On the classification of minimal ruled surfaces in pseudo-Euclidean space $\mathbb{R}^n_p$

Department of Mathematics and Information Sciences,
Tokyo Metropolitan University

Yuichiro Sato

Abstract

The study of ruled surfaces has a long history. In particular, there are many results on ruled minimal surfaces. One of the most famous facts is that the only non-planar, ruled minimal surface in $\mathbb{R}^n$ is the classical helicoid. O. Kobayashi classified spacelike minimal ruled surfaces of $\mathbb{R}^3_1$ ([1]). I. van de Woestijne classified timelike minimal ruled surfaces of $\mathbb{R}^3_1$ ([2]). And, H. Anciaux classified minimal ruled surfaces of $\mathbb{R}^n_p$ ([3]). But his proof is incomplete. This paper gives a complete classification of ruled minimal surfaces in $\mathbb{R}^n_p$.

Let $I \subset \mathbb{R}$ be an open interval including $0 \in \mathbb{R}$. Assume that $\gamma : I \to \mathbb{R}^n \setminus \{0\}$ is a C^∞-curve and $x : I \to \mathbb{R}^n$ is a C^∞-regular curve. Then, we define a mapping f by the following

$$f : I \times \mathbb{R} \ni (s,t) \longrightarrow \gamma(s)t + x(s) \in \mathbb{R}^n.$$

From now on we assume that f is an immersion. The image S of this mapping f

$$S := \{\gamma(s)t + x(s) \in \mathbb{R}^n \mid (s,t) \in I \times \mathbb{R}\}$$

is called a **ruled surface** in $\mathbb{R}^n$. Moreover we define the curve γ as a **direction curve** on S, and the curve x as a **base curve** on S. In particular, if the direction curve is parallel i.e. $\gamma(s) = \gamma_0$: constant, then we say that a given ruled surface is a **cylinder**. As the ambient space, we consider pseudo-Euclidean space $\mathbb{R}^n_p = (\mathbb{R}^n, \langle \cdot , \cdot \rangle_p := -\sum_{i=1}^p dx_i^2 + \sum_{i=p+1}^n dx_i^2)$. A C^∞-curve c in $\mathbb{R}^n_p$ is called a **null curve** if for any $s \in I$, it holds the condition $|c'(s)|_p^2 = 0$. Finally, a ruled surface S in $\mathbb{R}^n_p$ is **minimal** if the induced metric g on S is non-degenerate, and the mean curvature vector field $\vec{H}$ of S is identically vanishing

$$\text{i.e. } \det g = g_{11}g_{22} - g_{12}^2 \neq 0, \quad \vec{H} = \frac{1}{2}\frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{\det g} = 0.$$

From now on, we use the following notation:
O.S. : orthogonal system, O.N.S. : orthonormal system.

Theorem 1. *Let S be a non-planar, minimal ruled surface of pseudo-Euclidean space $\mathbb{R}^n_p$. Then, S is, up to isometry and scaling, locally congruent to an open subset of the following surfaces:*

1. *A minimal cylinder*
   $f(s,t) = \gamma_0 t + x(s)$,
   where γ_0 : null vector, $x(s)$: null curve s.t. $\langle \gamma_0, x'(s)\rangle_p \neq 0$

2. *An elliptic helicoid of the 1st kind*
   $f(s,t) = (\cos s e_1 + \sin s e_2)t + s e_3$,
   where $\{e_1, e_2, e_3\}$: O.N.S., $|e_1|_p^2 = |e_2|_p^2 = \pm 1, |e_3|_p^2 = \pm 1$

3. An elliptic helicoid of the 2nd kind
$f(s,t) = (\cos se_1 + \sin se_2)t + se_3,$
where $\{e_1, e_2, e_3\}$: *O.S.*, $|e_1|_p^2 = |e_2|_p^2 = \pm 1, |e_3|_p^2 = 0$

4. A hyperbolic helicoid of the 1st kind
$f(s,t) = (\cosh se_1 + \sinh se_2)t + se_3,$
where $\{e_1, e_2, e_3\}$: *O.N.S.*, $|e_1|_p^2 = -|e_2|_p^2 = \pm 1, |e_3|_p^2 = \pm 1$

5. A hyperbolic helicoid of the 2nd kind
$f(s,t) = (\cosh se_1 + \sinh se_2)t + se_3,$
where $\{e_1, e_2, e_3\}$: *O.S.*, $|e_1|_p^2 = -|e_2|_p^2 = \pm 1, |e_3|_p^2 = 0)$

6. A parabolic helicoid
$$f(s,t) = (t + s^2)e_1 + (\frac{s^3}{3} + st - s)e_2 + (\frac{s^3}{3} + st + s)e_3,$$
where $\{e_1, e_2, e_3\}$: *O.N.S.*, $|e_1|_p^2 = |e_2|_p^2 = -|e_3|_p^2 = \pm 1$

7. A minimal hyperbolic paraboloid
$f(s,t) = ste_1 + te_2 + se_3,$
where $\{e_1, e_2, e_3\}$: *O.S.*, $|e_1|_p^2 = 0, |e_2|_p^2 = \pm 1, |e_3|_p^2 = \pm 1$

Theorem 2. *There is no hyperbolic helicoid of the 2nd kind in Minkowski n-space* $\mathbb{R}_1^n$*. And, there is a hyperbolic helicoid of the 2nd kind but there is no elliptic helicoid of the 2nd kind in four dimensional pseudo-Euclidean space* $\mathbb{R}_2^4$ *with the neutral metric. We can summarize the existence by a table indicated below.*

	1.	*2.*	*3.*	*4.*	*5.*	*6.*	*7.*
$\mathbb{R}_0^3$	×	○	×	×	×	×	×
$\mathbb{R}_1^3$	○	○	×	○	×	○	×
$\mathbb{R}_1^4$	○	○	○	○	×	○	○
$\mathbb{R}_2^4$	○	○	×	○	○	○	○
$\mathbb{R}_1^n (n \geq 5)$	○	○	○	○	×	○	○
$\mathbb{R}_p^n (n \geq 5, 2 \leq p \leq n-2)$	○	○	○	○	○	○	○

Here the number of this table corresponds to that of the theorem 1.

References

[1] O. Kobayashi, "Maximal surfaces in the 3-dimensional Minkowski space L^3", Tokyo J. Math. 6 (1983), no. 2, 297–309.

[2] I. van de Woestijne, "Minimal surfaces of the 3-dimensional Minkowski space", in Geometry and Topology of submanifolds II., M. Boyom, J.-M. Morvan and L. Verstraelen Eds, World Scientific (1990), 344–369.

[3] H. Anciaux, "Minimal Submanifolds in Pseudo-Riemannian Geometry", World Scientific (2011).