CONTACT, PSEUDO-EINSTEIN HYPERSURFACES IN THE COMPLEX QUADRIC

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Abstract. In this survey article, first we introduce the classification of homogeneous hypersurfaces in some Hermitian symmetric spaces of rank 2. Next, by using the isometric Reeb flow, we give a complete classification for hypersurfaces M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2U_m)$, complex hyperbolic two-plane Grassmannians $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$, complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ and its dual $Q^{m*} = SO^o_{m,2}/SO_mSO_2$.

As a third, we introduce the classifications of contact hypersurfaces with constant mean curvature in the complex quadric Q^m and its noncompact dual Q^{m*} for $m \geq 3$. Finally we want to mention some classifications of real hypersurfaces in the complex quadrics Q^m with Ricci parallel, harmonic curvature, parallel normal Jacobi, pseud-Einstein, pseudo-anti commuting Ricci tensor and Ricci soliton etc.

Introduction

Let us denote by $(\bar{M}, g)$ a Riemannian manifold and $I(\bar{M}, g)$ a set of all isometries defined on $\bar{M}$. Here, a homogeneous submanifold of $(\bar{M}, g)$ is a connected submanifold M of $\bar{M}$ which is an orbit of some closed subgroup G of $I(\bar{M}, g)$. If the codimension of M is one, then M is called a *homogeneous hypersurface*. When M becomes a homogeneous hypersurface of $\bar{M}$, there exists some closed subgroup G of $I(\bar{M}, g)$ having M as an orbit. Since the codimension of M is one, the regular orbits of the action of G on $\bar{M}$ have codimension one, that is, the action of G on $\bar{M}$ is of cohomogeneity one. This means that the classification of homogeneous hypersurfaces is equivalent to the classification of cohomogeneity one actions up to orbit equivalence.

Hereafter let us note that HSSP denotes a Hermitian Symmetric Space. For HSSP with rank one we say that a complex projective space $\mathbb{C}P^m$, a complex hyperbolic space $\mathbb{C}H^m$. For HSSP of compact type with rank 2 we say $SU_{2+m}/S(U_2U_m)$, SO_8/U_4, $G_2(\mathbb{R}^{2+m})$, Sp_2/U_2 and $E_6/Spin_{10}U_1$, and for HSSP of non-compact type with rank 2 we can give $SU_{2,m}/S(U_2U_m)$, SO_8^*/U_4, $G_2^*(\mathbb{R}^{2+m})$, $Sp(2,\mathbb{R})/U_2$ and $E_6^{-14}/Spin_{10}U_1$ (See Helgason [12], [13]).

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1. Compact Hermitian Symmetric Space with rank 2

The study of real hypersurfaces in non-flat complex space forms or quaternionic space forms which belong to HSSP with rank 1 of compact type in section 1 is a classical topic in differential geometry. For instance, there have been many investigations for homogeneous hypersurfaces of type A_1, A_2, B, C, D and E in complex projective space $\mathbb{C}P^m$. They are completely classified by Cecil and Ryan [10], Kimura [19] and Takagi [57]. Here, explicitly, we mention that $A_1:$ Geodesic hyperspheres, $A_2:$ a tube around a totally geodesic complex projective spaces $\mathbb{C}P^k$, $B:$ a tube around a complex quadric Q^{m-1} and can be viewed as a tube around a real projective space $\mathbb{R}P^m$, $C:$ a tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ into $\mathbb{C}P^{2k+1}$ for some $k{\geq}2$, $D:$ a tube around the Plücker embedding into $\mathbb{C}P^9$ of the complex Grassmannian manifold $G_2(\mathbb{C}^5)$ of complex 2-planes in $\mathbb{C}^5$ and $E:$ a tube around the half spin embedding into $\mathbb{C}P^{15}$ of the Hermitian symmetric space SO_{10}/U_5.

From such a view point, we considered two natural geometric conditions for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ that the maximal complex subbundle $\mathcal{C}$ and a maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M, where the maximal complex subbundle $\mathcal{C}$ of the tangent bundle TM of M is defined by $\mathcal{C} = \{X{\in}TM | JX{\in}TM\}$, and the maximal quaternionic subbundle $\mathcal{Q}$ of TM is defined by $\mathcal{Q} = \{X{\in}TM | \mathfrak{J}X{\in}TM\}$ respectively. By using such conditions and the result in Alekseevskii [1], Berndt and Suh [3] proved the following

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the maximal complex subbundle $\mathcal{C}$ and a maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M if and only if*
(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or
(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

By using Theorem A, in a paper due to Berndt and Suh [4] we have given a complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with isometric Reeb flow as follows:

Theorem 1.1. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

2. Complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$

Now let us consider for the case that the Riemannian manifold $\bar{M}$ becomes a Riemannian symmetric spaces of non compact type with rank 1 or rank 2. As some examples of non compact type with rank 1 we say a real hyperbolic space $\mathbb{R}H^m = SO^0_{1,m}/SO_m$, a complex hyperbolic space $\mathbb{C}H^m = SU_{1,m}/S(U_1U_m)$, a quaternionic hyperbolic space $\mathbb{H}H^m = Sp_{1,m}/Sp_1Sp_m$, and a Caley projective plane $\mathbb{O}P^2 = F_4/Spin_9$. The study of homogeneous hypersurfaces in such a symmetric spaces of noncompact type with rank 1 was investigated in Berndt [5], Berndt and Tamaru [8].

Then by the argument asserted in section 1, we note that any homogeneous hypersurfaces in $SU_{2,m}/S(U_2U_m)$ becomes a tube around one singular orbit. By virtue of this fact and using geometric tools given in Helgason [12], [13], Eberlein [11] , Berndt and Suh [5] proved a characterization of homogeneous hypersurfaces in $SU_{2,m}/S(U_2U_m)$ as follows:

Theorem 2.1. *Let M be a connected real hypersurface in the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$, $m \geq 2$. Then the maximal complex subbundle $\mathcal{C}$ and a maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M if and only if M is congruent to an open part of one of the following hypersurfaces:*

(A) *a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$,*
(B) *a tube around a totally geodesic quaternionic hyperbolic space $\mathbb{H}H^n$ in $SU_{2,2}/S(U_2U_m)$, $m = 2n$,*
(C) *a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular.*

In this section we give a classification of all real hypersurfaces with isometric *Reeb flow* in complex hyperbolic two-plane Grassmann manifold $SU_{2,m}/S(U_2U_m)$ as follows (see Suh [45]):

Theorem 2.2. *Let M be a connected orientable real hypersurface in the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ or a horosphere whose center at infinity is singular.*

3. Isometric Reeb Flow in Complex Quadric $\mathbb{Q}^m$

The homogeneous quadratic equation $z_1^2 + \ldots + z_{m+2}^2 = 0$ on $\mathbb{C}^{m+2}$ defines a complex hypersurface Q^m in the $(m+1)$-dimensional complex projective space $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. The hypersurface Q^m is known as the m-dimensional complex quadric. The complex structure J on $\mathbb{C}P^{m+1}$ naturally induces a complex structure on Q^m which we will denote by J as well. We equip Q^m with the Riemannian metric g which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The 1-dimensional quadric Q^1 is isometric to the round 2-sphere S^2. For $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of rank two and its maximal sectional curvature is equal to 4. The 2-dimensional quadric Q^2 is isometric to the Riemannian product $S^2 \times S^2$.

For the complex projective space $\mathbb{C}P^m$ a full classification was obtained by Okumura in [29]. He proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_mU_1)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \ldots, m-1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_mU_2)$ the classification was obtained by Berndt and the author in [4]. We have proved that the Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. Finally, related to the isometric Reeb flow, we give a mention for our recent work due to Berndt and Suh [6]. In this lecture we want to investigate this problem for the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. In view of the previous two results a natural expectation could involve at least the totally geodesic $Q^{m-1} \subset Q^m$. But for real hypersurfaces in Q^m with isometric *Reeb flow* the situations are quite different from the above. Now we state the following.

Theorem 3.1. *(see* [6]*) Let M be a real hypersurface in the complex quadric Q^m, $m \geq 3$. Then the Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

4. Contact hypersurfaces in Complex Quadric $\mathbb{Q}^m$ and non-compact dual $\mathbb{Q}^{m^*}$

This section is a recent work due to Berndt and the author [7]. A contact manifold is a smooth $(2m-1)$-dimensional manifold M together with a one-form η satisfying $\eta \wedge (d\eta)^{m-1} \neq 0$, $m \geq 2$. The one-form η on a contact manifold is called a contact form. The kernel of η defines the so-called contact distribution $\mathcal{C}$ in the tangent bundle TM of M. Note that if η is a contact form on a smooth manifold M, then $\rho\eta$ is also a contact form on M for each smooth function ρ on M which is nonzero everywhere. The origin of contact geometry can be traced back to Hamiltonian mechanics and geometric optics. The standard example of a contact manifold is $\mathbb{R}^3$ together with the contact form $\eta = dz - y\,dx$.

Contact hypersurfaces in complex space forms of complex dimension $m \geq 3$ have been investigated and classified by Okumura [29] (for the complex Euclidean space $\mathbb{C}^m$ and the complex projective space $\mathbb{C}P^m$) and Vernon [58] (for the complex hyperbolic space $\mathbb{C}H^m$). In this paper we carry out a systematic study of contact hypersurfaces in Kähler manifolds. We will then apply our results to the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ and its noncompact dual space $Q^{m*} = SO^o_{m,2}/SO_mSO_2$ to prove the following two classifications:

Theorem 4.1. *(see* [7]*) Let M be a connected orientable real hypersurface with constant mean curvature in the complex quadric $Q^m = SO^o_{m+2}/SO_mSO_2$ and $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around a real form S^m of Q^m.*

When we consider a real hypersurface in complex hyperbolic quadric Q^{m*}, naturally we have one focal (singular) submanifold in Q^{m*}, which is different from the situation of Theorem 4.1. In this case we give a complete classification of contact real hypersurfaces in Q^{m*} as follows:

Theorem 4.2 (see [7]). *Let M be a connected orientable real hypersurface with constant mean curvature in the noncompact dual $Q^{m*} = SO^o_{m,2}/SO_mSO_2$ of the complex quadric and $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of one of the following contact hypersurfaces in Q^{n*}:*

(i) *the tube of radius $r \in \mathbb{R}_+$ around the totally geodesic $Q^{(m-1)*}$ in Q^{m*};*
(ii) *a horosphere in Q^{m*} whose center at infinity is determined by an $\mathfrak{A}$-principal geodesic in Q^{m*};*

(iii) *the tube of radius $r \in \mathbb{R}_+$ around a real form $\mathbb{R}H^m$ in Q^{m*}.*

5. Pseudo-Einstein real hypersurfaces in Complex Quadric $\mathbb{Q}^m$

A Riemannian manifold M is said to be *Einstein* if the Ricci tensor Ric is a scalar multiple of the Riemannian metric g on M, that is, $g(Ric(X), Y) = \lambda g(X, Y)$ for a smooth function λ and any vector fields X, Y tangent to M. Classically, Einstein hypersurfaces in real space forms have been studied by many differential geometers.

In complex space forms or in quaternionic space forms many differential geometers have discussed real Einstein hypersurfaces, complex Einstein hypersuraces or more generally real hypersurfaces with parallel Ricci tensor, that is $\nabla Ric = 0$, where ∇ denotes the Riemannian connection of M (see Cecil-Ryan [10], Kimura [18],[19], Romero [37], [38] and Martinez and Pérez [24]).

From such a view point Kon [21] has considered the notion of pseudo-Einstein real hypersurfaces M in complex projective space $\mathbb{C}P^m$ with Kähler structure J, which are defined in such a way that

$$Ric(X) = aX + b\eta(X)\xi,$$

where a, b are constants, $\eta(X) = g(\xi, X)$ and $\xi = -JN$ for any tangent vector field X and a unit normal vector field N defined on M. In [21] Kon has also given a complete classification of pseudo-Einstein real hypersurfaces in $\mathbb{C}P^m$ by using the work of Takagi [57] and proved that there do not exist Einstein real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$. Moreover, Kon [20] has considered a new notion of the Ricci tensor $\hat{Ric}$ in the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$

The notion of pseudo-Einstein was generalized by Cecil-Ryan [10] to any smooth functions a and b defined on M. By using the theory of tubes, Cecil-Ryan [10] have given a complete classification of such pseudo-Einstein real hypersurfaces and proved that there do not exist Einstein real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$.

On the other hand, Montiel [26] considered pseudo-Einstein real hypersurfaces in complex hyperbolic space $\mathbb{C}H^m$ and gave a complete classification of such hypersurfaces and also proved that there do not exist Einstein real hypersurfaces in $\mathbb{C}H^m$, $m \geq 3$.

For real hypersurfaces in quaternionic projective space $\mathbb{H}P^m$ the notion of pseudo Einstein was considered by Martinez and Pérez [24]. But in [32] Pérez proved that the unique Einstein real hypersurfaces in $\mathbb{H}P^m$ are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$ and $\cot^2 r = \frac{1}{2m}$.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The situation mentioned above is not so simple if we consider a real hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. This Riemannian symmetric space has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces M: That $[\xi] = \text{Span}\,\{\xi\}$ or $\mathcal{Q}^{\perp} = \text{Span}\,\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator, being $\xi = -JN$, $\xi_i = -J_iN$, $i = 1, 2, 3$, where N denotes a unit normal vector on M in $G_2(\mathbb{C}^{m+2})$ and $\{J_1, J_2, J_3\}$ a local basis of $\mathfrak{J}$.

A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be *pseudo-Einstein* if the Ricci tensor Ric of M satisfies

$$Ric(X) = aX + b\eta(X)\xi + c\sum_{i=1}^{3}\eta_i(X)\xi_i$$

for any constants a, b and c on M. In a paper due to Pérez, Suh and Watanabe [35] we have defined the notion of pseudo-Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with the assumption that b and c are *non-vanishing constants*. In this case the meaning of pseudo-Einstein is *proper pseudo-Einstein*. So in [35] we have given a complete classification of *proper Hopf* pseudo-Einstein as follows.

Theorem E. *Let M be a pseudo-Einstein Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then M is congruent to*

(a) a tube of radius r, $\cot^2\sqrt{2}r = \frac{m-1}{2}$, over $G_2(\mathbb{C}^{m+1})$, where $a = 4m+8$, $b+c = -2(m+1)$, provided that $c \neq -4$.

(b) a tube of radius r, $\cot r = \frac{1+\sqrt{4m-3}}{2(m-1)}$, over $\mathbb{H}P^m$, $m = 2n$, where $a = 8n+6$, $b = -16n+10$, $c = -2$.

For the real hypersurfaces of type (a) or of type (b) in Theorem A the constants b and c of pseudo-Einstein real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ never vanish at the same time on M, that is, at least one of them is non-vanishing at any point of M. As a direct consequence of Theorem A, we have also asserted that there are no Einstein Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Now let us consider the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ which is a Kähler manifold and a kind of Hermitian symmetric space of rank 2. For real hypersurfaces M in the complex quadric Q^m we have classified the isomeric Reeb flow which is defined by $\mathcal{L}_{\xi}g = 0$, where $\mathcal{L}_{\xi}$ denotes a Lie derivative along the Reeb direction ξ. The Lie invariant $\mathcal{L}_{\xi}g = 0$ along the direction ξ is equivalent to the commuting shape operator S of M in Q^m, that is, $S\phi = \phi S$. The tensor field ϕ on M is defined by $\phi X = JX - g(JX, N)N = JX - \eta(X)N$, so that ϕX is just the tangential component of JX. The classification of isomeric Reeb flow was mainly used in [49], [54] and [50]. Moreover, in order to give a complete classification of pseudo-Einstein hypersurfaces in the complex quadric Q^m we need the classification of isometric Reeb flow in Theorem E due to Berndt and Suh [4]

6. Pseudo-anti commuting Ricci tensor and Ricci soliton in Complex Quadric $\mathbb{Q}^m$

If the Ricci tensor Ric of a real hypersurface M in Q^m satisfies

$$Ric(X) = aX + b\eta(X)\xi,$$

for constants $a, b \in R$, then M is said to be *pseudo-Einstein*.

It is known that Einstein, pseudo-Einstein real hypersurfaces in the sense of Besse [9], Kon [22], and Cecil and Ryan [10], satisfy the condition of pseudo-anti commuting. Real hypersurfaces of type (B) in $\mathbb{C}P^m$, which is characterized by $S\phi + \phi S = k\phi, k{\neq}0$ and a tube over a totally real totally geodesic real projective space $\mathbb{R}P^n$, $m = 2n$, satisfy the formula of pseudo-anti commuting (see Yano and Kon [59]). Moreover, it can be easily checked that Einstein hyepersurfacs and some special kind of pseudo Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$, and hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$, which is a tube over a totally real totall geodesic quaternionic projective space $\mathbb{H}P^n$, $m = 2n$, satisfy this formula (see Pérez, Suh and Watanabe [35], Suh [40] and [43]).

In the complex quadric Q^m, Berndt and Suh [7] classified all of *contact* hypersurfaces in Q^m, which is defined by $S\phi + \phi S = k\phi, k{\neq}0$, and have given a characterization which is a tube of radius r around an m-dimensional totally real and totally geodesic sphere S^m in Q^m. All of these hypersurfaces in Hermitian symmetric spaces also satisfy the condition of pseudo-anti commuting.

Recently, we have known that a solution of the Ricci flow equation $\frac{\partial}{\partial t}g(t) = -2Ric(g(t))$ is given by

$$\frac{1}{2}(\mathfrak{L}_V g)(X,Y) + \mathrm{Ric}(X,Y) = \rho g(X,Y),$$

where ρ is a constant and $\mathfrak{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Morgan and Tian [25]). Then the solution is said to be a *Ricci soliton* with potential vector field V and Ricci soliton constant ρ, and surprisingly, it satisfies the pseudo-anti commuting condition $S\phi + \phi S = \kappa\phi$, where $\kappa = 2\rho$ is non-zero constant.

Now at each point $z \in M$ let us consider a maximal $\mathfrak{A}$-invariant subspace

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}$$

of T_zM, $z{\in}M$. Thus for a case where the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z{\in}M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}_z^\perp = \mathrm{Span}\ [A\xi, AN]$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z{\in}M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Then in this survey article we give a complete classification for pseudo-anti commuting real hypersurfaces in the complex quadric Q^m as follows:

Theorem 6.1. *(see* [52]*) Let M be a pseudo-anti commuting Hopf real hypersurfaces in the complex quadric Q^m, $m{\geq}3$. Then M is locally congruent to one of the following:*

(i) *M is an open part of a tube of radius r, $0 < r < \frac{\pi}{2\sqrt{2}}$, around a totally real and totally geodesic m-dimensional unit sphere S^m in Q^m, with $\mathfrak{A}$-principal unit normal.*
(ii) *M is an open part of a tube of radius r, $0 < r < \frac{\pi}{2}$, $r{\neq}\frac{\pi}{4}$, around a totally geodesic k-dimensional complex projective space $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$. Here the unit normal N is $\mathfrak{A}$-isotropic.*

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