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Applications of isoparametric foliation in spheres

WENJIAO YAN
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China;
Mathematical Institute, Graduate School of Sciences, Tohoku University, Sendai, 980-8578, Japan
e-mail : wjyan@bnu.edu.cn

Abstract. This paper gives a survey on the recent progress in the application of isoparametric foliation, especially in attacking Yau's conjecture on the first eigenvalue and Besse's problem on the generalization of Einstein condition.

1 Preliminaries

Research on classifications and applications of isoparametric foliation in spheres have been very active in the recent two decades. As is well known, an isoparametric hypersurface M^n in $S^{n+1}(1)$ is a hypersurface with constant principal curvatures. Let g be the number of distinct principal curvatures k_i ($k_1 > \cdots > k_g$) with multiplicity $m_i(i = 1, ..., g)$. A fundamental result of Münzner states that $g \in \{1, 2, 3, 4, 6\}$ and $m_i = m_{i+2}$ (subscripts mod g). When $g \leq 3$, the classification for isoparametric hypersurfaces are accomplished by E. Cartan, who proved them to be homogeneous; when $g = 4$, all isoparametric hypersurfaces are of OT-FKM type, or homogeneous with $(m_1, m_2) = (2, 2), (4, 5)$, except possibly for the case with $(m_1, m_2) = (7, 8)$; when $g = 6$, Abresch showed that the multiplicities $m_1 = m_2 = 1$ or 2, and then Dorfmeister-Neher and Miyaoka showed that they are homogeneous, namely, they can be characterized as the principal orbits of the isotropy representation of some rank 2 symmetric spaces.

2 Yau's conjecture on the first eigenvalue

Let M^n be a closed manifold. We know that Laplacian Δ is an elliptic operator and has a discrete spectrum:

$$\{0 = \lambda_0(M) < \lambda_1(M) \leqslant \lambda_2(M) \leqslant \cdots \leqslant \lambda_k(M), \cdots, \uparrow \infty\}.$$

When M^n is a minimal submanifold in S^N, Takahashi theorem implies that $\lambda_1(M) \leq n$. In this regard, S.T.Yau raised the following problem in his problem section in 1982:

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Conjecture([Yau]): Let M^n be a closed embedded minimal hypersurface in the unit sphere $S^{n+1}(1)$. Then

$$\lambda_1(M^n) = n.$$

Attacking this conjecture, Choi and Wang proved that $\lambda_1(M) \leq n/2$ in 1983. We restricted this problem to the isoparametric hypersurface, which is naturally embedded hypersurface. There are results of Muto-Ohnita-Urakawa (1984) and Kotani (1985) considering the homogeneous cases, and Muto (1988) considering a small part of inhomogeneous cases. Based on their results, we obtained a wide generalized result without using the classification of isoparametric hypersurfaces in spheres:

Theorem 2.1 ([TY1]). *Let M^n be a minimal isoparametric hypersurface in $S^{n+1}(1)$ with $g = 4$ and $m_1, m_2 \geqslant 2$. Then*

$$\lambda_1(M^n) = n$$

with multiplicity $n + 2$.

This theorem finally gives an affirmative answer to Yau's conjecture in the isoparametric case.

Another natural question in the isoparametric foliation is that *what is the first eigenvalues of the focal submanifolds M_1 and M_2?* As a matter of fact, focal submanifolds are minimal submanifolds in spheres. For the cases $g = 1, 2, 3$, it is trivial. For $g = 4$, we proved that

Theorem 2.2 ([TY1]). *Let M_1 be a focal submanifold of codimension $m_1 + 1$ in $S^{n+1}(1)$ with $g = 4$. If $\dim M_1 \geqslant \frac{2}{3}n + 1$, then*

$$\lambda_1(M_1) = \dim M_1 = m_1 + 2m_2.$$

We also would like to ask the following question: *Let M^d be a closed minimal submanifold in $S^{n+1}(1)$. If the dimension d of M^d satisfies $d \geqslant \frac{2}{3}n + 1$, then is it true that $\lambda_1(M^d) = d$?* To the best of our knowledge, there are no counter examples and the dimension condition is very important. For instance, Solomon found some eigenfunctions on the focal submanifold M_2 of OT-FKM type, which correspond to eigenvalues $4m_1$. However, if we require $m_1 < \frac{1}{2}m_2$ (for example $(m_1, m_2) = (2, 5)$), which implies that $d < \frac{2}{3}n + 1$, then $\lambda_1(M_2) \leqslant 4m_1 < \dim M_2$.

3 Besse's problem on the generalization of Einstein condition

According to [QTY], there are only two Einstein manifolds among all the known focal submanifolds of $g = 4$. It is natural to study the more general condition–Ricci parallel condition. A. Gray introduced two significant classes of Einstein-like Riemannian manifolds, i.e., $\mathcal{A}$ and $\mathcal{B}$ defined as follows, in which the Ricci tensor ρ is cyclic parallel and a Codazzi tensor, respectively. These two classes have been investigated extensively since then. Gray also showed that the intersection of $\mathcal{A}$ and $\mathcal{B}$ is the class of Ricci parallel manifolds. Then a further and natural question arises: *are the focal submanifolds of $g = 4$ Ricci parallel, $\mathcal{A}$-manifolds, or $\mathcal{B}$-manifolds ?*

On the other hand, it is well known that the D'Atri spaces (Riemannian manifolds with volume preserving geodesic symmetries) belong to the class $\mathcal{A}$. So the examples of $\mathcal{A}$-manifolds are not rare in the literature, but mostly are (locally) homogeneous. In this regard, Besse ([Bes], 16.56(i), pp.451) posed the following challenging problem as one of "some open problems" : *Find examples of $\mathcal{A}$-manifolds, which are neither locally homogeneous, nor locally isometric to Riemannian products and have non-parallel Ricci tensor.*

Aiming for this problem, Jelonek and Pedersen-Tod constructed $\mathcal{A}$-manifolds on S^1-bundles over locally non-homogeneous Kähler-Einstein manifolds, and on S^1-bundles over a $K3$ surface, from defining Riemannian submersion metric on the S^1-bundles. But their examples are not so satisfying, since they are not simply-connected, and the metrics are not natural enough.

Taking advantage of focal submanifolds, we find a series of simply-connected examples with natural metric for this open problem of Besse.

Theorem 3.1 ([TY2])**.** *All the focal submanifolds of isoparametric hypersurfaces in spheres with $g = 4$ are $\mathcal{A}$-manifolds, except possibly for the only unclassified case with $(m_1, m_2) = (7, 8)$.*

Theorem 3.2 ([TY2])**.** *For the focal submanifolds of isoparametric hypersurfaces in spheres with $g = 4$, we have*

(i) The M_1 of OT-FKM type is Ricci parallel if and only if $(m_1, m_2) = (2, 1), (6, 1)$, or it is diffeomorphic to $Sp(2)$ in the homogeneous case with $(m_1, m_2) = (4, 3)$; while the M_2 of OT-FKM type is Ricci parallel if and only if $(m_1, m_2) = (1, k)$.

(ii) For $(m_1, m_2) = (2, 2)$, the one diffeomorphic to $\widetilde{G}_2(\mathbb{R}^5)$ is Ricci parallel, while the other diffeomorphic to $\mathbb{C}P^3$ is not.

(iii) For $(m_1, m_2) = (4, 5)$, both are not Ricci parallel.

Proposition 3.3 ([TY2])**.** *The focal submanifolds of isoparametric hypersurfaces in spheres with $g = 4$ and $m_1, m_2 > 1$ are not Riemannian products.*

Proposition 3.4 ([TY2])**.** *The focal submanifolds M_1 of OT-FKM type with $(m_1, m_2) = (3, 4k)$ are not intrinsically homogeneous.*

Remark 3.5. *By Morse theory, one sees that if $m_1 > 1$ (resp. $m_2 > 1$), the focal submanifold M_2 (resp. M_1) is simply-connected. Combining Propositions 3.3, 3.4 with Theorems 3.1 and 3.2, we conclude that* *the focal submanifolds M_1 of OT-FKM type with $(m_1, m_2) = (3, 4k)$ are simply-connected $\mathcal{A}$-manifolds with non-parallel Ricci tensor, which are minimal submanifolds in spheres, but neither locally homogeneous, nor locally isometric to Riemannian products. Much more examples to the problem of Besse can be obtained in this way.*

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