Biharmonic maps on principal G-bundles

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§**1 Prehistory: Biharmonic functions.** Recall the works on biharmonic function by Lipman Bers. For a C^∞ function $U(x_1, y_1, x_2, y_2) = U(z_1, z_2)$, let

$$\Delta_1 U := \frac{\partial^2 U}{\partial {x_1}^2} + \frac{\partial^2 U}{\partial {y_1}^2}, \text{ and } \Delta_2 U := \frac{\partial^2 U}{\partial {x_2}^2} + \frac{\partial^2 U}{\partial {y_2}^2}.$$

Then, U is biharmonic if (i) $\Delta_1 U = \Delta_2 U = 0$, and (ii)

$$\frac{\partial^2 U}{\partial x_1 \partial x_2} + \frac{\partial^2 U}{\partial y_1 y_2} = 0, \quad \frac{\partial^2 U}{\partial x_1 y_2} - \frac{\partial^2 U}{\partial x_2 y_1} = 0.$$

It holds that $\Delta^2 U = (\Delta_1 + \Delta_2)^2 U = 0$. U is doubly harmonic if (i) only. Then, we have:

Theorem 1 *(L. Bers) If $U(z_1, z_2)$ is biharmonic on $\{(z_1, z_2) \in \mathbb{C}^2 | \, |z_1| < 1, \, |z_2| < 1\}$, and if there exist $c > 0$ and a sequence $\{{r_1}^\nu, \, {r_2}^\nu\}$ such that*

(i) $0 < {r_k}^\nu < 1 \quad (\nu = 1, 2, \ldots; k = 1, 2)$,

(ii) $\lim_{\nu \to \infty} {r_k}^\nu = 1 \quad (k = 1, 2)$, *and*

(iii) $\int_0^{2\pi} \int_0^{2\pi} | \, U({r_1}^\nu \, e^{i\,\theta_1}, {r_2}^\nu \, e^{i\,\theta_2}) \, | \, d\theta_1 \, d\theta_2 \leq c < \infty.$

Then, $U(z_1, z_2)$ can be written as:

$$U(z_1, z_2) = \int_0^{2\pi} \int_0^{2\pi} Q(z_1, z_2; \theta_1, \theta_2) \, u(\theta_1, \theta_2) \, d\theta_1 d\theta_2.$$

Here, the kernel function $Q(z_1, z_2; \theta_1, \theta_2)$ is given as:

$$Q(z_1, z_2; \theta_1, \theta_2) = \frac{i \, e^{i\,\theta_2}}{4\,\pi^2} \, \frac{\partial \, G(z_1, e^{i\,\theta_2})}{\partial \mathbf{n}(e^{i\,\theta_2})} \, P(z_2, e^{i\,\theta_1}),$$

$G(z, w)$ $(z \in \mathbb{D}, \, w \in \overline{\mathbb{D}})$, the Green kernel of the unit disc $\mathbb{D} := \{z \in \mathbb{C} | \, |z| < 1\}$, $\mathbf{n}(e^{i\,\theta})$, the inward unit normal of $\mathbb{D}$ at $e^{i\,\theta} \in \partial\mathbb{D}$, $P(z, e^{i\,\theta})$, $(z, e^{i\,\theta}) \in \mathbb{D} \times \partial\mathbb{D}$ is the Poisson kernel of $\mathbb{D}$,

$$P(z, e^{i\,\theta}) = P(s\, e^{i\,t}, e^{i\,\theta}) = \frac{1}{2\pi} \, \frac{1 - s^2}{1 - 2s \, \cos(t - \theta) + s^2}.$$

§**2. Introduction of biharmonic maps.** Consider an isometric immersion $f : (M^m, g) \hookrightarrow (\mathbb{R}^k, g_0)$ and $f(x) = (f_1(x), \cdots, f_k(x))$ $(x \in M)$. Then,

$$\Delta f := (\Delta f_1, \cdots, \Delta f_k) = m \, \mathbf{H},$$

Here, $\mathbf{H} := \frac{1}{m} \sum_{i=1}^m B(e_i, e_i)$, the mean curvature vector field, and $B(X, Y) := D^0_X(f_* Y) - f_*(\nabla_X Y)$, the second fundamental form.

Definition $f : (M^m, g) \hookrightarrow (\mathbb{R}^k, g_0)$ is minimal if $\mathbf{H} \equiv 0$.

Chen defined that f is biharmonic if $\Delta \mathbf{H} = \Delta(\Delta f) \equiv 0$.

Theorem 2 (Chen) *If $\dim M = 2$, any biharmonic surface is minimal.*

Chen's Conjecture: All biharmonic submanifolds in $(\mathbb{R}^k, g_0)$ are minimal.

For a C^∞ map $f : (M, g) \to (N, h)$, the energy functional is defined by

$$E(f) := \frac{1}{2}\int_M |df|^2\, v_g.$$

The first variation formula is:

$$\frac{d}{dt}\Big|_{t=0} E(f_t) = -\int_M \langle \tau(f), V\rangle\, v_g.$$

Here, $V_x = \frac{d}{dt}|_{t=0} f_t(x) \in T_{f(x)}N$, $(x \in M)$, and

$$\tau(f) = \sum_{i=1}^{m} B(f)(e_i, e_i), \quad B(f)(X, Y) = \nabla^N_{df(X)} df(Y) - df(\nabla_X Y), X, Y \in \mathfrak{X}(M).$$

$f : (M, g) \to (N, h)$ is harmonic if $\tau(f) = 0$. The second variation formula for the energy functional $E(\bullet)$ for a harmonic map $f : (M, g) \to (N, h)$ is:

$$\frac{d^2}{dt^2}\Big|_{t=0} E(f_t) = \int_M \langle J(V), V\rangle v_g,$$

where

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \quad \overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V, \ \ \mathcal{R}(V) := \sum_{i=1}^{m} R^N(V, df(e_i))df(e_i).$$

The k-energy functional due to Eells-Lemaire is

$$E_k(f) := \frac{1}{2}\int_M |(d+\delta)^k f|^2 v_g \ \ (k = 1, 2, \cdots).$$

Then, $E_1(f) = \frac{1}{2}\int_M |df|^2\, v_g$, $E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2\, v_g$. The first variation for $E_2(f)$ (G.Y. Jiang, Chin. Ann. Math. **7A** ('86), Note di Mat. **28** ('09), 209–232) is:

$$\frac{d}{dt}\Big|_{t=0} E_2(f_t) = -\int_M \langle \tau_2(f), V\rangle v_g,$$

$$\tau_2(f) := J(\tau(f)) = \overline{\Delta}\tau(f) - \mathcal{R}(\tau(f)).$$

A C^∞ map $f : (M, g) \to (N, h)$ is biharmonic if $\tau_2(f) = 0$. The second variation formula for $E_2(f)$ is given by

$$\frac{d^2}{dt^2}\Big|_{t=0} E_2(f_t) = \int_M \langle J_2(V), V\rangle v_g, \quad J_2(V) = J(J(V)) - \mathcal{R}_2(V),$$

$$\begin{aligned}\mathcal{R}_2(V) = {} & R^N(\tau(f), V)\tau(f) + 2\,\mathrm{tr} R^N(df(\cdot), \tau(f))\overline{\nabla}.V + 2\,\mathrm{tr} R^N(df(\cdot), V)\overline{\nabla}.\tau(f) \\ & + \mathrm{tr}(\nabla^N_{df(\cdot)} R^N)(df(\cdot), \tau(f))V + \mathrm{tr}(\nabla_{\tau(f)} R^N)(df(\cdot), V)df(\cdot).\end{aligned}$$

Theorem 3 (cf. [NUG]) *Let $f : (M, g) \to (N, h)$ be a biharmonic map of a complete Riemannian manifold (M, g) into another Riemannian manifold (N, h) of non-positive sectional curvature, with $E(f) = \frac{1}{2}\int_M |df|^2\, v_g < \infty$, and $E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2\, v_g < \infty$. Then, $f : (M, g) \to (N, h)$ is harmonic, i.e., $\tau(f) \equiv 0$.*

§3. Problems, examples and main results.

Problem 1. *Let $\pi : (P, g) \to (M, h)$ be a principal G-bundle. If π is biharmonic, is π harmonic ?*

Theorem 4. *Let $\pi : (P, g) \to (M, h)$, a compact principal G-bundle and the Ricci tensor of (M, h) is negative definite If π is biharmonic, then it is harmonic.*

Theorem 5. *Let $\pi : (P, g) \to (M, h)$ be a principal G-bundle & the Ricci tensor of (M, h) is non-positive. Assume that (P, g) is non-compact, complete, and π has the finite energy $E(\pi) < \infty$ and the finite bienergy $E_2(\pi) < \infty$. If π is biharmonic, then it is harmonic.*

Example 1 (cf. [LOu], p. 62) The inversion in the unit sphere $\phi : \mathbb{R}^n\backslash\{o\} \ni \mathrm{x} \mapsto \frac{\mathrm{x}}{|\mathrm{x}|^2} \in \mathbb{R}^n$ is a biharmonic morphism if $n = 4$. $\tau(\phi) = -\frac{4\,\mathrm{x}}{|\mathrm{x}|^4}$.

$\phi : (M, g) \to (N, h)$ is a biharmonic morphism if $f : U \subset N \to \mathbb{R}$ with $\phi^{-1}(U) \neq \emptyset$ biharmonic fct., $f \circ \phi : \phi^{-1}(U) \subset M \to \mathbb{R}$ is biharmonic.

Example 2 (cf. [LOu], p. 70) Let take $\beta = c_2\, e^{\int f(x)\, dx}$, $f(x) = \frac{-c_1\,(1+e^{c_1 x})}{1-e^{c_1 x}}$, $c_1,\ c_2 \in \mathbb{R}^*$.

$\pi : (\mathbb{R}^2{\times}\mathbb{R}^*, dx^2{+}dy^2{+}\beta^2(x)\,dt^2) \ni (x, y, t) \mapsto (x, y) \in (\mathbb{R}^2, dx^2{+}dy^2)$ gives a family of proper biharmonic (i.e., biharmonic but not harmonic) Riemannian submersions.

(Proof of Theorem 4) Let $P = P(M, G)$, a principal bundle. A compact Lie group G acts on P by $(G, P) \ni (a, u) \mapsto u \cdot a \in P$. The vertical subspace $G_u := \{A^*{}_u |\, A \in \mathfrak{g}\} \subset T_uP$, $\forall\ A \in \mathfrak{g}$, the fund. vector field $A^* \in \mathfrak{X}(P)$ def. by $A^*{}_u := \left.\frac{d}{dt}\right|_{t=0} u\exp(t\,A) \in T_uP.$

Assume a Riemannian metric g on P satisfies $R_a{}^*g = g$ for all $a \in G$. Then, we have

(a) $T_uP = G_u \oplus H_u$ (orthonormal decomposition.)

(b) $G_u = \{A^*{}_u |\, A \in \mathfrak{g}\}$, and

(c) $R_{a*}H_u = H_{u\cdot a}, \qquad a \in G,\ u \in P.$

Here $H_u \subset T_uP$ is the horizontal subspace.

The adapted Riemannian metri) is a Riemannian metric g on the total space P of a principal G-bundle $\pi : P \to M$,

$$g = \pi^*h + \langle\omega(\cdot), \omega(\cdot)\rangle,$$

where ω is a $\mathfrak{g}$-valued 1-form on P called a connection form, and $\langle\cdot, \cdot\rangle$ is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ satisfying that

$$\omega(A^*) = A, \qquad A \in \mathfrak{g},$$

$$R_a{}^*\omega = \mathrm{Ad}(a^{-1})\,\omega, \qquad a \in G.$$

Then, we have

$$g(X_u, Y_u) = h(\pi_*W_u, \pi_*Z_u) + \langle\, A,\ B\,\rangle,$$

for $X_u = A^*{}_u + W_u, Y_u = B^*{}_u + Z_u$, $(A, B \in \mathfrak{g}, W_u,\ Z_u \in H_u)$.

Assume that the projection $\pi : (P, g) \to (M, h)$ is biharmonic, $J(\tau(\pi)) \equiv 0$, where

$$\tau(\pi) := \sum_i \{\nabla^h_{e_i}\pi_*e_i - \pi_*(\nabla_{e_i}e_i)\}, \quad JV := \overline{\Delta}V - \mathcal{R}(V),$$

$$\overline{\Delta}V := -\sum_i \{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}V) - \overline{\nabla}_{\nabla_{e_i}e_i}V\}, \quad \mathcal{R}(V) := \sum_i R^h(V, \pi_*e_i)\pi_*e_i,$$

for $V \in \Gamma(\pi^{-1}TN)$. Here, $\{e_i\}$ is a locally defined orthonormal frame field on (P, g). Since $J(\tau(\pi)) = 0$,

$$\int_M \langle J(\tau(\pi)), \tau(\pi)\rangle\, v_g = \int_M \langle\overline{\nabla}^*\,\overline{\nabla}\,\tau(\pi), \tau(\pi)\rangle\, v_g - \int_M \sum_i \langle R^h(\tau(\pi), \pi_*e_i)\pi_*e_i, \tau(\pi)\rangle\, v_g$$

vanishes. Therefore, $\int_M \langle\overline{\nabla}\tau(\pi), \overline{\nabla}\tau(\pi)\rangle\, v_g$ is equal to

$$\int_M \sum_i R^h(\tau(\pi), e'_i)e'_i, \tau(\pi)\rangle\, v_g = \int_M \langle\rho^h(\tau(\pi)), \tau(\pi)\rangle\, v_g = \int_M \mathrm{Ric}^h(\tau(\pi))\, v_g,$$

where $\{e'_i\}$, a local orthonormal frame field and ρ^h is the Ricci tensor, $\mathrm{Ric}^h(X)$, $X \in TM$, is the Ricci curvature of (M, h). By the assumption that the Ricci curvature of (M, h) is negative definite, $\mathrm{Ric}^h(\tau(\pi)) \leq 0$, so that the right hand side is non-positive.

Since the left hand side of the above is non-negative, so that the both hand sides must vanish. Then, we have

$$\mathrm{Ric}^h(\tau(\pi)) \equiv 0 \quad \text{and} \quad \overline{\nabla}\,\tau(\pi) \equiv 0.$$

Let us define $\alpha \in A^1(M)$ by

$$\alpha(Y)(x) = \langle \tau(\pi)(u), Y_x \rangle, \ Y \in \mathfrak{X}(M),$$

$u \in P, x = \pi(u) \in M$. Then, for $Y,\ Z \in \mathfrak{X}(M)$,

$$\begin{aligned}(\nabla^h_Z \alpha)(Y) &= Z(\alpha(Y)) - \alpha(\nabla^h_Z Y) = Z\,\langle \tau(\pi), Y\rangle - \langle \tau(\pi), \nabla^h_Z Y\rangle \\ &= \langle \overline{\nabla}_Z \tau(\pi), Y\rangle + \langle \tau(\pi), \nabla^h_Z Y\rangle - \langle \tau(\pi), \nabla^h_Z Y\rangle = 0.\end{aligned}$$

Therefore, α is a parallel 1-form on (M, h). Our assumption is that the Ricci tensor of (M, h) is negative definite. Then, due to Bochner's theorem, α must vanish.

Bochner's theorem: *Let M be a compact Riemannian manifold with negative Ricci tensor. Then, it is well known that the following are equivalent:*

(i) there is no non-zero Killing vector field,

(ii) there is no non-zero parallel vector field,

(iii) there is no non-zero parallel 1-form on M.

Thus, X is a Killing vector field. i.e., $\tau(\pi) \equiv 0$, $\pi : (P, g) \to (M, h)$ is harmonic. Therefore, we obtain Theorem 4. □

§4 Principal G-bundles, proof of Theorem 5.

(The first step) Take a cut off function η on the total space (P, g) for a fixed point $p_0 \in P$ as follows:

$$0 \leq \eta \leq 1 \ (\text{on } P), \ \ \eta = 1 \ (\text{on } \ B_r(p_0) = \{p : \ d(p, p_0) < r\}),$$

$$\eta = 0 \ (\text{outside } \ B_{2r}(p_0)), \quad |\nabla \eta| \leq \frac{2}{r} \ (\text{on } \ P).$$

Let $\pi : (P, g) \to (M, h)$ be biharmonic. Then,

$$0 = J_2(\pi) = J_\pi(\tau(\pi)) = \overline{\Delta}\tau(\pi) - \textstyle\sum_{i=1}^p R^h(\tau(\pi), \pi_* e_i)\pi_* e_i. \tag{1}$$

Here, $\{e_i\}_{i=1}^p$ is a locally defined orthonormal frame field on (P, g) $(\dim P = p)$, and $\overline{\Delta}$ is the rough Laplacian: $\overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V = - \sum_i \{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V - \overline{\nabla}_{\nabla_{e_i} e_i} V\}$, $(V \in \Gamma(\pi^{-1}TM))$.

(The second step) By (1), we have

$$\int_P \langle \overline{\nabla}^* \overline{\nabla}\, \tau(\pi), \eta^2\, \tau(\pi)\rangle\, v_g = \int_P \eta^2 \langle \sum_i R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi)\rangle\, v_g. \tag{2}$$

Then, the right hand side of (2) is equal to

$$\begin{aligned}\int_P \eta^2 \sum_{i=1}^p \Big\langle R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi)\Big\rangle\, v_g &= \int_P \eta^2 \sum_{i=1}^m \Big\langle R^h(\tau(\pi), e'_i)e'_i, \tau(\pi)\Big\rangle\, v_g \\ &= \int_P \eta^2\, \mathrm{Ric}^h(\tau(\pi))\, v_g.\end{aligned}$$

Here, $\{e'_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, h), $\mathrm{Ric}^h(u)$ $(u \in TM)$ is the Ricci curvature of (M, h) which is non-positive by our assumption. Therefore, the left hand side of the above is non-positive.

(The third step) Then, we have

$$0 \geq \int_P \left\langle \overline{\nabla}^* \, \overline{\nabla} \, \tau(\pi), \eta^2 \, \tau(\pi) \right\rangle \, v_g = \int_P \left\langle \overline{\nabla} \, \tau(\pi), \overline{\nabla}(\eta^2 \, \tau(\pi)) \right\rangle \, v_g$$
$$= \int_P \sum_i \left\{ \eta^2 \left| \overline{\nabla}_{e_i} \tau(\pi) \right|^2 + e_i(\eta^2) \langle \overline{\nabla}_{e_i} \tau(\pi), \tau(\pi) \rangle \right\} v_g.$$

Here, the second term in the integrand in the above is $2\langle \eta \, \overline{\nabla}_{e_i} \tau(\pi), e_i(\eta) \, \tau(\pi) \rangle$.
Then, we have

$$\int_P \eta^2 \sum_{i=1}^p \left| \overline{\nabla}_{e_i} \tau(\pi) \right|^2 \leq -2 \int_P \sum_{i=1}^p \langle \eta \, \overline{\nabla}_{e_i} \tau(\pi), e_i(\eta) \, \tau(\pi) \rangle \, v_g = -2 \int_P \sum_{i=1}^p \langle V_i, W_i \rangle \, v_g.$$

Here, $V_i := \eta \, \overline{\nabla}_{e_i} \tau(\pi), W_i = e_i(\eta) \, \tau(\pi) \; (i = 1, \ldots, p)$.

$$0 \leq | \sqrt{\epsilon} \, V_i \pm \frac{1}{\sqrt{\epsilon}} \, W_i|^2 = \epsilon \, |V_i|^2 \pm \, 2 \, \langle V_i, W_i \rangle + \frac{1}{\epsilon} \, |W_i|^2,$$

$$\therefore \quad \mp 2 \, \langle V_i, W_i \rangle \leq \epsilon \, |V_i|^2 + \tfrac{1}{\epsilon} \, |W_i|^2. \qquad (\#)$$

Substituting (#) into the RHS of the above , and putting $\epsilon = \frac{1}{2}$,

$$\int_P \eta^2 \sum_{i=1}^p \left| \overline{\nabla}_{e_i} \tau(\pi) \right|^2 v_g \leq -2 \int_P \sum_{i=1}^p \langle V_i, W_i \rangle \, v_g$$
$$\leq \frac{1}{2} \int_P \sum_{i=1}^p \eta^2 \, |\overline{\nabla}_{e_i} \tau(\pi)|^2 \, v_g + 2 \int_P \sum_{i=1}^p e_i(\eta)^2 \, |\tau(\pi)|^2 \, v_g.$$

Therefore, we have

$$\int_P \eta^2 \sum_{i=1}^p \left| \overline{\nabla}_{e_i} \tau(\pi) \right|^2 v_g \leq 4 \int_P \sum_{i=1}^p |\, \nabla \eta|^2 \, |\tau(\pi)|^2 \, v_g \leq \frac{16}{r^2} \int_P |\tau(\pi)|^2 \, v_g. \qquad (\#\#)$$

(The fourth step) Tending $r \to \infty$ in (##), by completeness of (P, g) and

$$E_2(\pi) = \frac{1}{2} \int_P |\tau(\pi)|^2 \, v_g < \infty,$$

we have $\int_P \sum_{i=1}^p |\overline{\nabla}_{e_i} \, \tau(\pi)|^2 \, v_g = 0.$ We obtain $\overline{\nabla}_X \, \tau(\pi) = 0 \qquad (\forall \, X \in \mathfrak{X}(P)).$

Thus, $c = |\tau(\pi)|$ is constant $(\because)$ $X \, |\tau(\pi)|^2 = 2 \, \langle \overline{\nabla}_X \tau(\pi), \tau(\pi) \rangle = 0 \; (\forall \, X \in \mathfrak{X}(P))$.
In the case Vol$(P, g) = \infty$ and $E_2(\pi) < \infty$, we have $c = 0$.
$(\because)$ If $c \neq 0$, then $E_2(\pi) = \frac{1}{2} \int_P |\tau(\pi)|^2 \, v_g = \frac{c}{2}$ Vol$(P, g) = \infty$ which is a contradiction.
Thus, if Vol$(P, g) = \infty$, we have $c = 0,$ i.e., $\pi : \; (P, g) \to (M, h)$ is harmonic.
(The fifth step) In the case that $E(\pi) < \infty$ and $E_2(\pi) < \infty$, let us define a 1-form $\alpha \in A^1(P)$ by $\alpha(X) := \langle d\pi(X), \tau(\pi) \rangle, \quad (X \in \mathfrak{X}(P)).$ Then, we have

$$\int_P |\alpha| \, v_g = \int_P \Big(\sum_i |\alpha(e_i)|^2 \Big)^{1/2} v_g \leq \int_P |d\pi| \, |\tau(\pi)| \, v_g$$
$$\leq \Big(\int_P |d\pi|^2 \, v_g \Big)^{1/2} \Big(\int_P |\tau(\pi)|^2 \, v_g \Big)^{1/2} = 2 \, \sqrt{E(\pi) \, E_2(\pi)}.$$

For $\delta\alpha = -\sum_{i=1}^{p}(\nabla_{e_i}\alpha)(e_i) \in C^\infty(P)$, we have

$$
\begin{aligned}
-\delta\alpha &= \sum_i(\nabla_{e_i}\alpha)(e_i) = \sum_i\{e_i(\alpha(e_i)) - \alpha(\nabla_{e_i}e_i)\}\\
&= \sum_i\{e_i\,\langle d\pi(e_i), \tau(\pi)\rangle - \langle d\pi(\nabla_{e_i}e_i), \tau(\pi)\rangle\}\\
&= \langle\sum_i\{\overline{\nabla}_{e_i}d\pi(e_i) - d\pi(\nabla_{e_i}e_i)\}, \tau(\pi)\rangle + \sum_i\langle d\pi(e_i), \overline{\nabla}_{e_i}\tau(\pi)\rangle\\
&= \langle\tau(\pi), \tau(\pi)\rangle + \langle d\pi, \overline{\nabla}\tau(\pi)\rangle = |\tau(\pi)|^2
\end{aligned}
$$

since $\overline{\nabla}\tau(\pi) = 0$. By the above, we have

$$\int_P |\delta\alpha|\, v_g = \int_P |\tau(\pi)|^2\, v_g = 2\,E_2(\pi) < \infty.$$

By the completeness of (P, g), we can apply Gaffney's theorem,

$$0 = \int_P(-\delta\alpha)\, v_g = \int_P |\tau(\pi)|^2\, v_g.$$

Therefore, we obtain $\tau(\pi) = 0$, i.e., $\pi : (P, g) \to (M, h)$ is harmonic. □

§5 **Geometry of CR manifolds** Let us begin the CR formalism. I.e., an odd dimensional analogue of Kähler manifold: Let (M^{2n+1}, θ), a contact manifold of $(2n+1)$-dim., $T \in \mathfrak{X}(M)$, the characteristic vector field, $\theta(T) = 1$. $T_x(M) = H_x(M) \oplus \mathbb{R}T_x$, $(x \in M)$, and assume J is the complex str. on $H(M)$, and $J(H(M)) = H(M)$:

$J(JX) = -X$; $[X, Y] \in H(M)$ $(X, Y \in H(M))$.

Let g_θ, the Webster Riemannian metric on (M, θ), i.e., $g_\theta(X, Y) = d\theta(X, JY)$ $(X, Y \in H(M))$, $g_\theta(X, T) = 0$ $(x \in H(M))$, $g_\theta(T, T) = 1$. Then, (M, g_θ) is called a strictly pseudoconvex CR manif.

For two Riemannian manifolds (M^{2n+1}, g_θ), (N, h), and for $f \in C^\infty(M, N)$, let the pseudo energy be

$$E_b(f) = \frac{1}{2}\int_M \sum_{i=1}^{2n}(f^*h)(X_i, X_i)\, v_{g_\theta},$$

where $\{X_i\}$ is an orthonrmal frame field on $(H(M), g_\theta)$. The first variation formula is given by

$$\frac{d}{dt}\bigg|_{t=0} E_b(f_t) = -\int_M h(\tau_b(f), V)\, v_{g_\theta},$$

where $\tau_b(f) = \sum_{i=1}^{2n} B_f(X_i, X_i)$ is the pseudo tension field, and $B_f(X, Y)$ is the second fundamental form. Then the second variation formula is given as follows.

$$\frac{d^2}{dt^2}\bigg|_{t=0} E_b(f_t) = \int_M h(J_b(V), V)\, v_{g_\theta},$$

where $J_b(V) = \Delta_b V - \mathcal{R}_b(V)$, $\Delta_b V = -\sum_{i=1}^{2n}\{\overline{\nabla}_{X_i}(\overline{\nabla}_{X_i}V) - \overline{\nabla}_{\nabla_{X_i}X_i}V\}$, and $\mathcal{R}_b(V) = \sum_{i=1}^{2n} R^h(V, df(X_i))df(X_i)$. Here, $\overline{\nabla}$ is the induced connection of ∇^h, ∇ is the Tanaka-Webster connection. The pseudo bienergy is

$$E_{b,2}(f) = \frac{1}{2}\int_M h(\tau_b(f), \tau_b(f))\, v_{g_\theta}, \quad v_{g_\theta} = \theta \wedge (d\theta)^n.$$

The first variation formula of $E_{b,2}$ is

$$\frac{d}{dt}\Big|_{t=0} E_{b,2}(f_t) = -\int_M h(\tau_{b,2}(f), V)\, v_{g_\theta},$$

where $\tau_{b,2}(f)$ is the pseudo bitension field given by

$$\tau_{b,2}(f) = \Delta_b(\tau_b(f)) - \sum_{i=1}^{2n} R^h(\tau_b(f), df(X_i))df(X_i).$$

A C^∞ map $f : (M, g_\theta) \to (N, h)$ is pseudo biharmonic if $\tau_{b,2}(f) = 0$. A pseudo harmonic map is pseudo biharmonic.

The CR analogue of the generalized Chen's conjecture is: If (N, h) has non-positive curvature, then every pseudo biharmonic isometric immersion $f : (M, g_\theta) \to (N, h)$ must be pseudo harmonic.

Lemma (G.-Y. Jiang) Let $f : (M, g) \to (N, h)$ be an isometric immersion whose mean curvature vector field is parallel, i.e., $\overline{\nabla}^{\perp}_X \tau(f) = 0$ ($\forall\ X \in \mathfrak{X}(M)$). Then, we have

$$\begin{aligned}&\overline{\Delta}(\tau(f))\\ &= -\sum_{i,j} \langle \tau(f), R^h(df(e_i), df(e_j))df(e_j)\rangle\, df(e_i) + \sum_{i,j} \langle \tau(f), B_f(e_i, e_j)\rangle\, B_f(e_i, e_j).\end{aligned}$$

Recall $\tau_2(f) = \overline{\Delta}(\tau(f)) - \sum_j R^h(\tau(f), e_j)e_j$, and f is biharmonic if $\tau_2(f) = 0$. Here $\{e_i\}$ is a local orthon. frame field on (M, g).

Lemma Let $f : (M, g_\theta) \to (N, h)$, an admissible (i.e., $B_f(X, T) = 0$, $X \in H(M)$) isometric immer. whose pseudo mean curvature vector field is parallel, i.e., $\overline{\nabla}^{\perp}_X \tau_b(f) = 0$ ($\forall\ X \in H(M)$). Then, we have

$$\begin{aligned}\Delta_b(\tau_b(f)) &= -\sum_{i,j} \langle \tau_b(f), R^h(df(X_i), df(X_j))df(X_j)\rangle\, df(X_i)\\ &\quad - \sum_i \langle \tau_b(f), R^h(df(X_i), df(T))df(T)\rangle\, df(X_i)\\ &\quad + \sum_{i,j} \langle \tau(f), B_f(X_i, X_j)\rangle\, B_f(X_i, X_j).\end{aligned}$$

Here, recall $\tau_{b,2}(f) = \Delta_b(\tau_b(f)) - \sum_j R^h(\tau_b(f), X_j)X_j$, and f is biharmonic if $\tau_{b,2}(f) = 0$. Here, $\{X_i\}$ is a local orthonormal frame field on $H(M)$, T is the characteristic vector field of a strictly p.convex CR manifold (M, g_θ).

Theorem 6 *Let f be an isometric immersion of a CR manifold (M^{2n+1}, g_θ) into $S^{2n+2}(1)$, and $\overline{\nabla}^{\perp}_X \tau_b(f) = 0$ ($\forall\ X \in H(M)$) not harmonic. Then, f is pseudo biharmonic iff $|B_f|_{H(M)\times H(M)}|^2 = 2n$.*

Theorem 7 *Let f be an isom. immer. of a CR manifold (M^{2n+1}, g_θ) into the complex projective space $(\mathbb{P}^{n+1}(c), h, J)$ of holo. sect. curv. $c > 0$, and $\overline{\nabla}^{\perp}_X \tau_b(f) = 0$ ($\forall\ X \in H(M)$) not harmonic. Then, f is pseudo biharmonic if and only if either*

(1) $J(df(T)) \in df(TM)$ & $|B_f|_{H(M)\times H(M)}|^2 = \frac{(2n+3)c}{4}$, or

(2) $J(df(T)) \perp f(M)$ & $|B_f|_{H(M)\times H(M)}|^2 = \frac{2nc}{4}$.

§**6 Geometry of foliated Riemannian manifolds.** Let $\mathcal{F} = \cup_{\lambda\in\Lambda} L_\lambda$ be a foliation over a Riemannian manifold (M, g). For each leaf $L = L_\lambda$ ($\lambda \in \Lambda$) of $\mathcal{F}$, Let $Q = Q_\lambda := TM/L = TM/L_\lambda$, $\pi : TM \to Q = TM/L$, the projection, $L^\perp \subset TM$, the transversal subbundle, and $\sigma : Q \to L^\perp$, the corresponding bundle isomorphism.

Let ∇^M, the Levi-Civita connection of (M, g), and ∇, the transverse Levi-Civita connection on Q. Let φ, a foliated map of $(M, g, \mathcal{F})$ into $(M', g', \mathcal{F}')$, i.e., $\forall$ leaf L of $\mathcal{F}$, $\exists$ a leaf L' of $\mathcal{F}'$, $\varphi(L) \subset L'$. $\sigma : Q \to L^\perp$, a bundle map such that $\pi \circ \sigma = \mathrm{id}$. Let $d_T\varphi := \pi' \circ d\varphi \circ \sigma$; $Q \to Q'$ be a bundle map: $Q \xrightarrow{\sigma} L^\perp \subset TM \xrightarrow{d\varphi} TM' \xrightarrow{\pi'} Q'$. Here, $Q^* \subset T^*M$, $\pi : TM \to Q = TM/L$, $\pi' : TM' \to Q' = TM'/L'$. Then, it holds that $d_T\varphi \in \Gamma(Q^* \otimes \varphi^{-1}Q')$.

(First variation) (cf. Chiang-Wolak, Jung) The transversal energy is defined by $E_{\rm tr}(\varphi) := \frac{1}{2}\int_M |d_T\varphi|^2\, v_g$. For a C^∞ foliated variation $\{\varphi_t\}$ with $\varphi_0 = \varphi$ and $\frac{d\varphi_t}{dt}|_{t=0} = V \in \varphi^{-1}Q'$,

$$\frac{d}{dt}\bigg|_{t=0} E_{\rm tr}(\varphi_t) = -\int_M \langle V, \tau_{\rm tr}(\varphi)\rangle\, v_g.$$

Here, $\tau_{\rm tr}(\varphi)$ is the transversal tension field defined by

$$\tau_{\rm tr}(\varphi) := \sum_{a=1}^q (\widetilde{\nabla}_{E_a} d_T\varphi)(E_a).$$

Here, $\widetilde{\nabla}$ is the induced connection in $Q^*\otimes\varphi^{-1}Q'$ from the Levi-Civita connection of (M', g'), and $\{E_a\}_{a=1}^q$ is a local orthonormal frame field on Q.

A C^∞ foliated map $\varphi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is said to be transversally harmonic if $\tau_{\rm tr}(\varphi) \equiv 0$.

(Second variation formula) For every transversally harmonic map $\varphi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$, let $\varphi_{s,t} : M \to M'$ be any foliated variation of φ with $V = \frac{\partial\varphi_{s,t}}{\partial s}|_{(s,t)=(0,0)}$, $W = \frac{\partial\varphi_{s,t}}{\partial t}|_{(s,t)=(0,0)}$ and $\varphi_{0,0} = \varphi$, we have

$$\frac{\partial^2}{\partial s\partial t}\bigg|_{(s,t)=(0,0)} E_{\rm tr}(\varphi_{s,t}) = \int_M \langle J_{{\rm tr},\varphi}(V), W\rangle\, v_g,$$

Here, for $V \in \Gamma(\varphi^{-1}Q')$,

$$J_{{\rm tr},\varphi}(V) := \widetilde{\nabla}^*\widetilde{\nabla}V - \widetilde{\nabla}_\tau V - {\rm trace}_Q R^{Q'}(V, d_T\varphi)d_T\varphi$$
$$= -\sum_{a=1}^q (\widetilde{\nabla}_{E_a}\widetilde{\nabla}_{E_a} - \widetilde{\nabla}_{\nabla_{E_a}E_a})V - \sum_{a=1}^q R^{Q'}(V, d_T\varphi(E_a))d_T\varphi(E_a).$$

We want the condition to have $\int_M \langle\widetilde{\nabla}_\tau V, V\rangle\, v_g = 0$. The transversal bitension field $\tau_{{\rm tr},2}(\varphi)$ of a smooth foliated map φ is defined by $\tau_{2,{\rm tr}}(\varphi) := J_{{\rm tr},\varphi}(\tau_{\rm tr}(\varphi))$. The transversal bienergy $E_{2,{\rm tr}}$ of a smooth foliated map φ is defined by $E_{2,{\rm tr}}(\varphi) := \frac{1}{2}\int_M |\tau_{\rm tr}(\varphi)|^2\, v_g$. A smooth foliated map $\varphi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is said to be transversally biharmonic if $\tau_{2,{\rm tr}}(\varphi) \equiv 0$.

§**7 Rigidity of pseudo biharmonic maps.** We want to show

Theorem 8 *Let φ be a pseudo biharmonic map of a complete strictly pseudoconvex CR manifold (M, g_θ) into another Riemannian manifold (N, h) of non-positive curvature. If $E_{b,2}(\varphi) < \infty$ and $E_b(\varphi) < \infty$, then φ is pseudo harmonic.*

(Proof of Theorem 8) The proof of Theorem 8 is divided into four steps.

(The first step): Take a cut-off function η on M as

$$0 \le \eta(x) \le 1, \eta(x) = 1 \text{ on } B_r(x_0),\ \ \eta(x) = 0 \text{ outside } B_{2r}(x_0), \text{ and } |\nabla^{g_\theta}\eta| \le \tfrac{2}{r} \text{ on } M.$$

The pseudo bitension field $\tau_{b,2}(\varphi)$ of a map $\varphi : (M, g_\theta) \to (N, h)$ is:

$\tau_{b,2}(\varphi) = \Delta_b(\tau_b(\varphi)) - \sum_{i=1}^{2n} R^h(\tau_b(\varphi), d\varphi(X_i))d\varphi(X_i)$. For a pseudo biharmonic map $\varphi : (M, g_\theta) \to (N, h)$, because of $R^N \le 0$,

$$\int_M \langle\Delta_b(\tau_b(\varphi)), \eta^2\,\tau_b(\varphi)\rangle\, v_{g_\theta} = \int_M \eta^2 \sum_{i=1}^{2n}\langle R^h(\tau_b(\varphi), d\varphi(X_i))d\varphi(X_i), \tau_b(\varphi)\rangle\, v_{g_\theta} \le 0.$$

Here, $\Delta_b = (\overline{\nabla}^H)^*\,\overline{\nabla}^H$, where $\overline{\nabla}^H_X = \overline{\nabla}_{X^H}$, and $X = X^H + g_\theta(X, T)T$ $(X^H \in H(M))$ and $\overline{\nabla}$ is the induced connection on $\Gamma(\varphi^{-1}TN)$.

(The second step) Thus, we have

$$0 \geq \int_M \langle \Delta_b(\tau_b(\varphi)), \eta^2 \tau_b(\varphi)\rangle = \int_M \langle \overline{\nabla}^H \tau_b(\varphi), \overline{\nabla}^H(\eta^2 \tau_b(\varphi))\rangle$$

$$= \int_M \sum_{i=1}^{2n} \langle \overline{\nabla}_{X_i} \tau_b(\varphi), \overline{\nabla}_{X_i}(\eta^2\, \tau_b(\varphi))\rangle$$

$$= \int_M \{\eta^2 \langle \overline{\nabla}_{X_i}\tau_b(\varphi), \overline{\nabla}_{X_i}\tau_b(\varphi)\rangle + X_i(\eta^2)\langle \overline{\nabla}_{X_i}\tau_b(\varphi), \tau_b(\varphi)\rangle\}$$

$$= \int_M \eta^2 \left|\overline{\nabla}_{X_i}\tau_b(\varphi)\right|^2 + 2\int_M \langle \eta\, \overline{\nabla}_{X_i}\tau_b(\varphi), X_i(\eta)\, \tau_b(\varphi)\rangle.$$

Thus, letting $V_i := \eta\, \overline{\nabla}_{X_i}\tau_b(\varphi)$, $W_i := X_i(\eta)\, \tau_b(\varphi)$,

$$\int_M \eta^2 \left|\overline{\nabla}_{X_i}\tau_b(\varphi)\right|^2 \leq -2\int_M \langle \eta\, \overline{\nabla}_{X_i}\tau_b(\varphi), X_i(\eta)\, \tau_b(\varphi)\rangle = -2\int_M \sum_{i=1}^{2n} \langle V_i, W_i\rangle. \qquad (\#)$$

Use Cauchy-Schwarz inequality in (#), $\pm 2\langle V_i, W_i\rangle \leq \epsilon\, |V_i|^2 + \frac{1}{\epsilon}\, |W_i|^2$ ($\forall\ \epsilon > 0$).
We have

$$(\#) \leq \epsilon \int_M \sum_{i=1}^{2n} |V_i|^2 + \frac{1}{\epsilon}\int_M \sum_{i=1}^{2n} |W_i|^2.$$

Therefore, we have, putting, $\epsilon = \frac{1}{2}$,

$$\int_M \eta^2 \sum_{i=1}^{2n} \left|\overline{\nabla}_{X_i}\tau_b(\varphi)\right|^2 \leq \frac{1}{2}\int_M \sum_i \eta^2 \left|\overline{\nabla}_{X_i}\tau_b(\varphi)\right|^2 + 2\int_M \sum_i e_i(\eta)^2\, |\tau_b(\varphi)|^2.$$

Thus, we have

$$\int_M \eta^2 \sum_i \left|\overline{\nabla}_{X_i}\tau_b(\varphi)\right|^2 \leq 4\int_M |\nabla \eta|^2\, |\tau_b(\varphi)|^2 \leq \frac{16}{r^2}\int_M |\tau_b(\varphi)|^2. \qquad (*)$$

(The third step) By completeness, we can $r \to \infty$. $E_{b,2}(\varphi) = \frac{1}{2}\int_M |\tau_b(\varphi)|^2 < \infty$ implies that the right hand side of $(*)$ goes to zero if $r \to \infty$. Therefore, we have $\int_M \sum_{i=1}^{2n} \left|\overline{\nabla}_{X_i}\tau_b(\varphi)\right|^2 = 0$. Thus, we obtain $\overline{\nabla}_X \tau_b(\varphi) = 0$ ($\forall X \in H(M)$).

(The fourth step): Assume $E_b(\varphi) < \infty$ and $E_{b,2}(\varphi) < \infty$. Define a 1-form on M by

$$\alpha(X) := \begin{cases} \langle d\varphi(X), \tau_b(\varphi)\rangle & (X \in H(M)), \\ 0 & (X = T). \end{cases}$$

Then we have

$$\mathrm{div}(\alpha) = \textstyle\sum_j (\nabla^{g_\theta}_{X_j}\alpha)(X_j) + (\nabla^{g_\theta}_T \alpha)(T) = \sum_j \{X_j(\alpha(X_j)) - \alpha(\pi_H(\nabla^{g_\theta}_{X_j} X_j))\}$$
$$= \textstyle\sum_j \{X_j(\alpha(X_j)) - \alpha(\nabla_{X_j} X_j)\} = -\delta_b \alpha. \qquad (1)$$

And also

$$-\delta_b \alpha = X_j\, \langle d\varphi(X_j), \tau_b(\varphi)\rangle - \langle d\varphi(\nabla_{X_j} X_j), \tau_b(\varphi)\rangle$$
$$= \langle \overline{\nabla}_{X_j}(d\varphi(X_j)) - d\varphi(\nabla_{X_j} X_j), \tau_b(\varphi)\rangle + \langle d\varphi(X_j), \overline{\nabla}_{X_j}\tau_b(\varphi)\rangle$$
$$= \langle \tau_b(\varphi), \tau_b(\varphi)\rangle = |\tau_b(\varphi)|^2. \qquad (2)$$

Thus, we have

$$\int_M |\mathrm{div}(\alpha)| = \int_M |\tau_b(\varphi)|^2 = 2\, E_{b,2}(\varphi) < \infty.$$

Furthermore, we have

$$\int_M |\alpha| = \int_M \Big(\sum_j \langle d\varphi(X_j), \tau_b(\varphi)\rangle^2\Big)^{1/2} \le \int_M \Big(\sum_j |d\varphi(X_j)|^2\, |\tau_b(\varphi)|^2\Big)^{1/2}$$
$$= \int_M |d_b\varphi|\, |\tau_b(\varphi)| \le 2\sqrt{E_b(\varphi)}\sqrt{E_{b,2}(\varphi)} < \infty.$$

Then, we have $\int_M |\mathrm{div}(\alpha)| < \infty$ and $\int_M |\alpha| < \infty$. By Gaffney' s theorem, and completeness of (M, g), we have $0 = \int_M \mathrm{div}(\alpha) = \int_M |\tau_b(\varphi)|^2 = 2\, E_{b,2}(\varphi)$. I.e., $\tau_b(\varphi) = 0$. Thus, φ is pseudo harmonic. □

§**8 Rigidity of transversally biharmonic maps.**

The generalized Chen's conjecture for foliated Riemannian manifolds: *For any transversally biharmonic map from a foliated Riemannian manifold into another foliated Riemannian manifold whose transversally sectional curvature is non-positive. Then, it must be transversally harmonic.*

We want to show

Theorem 9 *Let φ be a C^∞ foliated map of a foliated Riemannian manifold $(M, g, \mathcal{F})$ into a foliated Riemannian manifold $(M', g', \mathcal{F}')$ satisfying the conservation law and transversally volume preserving. Assume that (M, g) is complete and the transversal sectional curvature of $(M', g', \mathcal{F}')$ is non-positive. Then, if φ is transversally biharmonic with finite transversal energy and finite transversal 2-energy, then φ is transversally harmonic.*

Let $\varphi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$, C^∞ fol. map. Let $\alpha(X, Y)$ $(X, Y \in \Gamma(L))$, s. fundamental form of $\mathcal{F}$ $\alpha(X, Y) = \pi(\nabla^Q_X Y)$, $(X, Y \in \Gamma(L))$, where $\pi : TM \to Q$, $Q = TM/L$, and L, the tangent bundle of $\mathcal{F}$. The tension field τ of $\mathcal{F}$ is $\tau = \sum_{i,j=1}^p g^{ij}\alpha(X_i, X_j)$, $(\{X_i\}_{i=1}^p$ spanns $\Gamma(L))$. Here, $\mathcal{F}$ is transversally volume preserving if $\mathrm{div}(\tau) = 0$, φ satisfies conservation law if $\{E_a\}$ $(a = 1, \dots, q)$, a local orthonormal frame field of $\Gamma(Q)$, $\mathrm{div}_{\widetilde{\nabla}} S(\varphi)(\cdot) = \sum(\widetilde{\nabla}_{E_a} S(\varphi))(E_a, \cdot) = 0$, where $S(\varphi) = \frac{1}{2}|d_T\varphi|^2 g_Q - \varphi^* g_{Q'}$ is the transversal stress-energy.

Gaffney's Theorem *Let (M, g), a complete Riemannian manifold, and X, a C^1 vector field on M.*

(1) If $\int_M |X|\, v_g < \infty$, and $\int_M \mathrm{div}(X)\, v_g < \infty$, then, $\int_M \mathrm{div}(X)\, v_g = 0$.

(2) If $f \in C^1(M)$, and X, a C^1 vector field on M satisfy

$\mathrm{div}(X) = 0$, $\int_M Xf\, v_g < \infty$, $\int_M |f|^2\, v_g < \infty$, *and* $\int_M |X|^2\, v_g < \infty$.

Then, we have: $\int_M Xf\, v_g = 0$.

We use the following lemma:

Lemma (S. D. Jung) For every C^∞ foliated map $\varphi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$, we have $\mathrm{div}_{\widetilde{\nabla}} S(\varphi)(X) = -\langle \tau_b(\varphi), d_T\varphi(X)\rangle$, $X \in \Gamma(Q)$. In particular, if φ satisfies the conservation law, i.e., $\mathrm{div}_{\widetilde{\nabla}} S(\varphi)(\cdot) = 0$, then $\langle \tau_b(\varphi), d_T\varphi(X)\rangle = 0$ $(X \in \Gamma(Q))$.

By Gaffney's theorem, we have

Lemma If $\mathcal{F}$ satisfies the transversally volume preserving, i.e., $\mathrm{div}(\tau) = 0$, where τ is the tension field of the second fundamental form of a foliation $\mathcal{F}$. Then $\int_M \tau(f)\, v_g = 0$, $(f \in V^\infty(M))$.

(Proof of Theorem 9) The proof of Theorem 9 is divided into six steps.

(The first step) Take a cut-off function η on M as

$0 \leq \eta(x) \leq 1,\ \ \eta(x) = 1$ on $B_r(x_0),\ \ \eta(x) = 0$ outside $B_{2r}(x_0),$ and $|\nabla^g \eta| \leq \frac{2}{r}$ on $M.$

The transversal tension field $\tau_{\rm tr}(\varphi)$ satisfies that

$\tau_{2,{\rm tr}}(\varphi) := J_{{\rm tr},\varphi}(\tau_{\rm tr}(\varphi)) = \ \widetilde{\nabla}^* \, \widetilde{\nabla} \tau_{\rm tr}(\varphi) - \widetilde{\nabla}_\tau \tau_b(\varphi) - {\rm tr}_Q R^{Q'}(\tau_{\rm tr}(\varphi), d_T\varphi) d_T\varphi = 0.$

Here $\widetilde{\nabla}$ is the induced connection on $\varphi^{-1}Q' \otimes T^*M.$

(The second step) For a transversally biharmonic map $\varphi:\ (M, g) \to (N, h),\ \mathcal{F},$ transv. volume preserv., ${\bf div}(\tau) = 0$, we have if $r \to \infty$,

$\int_M \langle \widetilde{\nabla}_\tau \tau_b(\varphi), \eta^2\, \tau_b(\varphi)\rangle \to \frac{1}{2}\int_M \tau\langle \tau_b(\varphi), \tau_b(\varphi)\rangle = 0.$

$\int_M \langle \widetilde{\nabla}^* \widetilde{\nabla}(\tau_{\rm tr}(\varphi)), \eta^2\, \tau_{\rm tr}(\varphi)\rangle\, v_g = \int_M \eta^2 \sum_{a=1}^q \langle R^{Q'}(\tau_{\rm tr}(\varphi), d_T\varphi(E_a)) d_T\varphi(E_a), \tau_{\rm tr}(\varphi)\rangle\, v_g$
≤ 0

since the transversal sectional curvature $K^{Q'}(\Pi_{\varphi,a})$ of $(M', g', \mathcal{F}')$ corresponding to each plane $\Pi_{\varphi,a}$ spanned by $\tau_{\rm tr}(\varphi)$ and $d_T\varphi(E_a)\quad (1 \leq a \leq q)$ is non-positive.

(The third step) Thus, we have

$$\begin{aligned}
0 \geq& \int_M \langle \widetilde{\nabla}^* \widetilde{\nabla}(\tau_{\rm tr}(\varphi)), \eta^2 \tau_{\rm tr}(\varphi)\rangle = \int_M \langle \widetilde{\nabla} \tau_{\rm tr}(\varphi), \widetilde{\nabla}(\eta^2 \tau_{\rm tr}(\varphi))\rangle \\
=& \int_M \sum_{a=1}^q \langle \widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi), \widetilde{\nabla}_{E_a}(\eta^2\, \tau_{\rm tr}(\varphi))\rangle \\
=& \int_M \{\eta^2\, |\widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi)|^2 + E_a(\eta^2) \langle \widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi), \tau_{\rm tr}(\varphi)\rangle\} \\
=& \int_M \eta^2 \left|\widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi)\right|^2 + 2\int_M \langle \eta\, \widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi), E_a(\eta)\, \tau_{\rm tr}(\varphi)\rangle.
\end{aligned}$$

By letting $V_a := \eta\, \widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi),\ W_a := E_a(\eta)\, \tau_{\rm tr}(\varphi),$

$$\int_M \eta^2 \left|\widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi)\right|^2 \leq -2\int_M \langle \eta\, \widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi), E_a(\eta)\, \tau_{\rm tr}(\varphi)\rangle = -2\int_M \sum_{a=1}^q \langle V_a, W_a\rangle. \quad (\#)$$

Use Cauchy-Schwarz inequality in (#): $\ \pm 2\langle V_a, W_a\rangle \leq \epsilon\, |V_a|^2 + \frac{1}{\epsilon}\, |W_a|^2 \quad (\forall\ \epsilon > 0).$
We have

$$(\#) \leq \epsilon \int_M \sum_{a=1}^q |V_a|^2 + \frac{1}{\epsilon}\int_M \sum_{a=1}^q |W_a|^2.$$

Therefore, we have, putting, $\epsilon = \frac{1}{2}$,

$$\int_M \eta^2 \sum_{a=1}^q \left|\widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi)\right|^2 \leq \frac{1}{2}\int_M \sum_a \eta^2 \left|\widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi)\right|^2 + 2\int_M \sum_a E_a(\eta)^2\, |\tau_{\rm tr}(\varphi)|^2.$$

Thus, we have

$$\int_M \eta^2 \ \sum_a \left|\widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi)\right|^2 \leq 4\int_M |\nabla \eta|^2\, |\tau_{\rm tr}(\varphi)|^2 \leq \frac{16}{r^2}\int_M |\tau_{\rm tr}(\varphi)|^2. \qquad (*)$$

(The fourth step) By completeness, we can $r \to \infty$. $E_{2,{\rm tr}}(\varphi) := \frac{1}{2}\int_M |\tau_{\rm tr}(\varphi)|^2 < \infty$ which implies that the right hand side of $(*)$ goes to zero if $r \to \infty$. Therefore, we have $\int_M \sum_{a=1}^q \left|\widetilde{\nabla}_{E_a} \tau_{\rm tr}(\varphi)\right|^2 = 0$. Thus, we have $\ \widetilde{\nabla}_X \tau_{\rm tr}(\varphi) = 0\ \ (\forall X \in Q)$.

(The fifth step): Define a 1-form α and a canonical vector field $\alpha^\#$ by $\alpha(X) := \langle d\varphi(\pi(X)), \tau_{\rm tr}(\varphi)\rangle,\quad (X \in \mathfrak{X}(M)),\ \langle \alpha^\#, Y\rangle := \alpha(Y),\quad (Y \in \mathfrak{X}(M))$. Let $\{E_i\}_{i=1}^p$ and

$\{E_a\}_{a=1}^q$ be locally defined orthonormal frame fields on leaves L and Q ($\dim L_x = p$, $\dim Q_x = q$, $x \in M$). Then, we have:

$$\mathrm{div}(\alpha^\#) = \sum_{i=1}^p g(\nabla^g_{E_i}\alpha^\#, E_i) + \sum_{a=1}^q g(\nabla^g_{E_a}\alpha^\#, E_a)$$
$$= \sum_{i=1}^p \{E_i(\alpha(E_i)) - \alpha((\nabla^g_{E_i}E_i))\} + \sum_{a=1}^q \{E_a(\alpha(E_a)) - \alpha(\nabla^g_{E_a}E_a)\} = -\delta_{\mathrm{tr}}\alpha. \qquad (1)$$

By $\widetilde{\nabla}_X\tau_{\mathrm{tr}}(\varphi) = 0$ ($\forall X \in Q$) and definition of α, we have

$$(1) = -\delta_{\mathrm{tr}}\alpha = \langle d\varphi(\pi(-\sum_{i=1}^p \nabla^g_{E_i}E_i)), \tau_{\mathrm{tr}}(\varphi)\rangle + \sum_{a=1}^q \{E_a\,\langle d\varphi(E_a), \tau_{\mathrm{tr}}(\varphi)\rangle$$
$$-\langle d\varphi(\pi(\nabla^g_{E_a}E_a)), \tau_{\mathrm{tr}}(\varphi)\rangle\}$$
$$= \langle d\varphi(\pi(-\sum_{i=1}^p \nabla^g_{E_i}E_i)), \tau_{\mathrm{tr}}(\varphi)\rangle$$
$$+ \sum_{a=1}^q \{\langle\widetilde{\nabla}_{E_a}(d\varphi(E_a)), \tau_{\mathrm{tr}}(\varphi)\rangle + \langle d\varphi(E_a), \widetilde{\nabla}_{E_a}\tau_{\mathrm{tr}}(\varphi)\rangle - \langle d\varphi(\pi(\nabla^g_{E_a}E_a)), \tau_{\mathrm{tr}}(\varphi)\rangle\}$$
$$= \langle d\varphi(\pi(-\sum_{i=1}^p \nabla^g_{E_i}E_i)) + \sum_{a=1}^q\{\widetilde{\nabla}_{E_a}(d\varphi(E_a)) - d\varphi(\pi(\nabla^g_{E_a}E_a))\}, \tau_{\mathrm{tr}}(\varphi)\rangle. \qquad (2)$$

(The sixth step): Assume $E_{\mathrm{tr}}(\varphi) < \infty$, and $E_{2,\mathrm{tr}}(\varphi) < \infty$. Since $\int_M \mathrm{div}(\alpha^\#)\, v_g = 0$, we have:

$$0 = \int_M \mathrm{div}(\alpha^\#)\, v_g = -\int_M \langle d\varphi(\pi(\sum_{i=1}^p \nabla^g_{E_i}E_i)), \tau_{\mathrm{tr}}(\varphi)\rangle\, v_g$$
$$+ \int_M \langle \sum_{a=1}^q \{\widetilde{\nabla}_{E_a}(d\varphi(E_a)) - d\varphi(\pi(\nabla^g_{E_a}E_a))\}, \tau_{\mathrm{tr}}(\varphi)\rangle\, v_g$$
$$= \int_M \langle \tau_{\mathrm{tr}}(\varphi) + d\varphi((\sum_{a=1}^q \nabla^g_{E_a}E_a)^\perp), \tau_{\mathrm{tr}}(\varphi)\rangle\, v_g. \qquad (3)$$

Because for the above last equality in (3), we used

$$\tau_{\mathrm{tr}}(\varphi) = \sum_{a=1}^q \{\widetilde{\nabla}_{E_a}(d\varphi(E_a)) - d\varphi(\nabla^g_{E_a}E_a)\}$$
$$= \sum_{a=1}^q \{\widetilde{\nabla}_{E_a}(d\varphi(E_a)) - d\varphi(\pi(\nabla^g_{E_a}E_a))\} - d\varphi((\sum_{a=1}^q \nabla^g_{E_a}E_a)^\perp).$$

Then, we have

$$(3) := \int_M \langle \tau_{\mathrm{tr}}(\varphi) + d\varphi((\sum_{a=1}^q \nabla^g_{E_a}E_a)^\perp), \tau_{\mathrm{tr}}(\varphi)\rangle\, v_g = \int_M \langle \tau_{\mathrm{tr}}(\varphi), \tau_{\mathrm{tr}}(\varphi)\rangle\, v_g. \qquad (4)$$

for $\varphi : (M, g) \to (N, h)$, satisfies the conservative law,

$$\langle d_T\varphi(X), \tau_{\mathrm{tr}}(\varphi)\rangle = 0 \qquad (X = (\sum_{a=1}^q \nabla^g_{E_a}E_a)^\perp \in \Gamma(Q)).$$

Here, $W^\perp$ is the Q-component of a vector field W on M relative to the decomposition $TM = L \oplus Q$. □

§**9. Legendrian submanifolds and Lagrangean submanifolds.** For Legendrian submanifolds and Lagrangean submanifolds let us recall:

Theorem 10 *Let M^m be an m-dimensional submanifold of a Sasakian manifold $(N^{2m+1}, h, J, \xi, \eta)$. Then, M is Legendrian in N if and only if $C(M) \subset C(N)$ is Lagrangian in a Kähler cone manifold $(C(N), \overline{h}, I)$.*

(Proof) M is Legendrian in N if and only if $h(\xi, X) = 0$ and $h(X, JY) = 0$ for all X, $Y \in \mathfrak{X}(M)$. The Kähler form of $C(N)$ is $\Omega = 2r\,dr \wedge \eta + r^2\, d\eta$ which satifies

$$\Omega(f_1\Phi + X, f_2\Phi + Y) = r^2\{h(\xi, f_1Y - f_2X) + h(X, JY)\}.$$

Thus, M is Legendrian if and only if the pullback of Ω to $C(M)$ vanishes. Namely, $C(M) \subset C(N)$ is Lagrangian. □

Theorem 11 *Let $\varphi : (M^m, g) \to N$, a Legendrian submanifold of a Sasakian manifold $(N^{2m+1}, h, J, \xi, \eta)$, and let $\overline{\varphi} : C(M) \ni (r, x) \mapsto (r, \varphi(x)) \in C(N)$, the Lagrangian submanifold of a Kähler cone manifold. Here, $\overline{g} = dr^2 + r^2 g$, $\overline{h} = dr^2 + r^2 h$. Then,*

(1) $\tau(\overline{\varphi}) = \frac{\tau(\varphi)}{r^2}$, i.e., $\overline{\varphi}$ is harmonic if and only if φ is harmonic.

(2) $\tau_2(\overline{\varphi}) := J_{\overline{\varphi}}(\tau(\overline{\varphi})) = \frac{J_\varphi(\tau(\varphi))}{r^4} + \frac{m\,\tau(\varphi)}{r^2} = \frac{\tau_2(\varphi)}{r^4} + \frac{m\,\tau(\varphi)}{r^2}$.

I.e., φ is harmonic if and only if $\overline{\varphi}$ is harmonic and φ is biharmonic if and only if $J_{\overline{\varphi}}(\tau(\overline{\varphi})) = m\,\tau(\overline{\varphi})$.

Corollary Let $\varphi : (M^m, g) \to N$ be a Legendrian submanifold of a Sasakian manifold $(N^{2m+1}, h, J, \xi, \eta)$, $\overline{\varphi} : C(M) \to C(N)$, the Lagrangian submanifold of a Kähler cone manifold. Then,

$\varphi : (M, g) \to N$ is proper biharmonic if and only is $\tau(\overline{\varphi})$ is an eigensection of $J_{\overline{\varphi}}$ with the eigenvalue m. Here, $J_{\overline{\varphi}}$ is an elliptic operator of the form:

$$J_{\overline{\varphi}}W := \Delta_{\overline{\varphi}}W - \sum_{i=1}^{m+1} R^{C(N)}(W, \overline{\varphi}_*\overline{e}_i)\overline{\varphi}_*\overline{e}_i, \; (W \in \Gamma(\overline{\varphi}^{-1}TC(N))),$$

and $R^{C(N)}$ is the curvature tensor of $(C(N), \overline{h})$.

§**10. Biharmonic maps and symplectic geometry.** Our question is as follows: What is a relation between biharmonic maps and symplectic geometry?

One can ask: "When are Lagrangian submanifolds biharmonic immersions into a symplectic manifold? "

Take as a symplectic manifold, a Kähler manifold: "When is its Lagrangian submanifold biharmonic immersion? "

Let (N, J, h) be a complex m-dimensional Kähler manifold, and consider a symplectic form on N by $\omega(X, Y) := h(X, JY)$, $X, Y \in \mathfrak{X}(N)$.

A real submanifold M in N of dimension m is called to be Lagrangian if the immersion $\varphi : M \to N$ satisfies that $\varphi^*\omega \equiv 0$, i.e.,

$$h_x(T_xM, J(T_xM)) = 0 \; (\forall\, x \in M).$$

Problem: When is $\varphi : (M, g) \to (N, J, h)$ biharmonic? Here, $g := \varphi^*h$.

Then, we have

Theorem 12 *(Maeta and Urakawa) Let (N, J, h), a Kähler manifold, and (M, g), a Lagrangian submanifold. Then, it is biharmonic if and only if*

$$\mathrm{Tr}_g(\nabla A_{\mathrm{H}}) + \mathrm{Tr}_g(A_{\nabla^\perp_\bullet \mathrm{H}}(\bullet)) - \sum \langle \mathrm{Tr}_g(\nabla^\perp_{e_i}B) - \mathrm{Tr}_g(\nabla^\perp_\bullet B)(e_i, \bullet), \mathrm{H}\rangle\, e_i = 0,$$

$$\Delta^\perp \mathrm{H} + \mathrm{Tr}_g B(A_{\mathrm{H}}(\bullet), \bullet) + \sum \mathrm{Ric}^N(J\mathrm{H}, e_i)Je_i - \sum \mathrm{Ric}(J\mathrm{H}, e_i)Je_i$$
$$- J\, \mathrm{Tr}_g A_{B(J\mathrm{H}, \bullet)}(\bullet) + m\, JA_{\mathrm{H}}(J\mathrm{H}) = 0.$$

where $m = \dim M$, and Ric, Ric^N are the Ricci tensors of (M, g), (N, h).

In particular, we have

Theorem 13 *(Maeta and Urakawa) If $(N, J, h) = N^m(4c)$, the complex space form of complex dim m, with constant holomorphic curvature $4c (< 0, \; = 0, \; > 0)$, and (M, g), a Lagrangian submanifold. Then it is biharmonic if and only if*

$$\mathrm{Tr}_g(\nabla A_{\mathrm{H}}) + \mathrm{Tr}_g(A_{\nabla^\perp_\bullet \mathrm{H}}(\bullet)) = 0, \quad \Delta^\perp \mathrm{H} + \mathrm{Tr}_g B(A_{\mathrm{H}}(\bullet), \bullet) - (m + 3)c\mathrm{H} = 0.$$

B.Y. Chen introduced the following two notions on Lagrangian submanifold M in a Kähler manifold N: H-umbilic: M is called H-umbilic if M has a local orthonormal frame field $\{e_i\}$ satisfying that

$$B(e_1, e_1) = \lambda\, Je_1, \quad B(e_1, e_i) = \mu\, Je_i, B(e_i, e_i) = \mu\, Je_1, \quad B(e_i, e_j) = 0 \; (i \neq j),$$

where $2 \leq i, j \leq m = \dim M$, B is the second f.f. of $M \hookrightarrow N$, and λ, μ are local functions on M.

PNMC: M has a parallel normalized mean curvature vector field if $\nabla^\perp(\frac{\mathrm{H}}{|\mathrm{H}|}) = 0.$

We have (cf. Maeta and Urakawa [MU])

Theorem 14 *Let $\varphi : M \to (N^m(4c), J, h)$ be a Lagrangian H-umbilic PNMC submanifold. Then, it is biharmonic iff $c = 1$ and $\varphi(M)$ is congruent to a submanifold of $P^m(4)$ given by*

$$\pi\Big(\sqrt{\frac{\mu^2}{1+\mu^2}}e^{-\frac{i}{\mu}x}, \sqrt{\frac{1}{1+\mu^2}}e^{i\mu x}y_1, \cdots, \sqrt{\frac{1}{1+\mu^2}}e^{i\mu x}y_m\Big)$$

where $x, y_i \in \mathbb{R}$ with $\sum_{i=1}^m y_i^2 = 1$. Here, $\pi : S^{2m+1} \to P^m(4)$ is the Hopf fibering, and $\mu = \pm\sqrt{\frac{m+5\pm\sqrt{m^2+6m+25}}{2m}}$, $(\lambda = (\mu^2 - 1)/\mu)$.

§11. **Bubbling phenomena of harmonic maps and biharmonic maps** For any $C > 0$, let $\mathcal{F} := \{\varphi : (M^m, g) \to (N^n, h)$ smooth harmonic $\mid \int_M |d\varphi|^m v_g \leq C\}$.

For any $C > 0$, let $\mathcal{F} := \{\varphi : (M^m, g) \to (N^n, h)$ smooth biharmonic $\mid \int_M |d\varphi|^m v_g \leq C$ & $\int_M |\tau(\varphi)|^2 v_g \leq C\}$.

Question: Are both $\mathcal{F}$ small or big? Our answer: a rather surprising: Both $\mathcal{F}$ are small!. I.e., both $\mathcal{F}$ cause bubblings, kinds of compactness.

More precisely, recall previous bubbling result of harmonic maps:

Theorem 15 *Let (M, g), (N, h) be compact Riem. manifold $\dim M \geq 3$. For any $C > 0$, let $\mathcal{F} := \{\varphi : (M^m, g) \to (N^n, h)$ smooth harmonic $\mid \int_M |d\varphi|^m v_g \leq C\}$. Then, for every $\{\varphi_i\} \in \mathcal{F}$, there exist $\mathcal{S} = \{x_1, \cdots, x_\ell\} \subset M$, and a harmonic map $\varphi_\infty : (M\backslash\mathcal{S}, g) \to (N, h)$ such that (1) $\varphi_{i_j} \to \varphi_\infty$ in the C^∞-topology on $M\backslash\mathcal{S}$ $(j \to \infty)$, (2) the Radon measures $|d\varphi_{i_j}|^m v_g$ converges to a measure given by $|d\varphi_\infty|^m v_g + \sum_{k=1}^\ell a_k\,\delta_{x_k}$ $(j \to \infty)$.*

Our bubbling of biharmonic maps with N. Nakauchi is :

Theorem 16 *(Bubbling) Let (M, g), (N, h) be compact Riem. mfds. $\dim M \geq 3$. For any $C > 0$, let $\mathcal{F} := \{\varphi : (M^m, g) \to (N^n, h)$ smooth biharmonic $\mid \int_M |d\varphi|^m v_g \leq C$ and $\int_M |\tau(\varphi)|^2 v_g \leq C\}$. Then, for every $\{\varphi_i\} \in \mathcal{F}$, there exist $\mathcal{S} = \{x_1, \cdots, x_\ell\} \subset M$,* and a biharmonic map $\varphi_\infty : (M\backslash\mathcal{S}, g) \to (N, h)$ such that (1) $\varphi_{i_j} \to \varphi_\infty$ in the C^∞-topology on $M\backslash\mathcal{S}$ $(j \to \infty)$, (2) the Radon measure $|d\varphi_{i_j}|^m v_g$ converges to a measure $|d\varphi_\infty|^m v_g + \sum_{1\leq k\leq\ell} a_k\,\delta_{x_k}$ $(j \to \infty)$.

§12. **Joint works with N. Koiso.** We state a joint work with N. Koiso and H. Urakawa: Let $\varphi : M^m \looparrowright (\mathbb{R}^{m+1}, g_0)$, a biharm. hypersurface, λ_i, the principal curvature, $(i = 1, \cdots, m)$, v_i, the unit principal curvature vector fields. Let $\tau := \sum \lambda_i$. Then, $-\frac{\tau}{2}$ is a simple principal curvature, say $\lambda_m = -\frac{\tau}{2}$. Then, we have

Theorem 17 *(Koiso-Urakawa) Let $\varphi : M^m \looparrowright (\mathbb{R}^{m+1}, g_0)$, a biharmonic hypersurface, with $\lambda_i \neq \lambda_j$ $(i \neq j)$, and $g(\nabla_{v_i} v_j, v_k) \neq 0$ $(\forall i, j, k = 1, \cdots, m-1)$, ∇, the induced connection with respect to the induced metric g. Then, M is minimal.*

Theorem 18 (Koiso-Urakawa) *Every Riemannian manifold (M, g) can be embedded as a biharmonic but not minimal hypersurface in a Riemannian manifold,*

$(M\times\mathbb{R}, \overline{g}(t) := g(t)+dt^2)$ with $g(0) = g$. Here $g(t)$ is a solution of the system of the ordinary differential equation's: $\alpha = -\frac{1}{2}\,g'(t)$, $\beta = -\frac{1}{2}\,g''(t) + \frac{1}{4}\,\mathrm{C}_{g(t)}\,(g'(t)\otimes g'(t))$. Here $g'(t)(X, Y) = \partial g(t)(X, Y)/\partial t$, and $\mathrm{C}_{g(t)}(\cdot)$, is the contraction, $\alpha(X, Y) = \overline{g}(\overline{\nabla}_X Y, N)$ $(X, Y \in \mathfrak{X}(M))$, $N = \partial/\partial t$, is the unit normal vector field along M at $t = 0$, and $\beta(X, Y) := \overline{g}(0)(\overline{R}(N, X)Y, N)$.

§13. Classification of biharmonic homogeneous submanifolds in compact symmetric spaces.

Theorem 19 *Let* (G, K_1, K_2) *be any commutative symmetric triad, i.e.,* G*, a compact simple Lie group,* G/K_i $(i = 1, 2)$*, compact symmetric space, two involutions* θ_i*,* $\theta_1\theta_2 = \theta_2\theta_1$*, and* K_2*,* K_1 *act on* G/K_1*,* G/K_2*, of cohomogeneity one, respectively.*

Then, K_2*-orbit, proper biharmonic if and only if* K_1*-orbit, proper biharmonic.*

Furthermore, we have:

Case 1: 3 cases.

· $(SO(1 + b + c), SO(1 + b) \times SO(c), SO(b + c))$,

· $(SU(4), S(U(2) \times U(2)), Sp(2))$,

· $(Sp(2), U(2), Sp(1) \times Sp(1))$.

In each case, there exists a unique proper biharmonic hypersurfaces K_2*-orbit in* G/K_1*.*

Case 2: 7 cases.

· $(\mathrm{SO}(2 + 2q),\ \mathrm{SO}(2) \times \mathrm{SO}(2q),\ \mathrm{U}(1 + q))\quad (q > 1)$,

· $(\mathrm{SU}(1 + b + c),\ \mathrm{S}(\mathrm{U}(1 + b) \times \mathrm{U}(c)), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(b + c))\quad (b \geq 0,\ c > 1)$,

· $(\mathrm{Sp}(1 + b + c),\ \mathrm{Sp}(1 + b) \times \mathrm{Sp}(c), \mathrm{Sp}(1) \times \mathrm{Sp}(b + c))\quad (b \geq 0,\ c > 1)$,

· $(\mathrm{SO}(8),\ \mathrm{U}(4),\ \mathrm{U}(4)')$,

· $(\mathrm{E}_6,\ \mathrm{SO}(10) \cdot \mathrm{U}(1),\ \mathrm{F}_4)$,

· $(\mathrm{SO}(1 + q),\ \mathrm{SO}(q),\ \mathrm{SO}(q))\quad (q > 1)$,

· $(\mathrm{F}_4,\ \mathrm{Spin}(9),\ \mathrm{Spin}(9))$.

In these cases, there exists a unique proper biharmonic hypersurface orbit of K_2*-action on* G/K_1*.*

Case 3: 8 cases.

· $(\mathrm{SO}(2c),\ \mathrm{SO}(c) \times \mathrm{SO}(c),\ \mathrm{SO}(2c - 1))\quad (c > 1)$,

· $(\mathrm{SU}(4),\ \mathrm{Sp}(2),\ \mathrm{SO}(4))$,

· $(\mathrm{SO}(6),\ \mathrm{U}(3),\ \mathrm{SO}(3) \times \mathrm{SO}(3))$,

· $(\mathrm{SU}(1 + q),\ \mathrm{SO}(1 + q),\ \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(q)))\quad (q > 1)$,

· $(\mathrm{SU}(2 + 2q),\ \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2q)),\ \mathrm{Sp}(1 + q))\quad (q > 1)$,

· $(\mathrm{Sp}(1 + q),\ \mathrm{U}(1 + q),\ \mathrm{Sp}(1) \times \mathrm{Sp}(q))\quad (q > 1)$,

· $(\mathrm{E}_6,\ \mathrm{SU}(6) \cdot \mathrm{SU}(2),\ \mathrm{F}_4)$,

· $(\mathrm{F}_4,\ \mathrm{Sp}(3) \cdot \mathrm{Sp}(1),\ \mathrm{Spin}(9))$.

In this case, for all biharmonic regular orbits of K_2*-action on* G/K_1 *(same as,* K_1*-action on* G/K_2*) is minimal.*

Theorem 20 *Assume that* (G, K_1, K_2) *is a commutative compact symmetric triad with* $\dim \mathfrak{a} = 1$*. Then, all biharmonic regular orbits for* $(K_2 \times K_1)$*-actions on* G *are classified as follows: All cases admitting regular orbits of the* $(K_2 \times K_1)$*-action on* G *which there exist two distinct proper biharmonic hypersurfaces, are one of the 15 cases in the following list.*

(1) All (G, K_1, K_2) *which have* $\exists_2$ *proper biharmonic hypersurfaces*

· $(SO(1 + b + c), SO(1 + b) \times SO(c), SO(b + c))$

· $(SU(4), Sp(2), SO(4))$ · $(SU(4), S(U(2) \times U(2)), Sp(2))$

· $(Sp(2), U(2), Sp(1) \times Sp(1))$

· $(SO(2 + 2q), SO(2) \times SO(2q), U(1 + q))\quad (q > 1)$

· $(SU(1 + b + c), S(U(1 + b) \times U(c)), S(U(1) \times U(b + c))$

· $(Sp(1 + b + c), Sp(1 + b) \times Sp(c), Sp(1) \times Sp(b + c))$

· $(SO(1 + q), SO(q), SO(q))\quad (q > 1)$

· $(SU(1 + q), SO(1 + q), S(U(1) \times U(q)))\quad (q > 52)$

· $(SU(2+2q), S(U(2)\times U(2q)), Sp(1+q))$ $(q > 1)$
· $(Sp(1+q), U(1+q), Sp(1)\times Sp(q))$ $(q = 2,\ q > 45)$
· $(E_6, SO(10)\cdot U(1), F_4)$
· $(F_4, Spin(9), Spin(9))$
· $(F_4, Sp(3)\cdot Sp(1), Spin(9))$
· $(SO(8), U(4), U(4)')$.

(2) (G, K_1, K_2), *any biharmnic regular orbit of the* $(K_2\times K_1)$*-action on* G *is harmonic* Recall the action of $K_2\times K_1$ on G is $(k_2, k_1)\cdot x := k_2 x k_1{}^{-1}$ $(k_2\in K_2,\ k_1\in K_1,\ x\in G)$.

(2-1) $(SO(6), U(3), SO(3)\times SO(3))$,
(2-2) $(SU(1+q), SO(1+q), S(U(1)\times U(q)))$ $(52\geq q > 1)$,
(2-3) $(Sp(1+q), U(1+q), Sp(1)\times Sp(q))$ $(45\geq q > 2)$,
(2-4) $(E_6, SU(6)\cdot SU(2), F_4)$.

For compact symmetric triads (G, K_1, K_2) whose K_2-action on G/K_1 is cohomogeneity two, we have :

Theorem 21 *Let* (G, K_1, K_2)*, a compact symmetric triad whose the* K_2*-action on* G/K_1 *is of cohomogeneity two. Then, all singular orbit types are divided into one of the following three cases: (Note the codimension of all such orbits of* K_2 *in* $G/K_1\geq 2$*).*

(i) *There exists a unique proper biharmonic orbit,*
(ii) *there exist two proper biharmonic orbits,*
(iii) *any biharmonic orbit is harmonic.*

Theorem 22 *All the compact symmetric triads* (G, K_1, K_2)*, the* K_2*-action on* G/K_1 *is cohomogeneity two as follows:*

(1) $\mathbf{A}_2$*:* **12** *cases* **(ii)**,
(2) $\mathbf{B}_2$*:* **6** *cases* **(ii)**,
(3) $\mathbf{C}_2$*:* **15** *cases* **(ii)**,
(4) $\mathbf{BC}_2$*:* **12** *cases* **(ii)**,
(5) $\mathbf{G}_2$*:* **4** *cases* **(ii)** *and* **2** *cases* **(iii)**,
(6) $\mathbf{I}$-$\mathbf{B}_2$*:* **2** *cases* **(i)**, **4** *cases in* **(ii)**,
(7) $\mathbf{I}$-$\mathbf{C}_2$*:* **4** *cases* **(i)** *and* **8** *cases* **(ii)**,
(8) $\mathbf{I}$-$\mathbf{C}_2$*:* **4** *cases* **(i)** *and* **8** *cases in* **(ii)**,
(9) $\mathbf{I}$-$\mathbf{BC}_2$-$\mathbf{A}_1^2$*:* **9** *cases* **(ii)**,
(10) $\mathbf{II}$-$\mathbf{BC}_2$*:* **9** *cases* **(iii)**,
(11) $\mathbf{I}$-$\mathbf{BC}_2$-$\mathbf{B}_2$*:* **4** *cases* **(ii)** *and* **5** *cases in* **(iii)**,
(12) $\mathbf{III}$-$\mathbf{A}_2$*:* **9** *cases* **(iii)**,
(13) $\mathbf{III}$-$\mathbf{B}_2$*:* **3** *cases* **(iii)**,
(14) $\mathbf{III}$-$\mathbf{C}_2$*:* **2** *cases* **(i)** *and* **7** *cases in* **(iii)**,
(15) $\mathbf{III}$-$\mathbf{BC}_2$*:* **9** *cases* **(iii)**,
(16) $\mathbf{III}$-$\mathbf{G}$*:* **2** *cases* **(iii)**.

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