Minimal surfaces in the product space $N^3(c) \times R$

Makoto SAKAKI

Graduate School of Science and Technology, Hirosaki University
Hirosaki 036-8561, Japan
E-mail: sakaki@hirosaki-u.ac.jp (new from March 1 in 2015)

Abstract. We discuss the structure equations and the curvature ellipse of minimal surfaces in the product space $N^3(c) \times R$, where $N^3(c)$ is the 3-dimensional simply connected space form of constant curvature c. We also give some related problems.

1. Introduction

Let $N^n(c)$ denote the n-dimensional simply connected space form of constant curvature c. That is, $N^n(c)$ is either the sphere $S^n(c)$, the hyperbolic space $H^n(c)$, or the Euclidean space R^n. There are many results on the geometry of surfaces in $N^3(c)$. A natural generalization is the geometry of surfaces in $N^4(c)$. Another generalization is that of surfaces in $N^3(c) \times R$, from which, we expect to find a new view point to the geometry of surfaces in $N^3(c)$.

From about 2002, Abresch, Rosenberg, Meeks and so on have begun to study surfaces in $N^2(c) \times R$ (cf. [1], [3], [4], [7], [8]). The higher codimensional cases have been studied more recently, by Fetcu, Rosenberg and so on (cf. [2], [5], [6], [9], [10], [11]).

For a surface M in a Riemannian manifold, the curvature ellipse at $p \in M$ is defined by

$$E(p) = \{h(X, X) | X \in T_pM, |X| = 1\},$$

where h is the second fundamental form. We say that M is isotropic if the curvature ellipse is a circle at any point.

Here, using the method of moving frames, and writing the structure equations with connection forms, we discuss the curvature ellipse of minimal surfaces in $N^3(c) \times R$. The results are stated as follows:

Theorem 1 ([11]). *Let M be a minimal surface in $N^3(c) \times R$ where $c \neq 0$. If M is isotropic, then M is totally geodesic.*

Remark. When $c = 0$, a minimal surface in R^4 is isotropic if and only if it is a holomorphic curve in $C^2 = R^4$.

Theorem 2 ([11]). *There exists no minimal surface in $N^3(c) \times R$ where $c \neq 0$ such that the semi-major axis and the semi-minor axis of the curvature ellipse are both positive constant.*

Remark. Except for the Clifford torus in $S^3(c)$, there are minimal surfaces in $S^3(c) \times R$ such that the semi-major axis of the curvature ellipse is positive constant and the semi-minor axis of the curvature ellipse is identically zero (cf. [2]).

2. On the structure equations

Let $\{e_A\}_{1 \leq A \leq 4}$ be a local orthonormal frame field in $N^3(c) \times R$, and $\{\omega^A\}$ the dual coframe field. The connection forms $\{\omega_B^A\}$ are given by

$$de_B = \sum_A \omega_B^A e_A.$$

The structure equations are

$$d\omega^A = -\sum_B \omega_B^A \wedge \omega^B,$$

$$d\omega_B^A = -\sum_C \omega_C^A \wedge \omega_B^C + \frac{1}{2} \sum_{C,D} \bar{R}_{ABCD} \omega^C \wedge \omega^D.$$

Here $\bar{R}$ is the curvature tensor of $N^3(c) \times R$, which satisfies

$$\bar{R}(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,\xi\rangle\langle Z,\xi\rangle X + \langle X,\xi\rangle\langle Z,\xi\rangle Y$$
$$+\langle X,Z\rangle\langle Y,\xi\rangle\xi - \langle Y,Z\rangle\langle X,\xi\rangle\xi\},$$

and ξ is the unit vector to the factor R.

Let M be a surface in $N^3(c) \times R$. We choose $\{e_A\}$ so that $\{e_i\}_{1 \leq i \leq 2}$ are tangent to M and $\{e_\alpha\}_{3 \leq \alpha \leq 4}$ are normal to M. Then, along M, we have

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j,$$

where h_{ij}^α are the components of the second fundamental form. The Gaussian curvature K and the normal curvature K_ν are given by

$$d\omega_2^1 = K\omega^1 \wedge \omega^2, \quad d\omega_4^3 = K_\nu \omega^1 \wedge \omega^2,$$

respectively. We decompose ξ as

$$\xi = T + \eta,$$

where T is tangent to M and η is normal to M.

Now let M be a minimal surface in $N^3(c) \times R$. If either the curvature ellipse is not a circle at any point, or the curvature ellipse is a circle of positive radius at any point, then we can choose the frame field $\{e_A\}$ so that

$$(h_{ij}^3) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix},$$

for some smooth functions a and b with $|a| \geq |b|$. Then

$$\omega_1^3 = a\omega^1, \quad \omega_2^3 = -a\omega^2, \quad \omega_1^4 = b\omega^2, \quad \omega_2^4 = b\omega^1.$$

The Gauss and Ricci equations become

$$K = c(1 - |T|^2) - a^2 - b^2, \quad K_\nu = 2ab.$$

As the Codazzi equation, by the covariant derivatives of ω_i^α, we have four PDEs for a and b. Since it is the product case, the Gauss, Codazzi and Ricci equations are not necessary and sufficient. There are other conditions. In fact, from $\bar{\nabla}\xi = 0$ where $\bar{\nabla}$ is the Levi-Civita connection of $N^3(c) \times R$, we have eight PDEs for the components of T and η.

Using those fourteen equations in the proof by contradiction for Theorems 1 and 2, we can get the conclusions.

3. Some related problems

In the proof by contradiction for Theorem 2, the notion of constant angle surfaces plays an important role. A surface in $N^3(c) \times R$ is called a constant angle surface (or a helix surface) if the tangent planes make a constant angle with ξ, which is equivalent to that $|T|$ is constant (cf. [2], [4], [5], [9]).

By [2], for any minimal constant angle surface in $S^3(c) \times R$ with $0 < |T| < 1$, the semi-major axis of the curvature ellipse is positive constant and the semi-minor axis of the curvature ellipse is identically zero. It is not certain if the converse is true. So we can consider the following:

Question. Except for minimal constant angle surfaces, does there exist a minimal surface in $N^3(c) \times R$ with $c \neq 0$ such that the semi-major axis of the curvature ellipse is positive constant and the semi-minor axis of the curvature ellipse is identically zero?

We shall give some other related problems.

Problem 1. For minimal surfaces in $N^3(c) \times R$ $(c \neq 0)$, find the reduced partial differential equations that are equivalent to the structure equations.

Problem 2. Discuss isotropic surfaces in $N^3(c) \times R$ $(c \neq 0)$.

Problem 3. Discuss isotropic minimal surfaces in $N^4(c) \times R$ $(c \neq 0)$.

Problem 4. Discuss minimal 2-spheres in $S^3(c) \times R$.

Problem 5. Classify minimal surfaces with constant Gaussian curvature in $N^3(c) \times R$ $(c \neq 0)$.

Problem 6. Discuss surfaces in indefinite product spaces or warped products.

Remark. See [3] for Problems 1 and 5 in the case of minimal surfaces in $N^2(c) \times R$ $(c \neq 0)$.

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