THE PARALLELISMS FOR THE SHAPE OPERATOR OF REAL HYPERSURFACES IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIANS

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As examples of Hermitian symmetric spaces of rank 2 we can give Riemannian symmetric spaces $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ and $Q^m = SO_{m+2}/SO_mSO_2$, which are said to be complex two-plane Grassmannians and complex quadric, respectively. Recently, the second author has focused on research for hypersurfaces of Q^m (see [7, 8]). On the other hand, as another kind of Hermitian symmetric space with rank 2 of noncompact type, we can give an example of *complex hyperbolic two-plane Grassmannians* $G_2^*(\mathbb{C}^{m+2}) = SU_{m,2}/S(U_2U_m)$ which is a set of all complex two-dimensional linear subspaces in indefinite complex Euclidean spaces $\mathbb{C}_2^{m+2}$. Then it is known that $G_2^*(\mathbb{C}^{m+2})$ has both a Kaehler structure J and a quaternionic Kaehler structure $\mathfrak{J}$ not containing J, for details we refer to [5, 6]. In particular, when $m = 1$, $G_2^*(\mathbb{C}^3)$ is isometric to the two-dimensional complex hyperbolic space $\mathbb{C}H^2$ with constant holomorphic sectional curvature -4. When $m = 2$, we note that the isomorphism $\mathrm{SO}_{4,2} \simeq \mathrm{SU}_{2,2}$ yields an isometry between $G_2^*(\mathbb{C}^4)$ and the indefinite real Grassmann manifold $G_2^*(\mathbb{R}_2^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}_2^6$. For this reason we assume $m \geq 3$ from now on.

On a real hypersurface M in $G_2^*(\mathbb{C}^{m+2})$, we can naturally consider two geometric conditions that the 1-dimensional distribution $\mathcal{C}^\perp = \mathrm{Span}\{\xi\}$ and the 3-dimensional distribution $\mathcal{Q}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M. Here the almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. And a real hypersurface such that $A\mathcal{C}^\perp \subset \mathcal{C}^\perp$ is called by *Hopf hypersurface*. The *almost contact 3-structure* vector fields ξ_ν for the 3-dimensional distribution $\mathcal{Q}^\perp$ of M in $G_2^*(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where J_ν denotes a canonical local basis of a quaternionic Kaehler structure $\mathfrak{J}$, such that $T_xM = \mathcal{Q} \oplus \mathcal{Q}^\perp$, $x \in M$. In addition, a real hypersurface of $G_2^*(\mathbb{C}^{m+2})$ satisfying $g(A\mathcal{Q}, \mathcal{Q}^\perp) = 0$ (i.e. $A\mathcal{Q}^\perp \subset \mathcal{Q}^\perp$ or $A\mathcal{Q} \subset \mathcal{Q}$, resp.) is said to be a $\mathcal{Q}^\perp$*-invariant hypersurface*.

In a paper due to Suh [5] we have introduced the following theorem.

Theorem A. *Let M be a connected real hypersurface in $G_2^*(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $\mathcal{C}^\perp$ and $\mathcal{Q}^\perp$ are invariant under the shape operator of M if and only if M is congruent to an open part of one of the following hypersurfaces:*

($\mathcal{T}_A$) *a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$;*

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[2] *Key words* : Hopf hypersurfaces; complex hyperbolic two-plane Grassmannians; complex two-plane Grassmannians; cyclic parallelism; Reeb parallelism; shape operator; geodesic Reeb flow.

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($\mathcal{T}_B$) *a tube around a totally geodesic* $\mathbb{H}H^n$ *in* $SU_{2,m}/S(U_2U_m)$, $m = 2n$;

($\mathcal{H}_A$) *a horosphere in* $SU_{2,m}/S(U_2U_m)$ *whose center at infinity is singular and of type* $JX \in \mathfrak{J}X$;

($\mathcal{H}_B$) *a horosphere in* $SU_{2,m}/S(U_2U_m)$ *whose center at infinity is singular and of type* $JX \perp \mathfrak{J}X$;

or the following exceptional case holds:

($\mathcal{E}$) *The normal bundle* νM *of* M *consists of singular tangent vectors of type* $JX \perp \mathfrak{J}X$. *Moreover,* M *has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = (\mathcal{C} \cap \mathcal{Q})^\perp, \quad T_\gamma = J\mathcal{Q}^\perp, \quad T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ *is another (possibly nonconstant) principal curvature function, then we have* $T_\mu \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}$, $JT_\mu \subset T_\lambda$ *and* $\mathfrak{J}T_\mu \subset T_\lambda$.

In this talk, we consider a new concept as the generalization of parallelism with respect to the shape operator $\tilde{A}$ of a submanifold $\tilde{M}$ in a Riemannian manifold. Actually, if the shape operator $\tilde{A}$ of $\tilde{M}$ satisfies

$$(*) \qquad \begin{aligned} &\mathfrak{S}_{X,Y,Z \in T\tilde{M}}\, g((\tilde{\nabla}_X \tilde{A})Y, Z) \\ &\quad = g((\tilde{\nabla}_X \tilde{A})Y, Z) + g((\tilde{\nabla}_Y \tilde{A})Z, X) + g((\tilde{\nabla}_Z \tilde{A})X, Y) = 0 \end{aligned}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, then $\tilde{M}$ is said to be *cyclic parallel*. When $\tilde{M}$ is a real hypersurface in complex space form $M^n(c)$ with constant holomorphic sectional curvature $4c$, $c \neq 0$, $n \geq 3$, the cyclic parallelism is equivalent to the condition

$$(\nabla_X A)Y = -c\{\eta(Y)\phi X + g(\phi X, Y)\xi\}$$

for any vector fields X and Y tangent to $\tilde{M}$. Maeda ([4]) and Chen, Ludden and Montiel ([1]) classified real hypersurfaces in $M^n(c)$, $c \neq 0$, under this condition.

By these motivations we consider the cyclic parallelism for real hypersurfaces in complex hyperbolic two-plane Grassmannians $G_2^*(\mathbb{C}^{m+2})$ and prove the following theorem.

Theorem 1. *There does not exist any cyclic parallel hypersurface in complex hyperbolic two-plane Grassmannians* $G_2^*(\mathbb{C}^{m+2})$, $m \geq 3$, *with non-vanishing geodesic Reeb flow.*

Moreover, by virtue of the equation of Codazzi on a real hypersurface in $G_2^*(\mathbb{C}^{m+2})$ we see that if the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$, then a cyclic parallel hypersurface satisfies the condition of Reeb parallelism. So, we obtain:

Theorem 2. *Let* M *be a real hypersurfce in complex hyperbolic two-plane Grassmannians* $G_2^*(\mathbb{C}^{m+2})$, $m \geq 3$, *with non-vanishing geodesic Reeb flow. Then the shape operator* A *of* M *is Reeb parallel if and only if* M *is locally congruent to an open part of one of the following two hypersurfaces:*

($\mathcal{T}_A$) *a tube around some totally geodesic* $G_2^*(\mathbb{C}^{m+1}) = SU_{2,m-1}/S(U_2U_{m-1})$ *in* $G_2^*(\mathbb{C}^{m+2})$ *or*

($\mathcal{H}_A$) *a horospher whose center at infinity is singular and of type* $JN \in \mathfrak{J}N$.

Finally, in [2] it has studied about the Reeb parallelism of a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. As a lemma for this result we observe the relation between the cyclic parallelism and the Reeb parallelism for a real hypersurface in $G_2(\mathbb{C}^{m+2})$. And by virtue of this result we obtain:

Theorem 3. *There does not exist any cyclic parallel hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with non-vanishing geodesic Reeb flow.*

Throughout this talk, we use some references [5] and [6] (or [2] and [3], resp.) to recall the Riemannian geometry of $G_2^*(\mathbb{C}^{m+2})$ (or $G_2(\mathbb{C}^{m+2})$, resp.) and some fundamental formulas including the Codazzi and Gauss equations for a real hypersurface in $G_2^*(\mathbb{C}^{m+2})$ (or $G_2(\mathbb{C}^{m+2})$, resp.).

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