THE EXISTENCE CONDITIONS OF REAL HYPERSURFACES IN COMPLEX GRASSMANNAINS OF RANK TWO

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A Hermitian symmetric space (HSS) is a Hermitian manifold which at every point is an isolated fixed point of an involutive holomorphic isometry [5]. First studied by Elie Cartan, they form a natural generalization of the notion of Riemannian symmetric space from real manifolds to complex manifolds. Briefly, HSS is symmetric space equiped with Kähler structure. In the early, 20 century, Elie Cartan [4] classified the submanifolds in Riemannian Symmetric spaces which gives remarkable properties and turned them into a main subject in differential geometry and other parts of pure mathematics. At that time, he and other geometer used to study real hypersurfaces in Hermitian symmetric spaces (HSS) of rank one (see [1], and so on). The rank of semisimple symmetric space $\bar{M} = G/K$ is the dimension of maximal Abelian subspace of $\mathfrak{m}$ in some Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of Lie algebra $\mathfrak{g}$ of $G = I^o(\bar{M})$ (see [2]). Complex projective space $\mathbb{C}P^n = SU_{n+1}/S(U_1{\cdot}U_n)$, complex hyperbolic space $\mathbb{C}H^n = SU_{n,1}/S(U_1{\cdot}U_n)$ are typical examples of HSS of rank one. In 21 century, Berndt, Suh, al. [3] generalized these theory into the case of rank 2.

The complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$ which is compact complex Grassmannians of rank 2. Another one is complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$ the set of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$ which is non-compact complex Grassmannians of rank 2. These Riemannian symmetric spaces have remarkable geometrical structures. These are the unique compact (non-compact), irreducible, Kähler, quaternionic Kähler manifold with positive (negative) scalar curvature.

Characterizing typical model spaces of real hypersurfaces in $\bar{M}$ under certain geometric conditions is one of our main interests. In this paper, we assign some meaningful geometric condition on real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ (or $G_2^*(\mathbb{C}^{m+2})$).

The following results are theorems of the presentation.

Theorem 1. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $(R_\xi\phi)S = S(R_\xi\phi)$. If the Reeb curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ, then M is locally congruent to type (A).*

Theorem 2. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $(\bar{R}_N\phi)S = S(\bar{R}_N\phi)$. If the Reeb curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ, then M is locally congruent to type (A).*

We consider the notion of GTW Reeb parallelism for the Ricci tensor S on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. The Ricci tensor S is said to be *GTW Reeb parallel* if the covariant derivative in GTW connection $\widehat{\nabla}^{(k)}$ of S along the Reeb direction vanishes, that is, $\widehat{\nabla}^{(k)}_\xi S = 0$. With this condition, we assert:

Theorem 3. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\alpha \neq 2k$. The Ricci tensor S on M is GTW Reeb parallel if and only if M is locally congruent to one of the following:*

(i) *type (A) with radius $r \neq \frac{1}{2\sqrt{2}} \cot^{-1}(\frac{k}{\sqrt{2}})$, or*

(ii) *type (B) with radius $r = \frac{1}{2} \cot^{-1}(\frac{-k}{4(2n-1)})$.*

The following results are theorems of the presentation.

We consider a condition weaker than parallel Ricci tensor for real hypersurfaces M in $SU_{2,m}/S(U_2{\cdot}U_m)$. From such a point of view, we study a restricted parallelism for the Ricci tensor S of M in $SU_{2,m}/S(U_2{\cdot}U_m)$, namely, Reeb parallel Ricci tensor. That is, the Ricci tensor S satisfies

$$(\nabla_\xi S)X = 0. \tag{*}$$

With these hypothesis, we assert:

Theorem 4. *Let M be a Hopf hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, with $\alpha = g(A\xi, \xi) \neq 0$. The Ricci tensor S of M is Reeb parallel if and only if M is locally congruent to one of the following:*

(i) *a tube over a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or*

(ii) *a horosphere in $SU_{2,m}/S(U_2{\cdot}U_m)$ where center at infinity is singular and of type $JX \in \mathfrak{J}X$.*

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