CLASSIFICATION OF REAL HYPERSURFACES IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNAINS RELATED TO THE RICCI TENSOR

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In the geometry of real hypersurfaces in complex space forms $M_m(c)$ or in quaternionic space forms $Q_m(c)$ Kimura [1] and [2] (resp. Pérez and Suh [4]) considered real hypersurfaces in $M_n(c)$ (resp. in $Q_m(c)$) with commuting Ricci tensor, that is, $S\phi = \phi S$, (resp. $S\phi_i = \phi_i S$, $i = 1, 2, 3$)where S and ϕ (resp. S and ϕ_i, $i = 1, 2, 3$) denote the Ricci tensor and the structure tensor of real hypersurfaces in $M_m(c)$ (resp. in $Q_m(c)$).

In [1] and [2], Kimura has classified that a Hopf hypersurface M in complex projective space $P_m(\mathbb{C})$ with commuting Ricci tensor is locally congruent to of type (A), a tube over a totally geodesic $P_k(\mathbb{C})$, of type (B), a tube over a complex quadric Q_{m-1}, $\cot^2 2r = m-2$, of type (C), a tube over $P_1(\mathbb{C})\times P_{(m-1)/2}(\mathbb{C})$, $\cot^2 2r = \frac{1}{m-2}$ and n is odd, of type (D), a tube over a complex two-plane Grassmannian $G_2(\mathbb{C}^5)$, $\cot^2 2r = \frac{3}{5}$ and $n = 9$, of type (E), a tube over a Hermitian symmetric space $SO(10)/U(5)$, $\cot^2 2r = \frac{5}{9}$ and $m = 15$.

On the other hand, in a quaternionic projective space $\mathbb{Q}P^m$ Pérez and Suh [4] have classified real hypersurfaces in QP^m with commuting Ricci tensor $S\phi_i = \phi_i S$,$i = 1, 2, 3$, where S (resp. ϕ_i) denotes the Ricci tensor(resp. the structure tensor) of M in $\mathbb{Q}P^m$, is locally congruent to of A_1, A_2-type, that is, a tube over $\mathbb{Q}P^k$ with radius $0 < r < \frac{\pi}{2}$, $k\in\{0, \cdots, m-1\}$. The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where J_i, $i = 1, 2, 3$, denote a quaternionic Kähler structure of $\mathbb{Q}P^m$ and N a unit normal field of M in $\mathbb{Q}P^m$. Moreover, Pérez and Suh [3] have considered the notion of $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where R denotes the curvature tensor of a real hypersurface M in $\mathbb{Q}P^m$, and proved that M is locally congruent to a tube of radius $\frac{\pi}{4}$ over $\mathbb{Q}P^k$.

Let us denote by $SU_{2,m}$ the set of $(m+2)\times(m+2)$-indefinite special unitary matrices and U_m the set of $m\times m$-unitary matrices. Then the Riemannian symmetric space $SU_{2,m}/S(U_2 U_m)$, $m \geq 2$, which consists of complex two-dimensional subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, has a remarkable feature that it is a Hermitian symmetric space as well as a quaternionic Kähler symmetric space. In fact, among all Riemannian symmetric spaces of noncompact type the symmetric spaces $SU_{2,m}/S(U_2 U_m)$, $m \geq 2$, are the only ones which are Hermitian symmetric and quaternionic Kähler symmetric.

When the Ricci tensor S satisfies the formula $S\phi + \phi S = 2k\phi$, on M in $SU_{2,m}/S(U_2{\cdot}U_m)$, we say M has a *pseudo anti-commuting Ricci tensor*. We give a complete classification of real hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$ satisfying the notion of *pseudo anti-commuting Ricci tensor* as follows:

Main Theorem 1. *Let M be a Hopf real hypersurface in $SU(2,m)/S(U(2){\cdot}U(m))$ with pseudo anti-commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to one of the following:*

(B) *a tube around a totally geodesic $\mathbb{H}H^n$ in $SU(2,2n)/S(U(2)\times U(2n))$, $m = 2n$;*

(C_2) *a horosphere in $SU(2,m)/S(U(2)\times U(m))$ whose center at infinity is singular and of type $JN \perp \mathfrak{J}N$;*

(D) *The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$. Moreover, M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \sqrt{2}\ ,\ \gamma = 0\ ,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C}\cap\mathcal{Q})\ ,\ T_\gamma = J(TM\ominus\mathcal{Q})\ ,\ T_\lambda \subset \mathcal{C}\cap\mathcal{Q}\cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then we have $T_\mu \subset \mathcal{C}\cap\mathcal{Q}\cap J\mathcal{Q}$, $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$.

Now let us recall an n-dimensional Riemannian manifold (M,g) is said to be a *Ricci soliton*, If there exists a sooth vector field $V \in T_pM$ that satisfies $\frac{1}{2}(\mathfrak{L}_V g)(X,Y) + Ric(X,Y) = 2kg(X,Y)$ for any $X,Y \in TM$. As an application of our Main Theorem 1 to the Ricci soliton problem, we give another Main Theorem 2 as follows:

Main Theorem 2. *There do not exist any Ricci soliton (M,g,ξ,k) on Hopf real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$.*

References

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