[*]$ is differentiation of function $*$. From (A), we see that if $|\tau|^2$ takes local maximum at some point, then M is minimal. In particular, τ is not constant. From (B), if there are no λ_i with $\tau + 2\lambda_i = 0$, then τ is constant. Hence there exists λ_i with $\tau + 2\lambda_i = 0$. We may assume $\tau + 2\lambda_n = 0$.

Since τ is not constant, equation: $\tau = c$ defines a hypersurface F in M at generic points. We call F a **characteristic hypersurface** of M. Put $n_1 = n-1$. The vectors $\{v_i\}_{1 \leq i \leq n_1}$ consist an orthonormal tangent frame field on F. Put $\mu_i := g(\nabla_{v_i} v_i, v_n)$, which turns out to be principal curvatures of F in M. From (A), (B) and Gauss, Codazzi equation, we can derive the following ODE.

Proposition 2.1. *λ_i and μ_i satisfy the over-determined ODE:*

$$(\#)\qquad \begin{array}{ll} \text{(D)} & (\lambda_i)' = (\frac{1}{2}\tau + \lambda_i)\mu_i, \quad (\mu_i)' = (\mu_i)^2 - \frac{1}{2}\tau\lambda_i, \\ \text{(T)} & -\tau'' + \tau'\sum_{i<n}\mu_i + \tau\big(\frac{1}{4}\tau^2 + \sum_{i<n}(\lambda_i)^2\big) = 0, \end{array}$$

where $*' = v_n[*]$ and $\tau := (2/3)\sum_{i<n}\lambda_i$.

Remark 2.2. This proposition holds even if principal curvatures are not simple.

Since ODE (#) is algebraic, we get the following

Proposition 2.3. *Solutions $(\lambda_i, \mu_i) \in \mathbf{R}^{2n_1}$ to (#) runs in the zero-set of a homogeneous polynomial P_3 of degree 3. Put $P_{k+1} := (P_k)'$. The set S of initial data of solutions to (#) becomes an algebraic manifold $\cap_{k=3}^{\infty}(P_k)^{-1}(0)$.*

Conjecture 2.4. $S = \cap_{k\geq 3}(P_k)^{-1}(0) \subset \tau^{-1}(0)$, and so Chen's conjecture is true.

3. Proof of Theorem N

Based on ODE (#), we prove theorem N. First, we prepare the following

Lemma 3.1. *$\lambda_n = -\tau/2$ is simple. If $\lambda_i \equiv \lambda_j$, then $\mu_i \equiv \mu_j$.*

Therefore, solutions to (#) are considered as curves in $\mathbf{C}^4(\lambda_1, \lambda_2, \mu_1, \mu_2)$. Let m_i be the multiplicity of λ_i. We denote by π the projection $\mathbf{C}^4 \to \mathbf{C}^2(\lambda_1, \lambda_2)$, and by p the projection $\mathbf{C}^2 \setminus \{0\} \to P^1(\mathbf{C})$. The set S becomes an algebraic manifold in $\mathbf{C}^4$. Therefore, $p(\pi(S))$ is whole $P^1(\mathbf{C})$ or a finite point set.

On the other hand we can show, may be using a computer, that

Step 1. $(m_2 + 3, -m_1) \notin \pi(S)$.

Thus, $p(\pi(S))$ is a finite point set, and the ratio λ_2/λ_1 is constant along each solution to (#).

Step 2. Any solution to (#) with constant ratio λ_2/λ_1 is in $\tau^{-1}(0)$.

Q.E.D.

4. Proof of Theorem I

To prove Theorem I, we have to analyze the characteristic submanifold F.

Definition 4.1. Put $J = \{\{i, j\} \mid 1 \leq i, j \leq n_1,\ i \neq j\}$. If a distinct triplet $\{i, j, k\}$ satisfies $g(\nabla_{v_i} v_j, v_k) \neq 0$, then we define $\{i, j\} \sim \{j, k\} \sim \{i, k\}$. Let $\sim_J$ be the equivalence relation on J generated by $\sim$. If all $\{i, j\} \in J$ are equivalent under $\sim_J$, the frame field $\{v_i\}$ is **irreducible**.

Remark. It is weaker than the assumption of Theorem 1.3.

Definition 4.2. If there exist functions φ, ψ on M s.t. $\mu_i = \varphi\lambda_i + \psi$ for $\forall i \leq n_1$, then $\{\lambda_i\}$ and $\{\mu_i\}$ are **linearly related**.

Lemma 4.3. *We assume that $n_1 \geq 3$. (1) If the frame field $\{v_i\}$ is irreducible, then $\{\lambda_i\}$ and $\{\mu_i\}$ are linearly related, and $\lambda_i, \mu_i, \varphi, \psi$ are constant on each characteristic hypersurface F. (2) If φ or ψ is constant in t, then $\tau \equiv 0$.*

Theorem 4.4 (I)**.** *There are no non-minimal bi-harmonic hypersurfaces M^n in E^{n+1} with following properties:*
(1) $\{\lambda_i\}$ are simple . (2) $\{v_i\}_{i\leq n_1}$ is irreducible.

To prove Theorem 4.4, we need simple, but length calculation. Last equation to prove Theorem I is

$$
\begin{aligned}
&12n_1(n_1-1)\psi^2\varphi^3(1+\varphi^2)^2 \\
&\times\{-105(12+n_1+3(n_1)^2)+(-5901+875n_1+1026(n_1)^2)\varphi^2 \\
&\quad -2(-351+1264n_1+567(n_1)^2)\varphi^4+8(-159+34n_1+45(n_1)^2)\varphi^6\} \\
&\times\{-7(33-17n_1)^2(27+3n_1-13(n_1)^2-6(n_1)^3+5(n_1)^4) \\
&\quad +(-2755134+4210164n_1-839475(n_1)^2-1289439(n_1)^3+362329(n_1)^4+269159(n_1)^5-100964(n_1)^6)\varphi^2 \\
&\quad +3(-3211164+5957928n_1-3766311(n_1)^2+651168(n_1)^3+904142(n_1)^4-789504(n_1)^5+167725(n_1)^6)\varphi^4 \\
&\quad -(2628288+11059011n_1-21744558(n_1)^2+1018458(n_1)^3+11702488(n_1)^4-5345493(n_1)^5+825166(n_1)^6)\varphi^6 \\
&\quad +(-20731545+44245224n_1-22777452(n_1)^2-9103320(n_1)^3+13627127(n_1)^4-5693096(n_1)^5+576422(n_1)^6)\varphi^8 \\
&\quad -12(-683640+1720305n_1-1262466(n_1)^2-367722(n_1)^3+758200(n_1)^4-199503(n_1)^5+13322(n_1)^6)\varphi^{10} \\
&\quad -4(843453-2056212n_1+731808(n_1)^2+834336(n_1)^3-446779(n_1)^4+56460(n_1)^5+1094(n_1)^6)\varphi^{12} \\
&\quad +16(9-7n_1)^2(-9+2(n_1)^2)(-43-26n_1+5(n_1)^2)\varphi^{14}\}.
\end{aligned}
$$

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