Maximal antipodal subgroups of the compact Lie group G_2 of exceptional type

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1 Maximal antipodal sets of a symmetric space

Let M be a connected compact Riemannian symmetric space. And $I(M)_0$ the identity connected component of the isometry group. For $x \in M$, the geodesic symmetry at x is denoted as s_x.

Definition (Chen-Nagano [1])

(1) An *antipodal set* in M is defined to be a subset A of M such that $s_x y = y$ for all $x, y \in A$.

(2) The *2-number* $\sharp_2 M$ of M is defined to be the supremum of cardinality $\sharp A$ of an antipodal set A in M.

(3) A *great* antipodal set A_2 in M is defined to be an antipodal set in M such that $\sharp A_2 = \sharp_2 M$.

(4) An antipodal set A in M is said to be *maximal* iff $A' = A$ for all antipodal subset A' in M such that $A' \supseteq A$.

(5) Two antipodal sets A, A' in M are said to be *congruent* iff $\alpha A = A'$ for some $\alpha \in I(M)_0$.

2 Poles and polars of a symmetric space

For $x \in M$, put $F(s_x, M) := \{y \in M \mid s_x y = y\}$. Then $F(s_x, M)\backslash\{x\} = \{o_i \mid 1 \leq i \leq a\} \cup (\cup_{j=1}^{b} M_j^+)$ as a disjoint union of some *poles* (*i.e.*, zero-dimensional connected components) $\{o_i \mid 1 \leq i \leq a\}$ and *polars* (*i.e.*, positive-dimensional connected components) M_j^+ $(1 \leq j \leq b)$ for some non-negative integers a, b, where $a = 0$ or $b = 0$ means that $\{o_i \mid 1 \leq i \leq a\}$ or $\cup_{j=1}^{b} M_j^+$ is an empty set, respectively.

Lemma 1. *For $x \in M$, if $b = 1$ and $a = 0$ or 1, then the assignment*

$$A_1 \mapsto A'_1 := \{x\} \cup \{o_i \mid 1 \leq i \leq a\} \cup A_1$$

from the set of all maximal antipodal sets in M_1^+ to that in M induces a surjection between their congruent class.

Proof. Let A be a maximal antipodal set in M containing x. Then $A_1 := A \setminus \{x, o_i \mid 1 \leq i \leq a\} \subseteq F(s_x, M) \setminus \{x, o_i \mid 1 \leq i \leq a\} = M_1^+$ as a maximal antipodal set in M_1^+ such that $A'_1 = A$. □

3 Maximal antipodal subgroups of a Lie group

Let M be a connected compact Lie group being a Riemannian symmetric space by a bi-invariant metric on M. Then any two conjugate subgroups of M are congruent in M, and vice varsa if M is a simple Lie group.

Remark (Chen-Nagano[1, Remarks 1.2, 1.3]). *Any maximal antipodal set A in M containing the unit element e is a discrete abelian subgroup of M, which is isomorphic to $(\boldsymbol{Z}_2)^t$ with $2^t < \infty$.*

4 Connected Lie group G_2 of exceptional type

Let G_2 be a connected compact simple Lie group of type G_2. And $S^2 \cdot S^2$ the quotient space $(S^2 \times S^2)/\boldsymbol{Z}_2$ of $S^2 \times S^2$ by a natural action of $\boldsymbol{Z}_2 := \{\pm(1,1)\}$ on $S^2 \times S^2$ [1, 3.8]. Then the following theorem 1 was given by Nagano without proof.

Theorem 1 (Nagano [2, p.66]). Put $M := G_2$. Then $F(s_e, G_2)\backslash\{e\} = M_1^+ \cong G_2/SO(4)$. For $o \in M_1^+$, $F(s_o, M_1^+)\backslash\{o\} = M_{1,1}^+ \cong S^2 \cdot S^2$.

Lemma 2. *Put $M := S^2 \cdot S^2 \ni [\vec{x}, \vec{y}] := \{\pm(\vec{x}, \vec{y})\}$ and $x_{\pm i} := [\vec{e}_i, \pm\vec{e}_i]$ $(i = 1, 2, 3)$ for an arbitary orthonormal frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $\boldsymbol{R}^3$. Then any maximal antipodal set in M is congruent to $A := \{x_{\pm i} \mid i = 1, 2, 3\}$.*

Proof. $F(s_{x_1}, M) \setminus \{x_1\} = \{x_{-1}\} \cup M_1^+$; $M_1^+ := (S^2 \cap \vec{e}_1^{\perp})^2 / \boldsymbol{Z}_2$. By virtue of Lemma 1 ($a = 1$), any maximal antipodal set in M is congruent to $A'_1 := \{x_{\pm 1}\} \cup A_1$ for some maximal antipodal set A_1 containing x_2 in M_1^+. Then $A_1\backslash\{x_2\} \subseteq \{x_{-2}\} \cup (S^2 \cap \vec{e}_1^{\perp} \cap \vec{e}_2^{\perp})^2/\boldsymbol{Z}_2 = \{x_{-2}, x_{\pm 3}\}$, so that $A'_1 \subseteq A$ which is antipodal. Since A'_1 is maximal, $A'_1 = A$. □

By virtue of Lemma 1, the following result is then obtained.

Theorem 2 ([6]). *Let A be the maximal antipodal set in $(S^2 \times S^2)/\boldsymbol{Z}_2$ defined in Lemma 2. Moreover, let $\varphi : (S^2 \times S^2)/\boldsymbol{Z}_2 \longrightarrow M_{1,1}^+$ be an isometry giving an isometry $(S^2 \times S^2)/\boldsymbol{Z}_2 \cong M_{1,1}^+$ mentioned in Theorem 1. Put $B := \varphi(A)$, $B' := \{o\} \cup B$ and $B'' := \{e, o\} \cup B$. Then*

(1) *Any maximal antipodal set in M_1^+ is congruent to B'; and*

(2) *Any maximal antipodal subgroup of G_2 is conjugate to B''.*

5 Explicit description of G_2

The explicit description of G_2 is given after Yokota as follows: Let $\boldsymbol{H}$ be the quaternions with the unit element 1 and the Hamilton's triple $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ with the conjugation $\bar{b} := b_0 1 - b_1 \boldsymbol{i} - b_2 \boldsymbol{j} - b_3 \boldsymbol{k}$ ($b = b_0 1 + b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k} \in \boldsymbol{H}$). By Cayley-Dickson process, the octanions are given as $\boldsymbol{O} := \boldsymbol{H} \times \boldsymbol{H}$ with the $\boldsymbol{R}$-bilinear product $xy := (mn - \bar{b}a, a\bar{n} + bm)$ for $x = (m, a)$ and $y = (n, b) \in \boldsymbol{O}$. By the octanionic conjugation $\bar{x} := (\bar{m}, -a) \in \boldsymbol{O}$, a positive-definite $\boldsymbol{R}$-bilinear inner product is defined as $(x \mid y) := (x\bar{y} + y\bar{x})/2 \in \boldsymbol{R}$. Put

$$G := \{\alpha \in GL_{\boldsymbol{R}}(\boldsymbol{O}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}$$

as the automorphism group of the $\boldsymbol{R}$-algebra $\boldsymbol{O}$. Then $\alpha 1 = 1$, $\overline{\alpha x} = \alpha \bar{x}$ and $(\alpha x \mid \alpha y) = (x \mid y)$ for $\alpha \in G$ and $x, y \in \boldsymbol{O}$. Moreover, put $\mathrm{Im}\boldsymbol{O} := \{x \in \boldsymbol{O} \mid \bar{x} = -x\} \cong \boldsymbol{R}^7$, $S^6 := \{x \in \mathrm{Im}\boldsymbol{O} \mid (x \mid x) = 1\} \ni (\boldsymbol{i}, 0)$ and $H := \{\alpha \in G \mid \alpha(\boldsymbol{i}, 0) = (\boldsymbol{i}, 0)\}$.

Proposition 1. (1) *G acts transitively on S^6 such that $H \cong SU(3)$, so that $G/H \cong S^6$. As the result, G is a connected and simply connected 14-dimensional compact Lie group.*

(2) *Take an isomorphism $f : SU(3) \longrightarrow H$ given by (1). If T^2 is a maximal torus of $SU(3)$, then $G = \cup_{\alpha \in G}\, \alpha f(T^2)\alpha^{-1}$. As the result,* $\mathrm{rank}\ G = \mathrm{rank}\ H = 2$.

Proof. (1) The first part was directly proved by Yokota [3, pp.250–251]. The last part follows from the first one.

(2) Since G is connected, $G \subseteq SO(\mathrm{Im}\boldsymbol{O}) \cong SO(7)$. Since any element of $SO(7)$ admits a fixed-point in S^6, any $\alpha \in G$ admits some $p \in S^6$ such that $\alpha p = p$. By (1), $\beta p = (\boldsymbol{i}, 0)$ for some $\beta \in G$. Then $(\beta\alpha\beta^{-1})(\boldsymbol{i}, 0) = (\boldsymbol{i}, 0)$. Hence, $\beta\alpha\beta^{-1} = f(A)$ for some $A \in SU(3)$. For some $B \in SU(3)$, $BAB^{-1} \in T^2$. Hence, $(f(B)\beta)\ \alpha\ (f(B)\beta)^{-1} \in f(T^2)$. □

Put $Sp(1) := \{q \in \boldsymbol{H} \mid |q| = 1\}$, $\psi : Sp(1) \times Sp(1) \longrightarrow GL_{\boldsymbol{R}}(\boldsymbol{O})$;

$$\psi(p, q)(m, a) := (qm\bar{q}, pa\bar{q}).$$

Moreover, put $e = \psi(1,1), \gamma := \psi(1,-1)$, $G^\gamma := \{\alpha \in G \mid \alpha\gamma = \gamma\alpha\}$. An explicit description of the polar decomposition of the automorphism group of the real split octanions was given by Yokota [4], by which the following proposition 2 was also obtained (cf. [5, 1.3.3, 1.3.4] for a precise proof).

PROPOSITION 2 (1) $\psi(Sp(1) \times Sp(1)) = G^\gamma$,

(2) ker $\psi = \{\pm(1,1)\}$, $G^\gamma \cong SO(4)$.

COROLLARY. *G is a connected, simply connected, compact, simple Lie group of type G_2 with $z(G) = \{e\}$.*

Proof. (1) $z(G) = \{e\}$ (Yokota, arXiv:0902.0431v1, Theorem 1.11.1): In fact, $z(G) \subset z(G^\gamma) = z(\psi(Sp(1) \times Sp(1))) = \{\psi(1,\pm 1)\} = \{e, \gamma\}$ and $\gamma \notin z(G)$ by dim$G^\gamma = 6 < 14 =$ dim$G$. (2) By the step (1) and Proposition 1 (2), G is semisimple of type $A_1 \oplus A_1$, A_2 or G_2 of dimension 6, 8 or 14. Hence, G is simple of type G_2 by Proposition 1 (1). □

6 Explicit description of polars in G_2

By Corollary, G is denoted also as G_2. By explicit description of polars in G_2, the results of Theorems 1 and 2 are directly examined as follows:

THEOREM 3 ([6]).
(1) $F(s_e, G)\backslash\{e\} = M_1^+ = \{g\gamma g^{-1} \mid g \in G\} \cong G_2/SO(4)$.
(2) *For $o := \gamma \in M_1^+$, $F(s_o, M_1^+)\backslash\{o\} = M_{1,1}^+$ and*

$$M_{1,1}^+ = \{\psi(p,q) \mid p^2 = q^2 = -1\} \cong (S^2 \times S^2)/\boldsymbol{Z}_2.$$

(3) *Any maximal antipodal set in $M_{1,1}^+$ is congruent to*

$$B := \{\psi(p, \pm p) \mid p = \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}.$$

(4) *Any maximal antipodal set in M_1^+ is congruent to*

$$B' := \{\psi(1,-1)\} \cup B.$$

(5) *Any maximal antipodal subgroup of G_2 is conjugate to*

$$B'' := \{\psi(1,\pm 1)\} \cup B.$$

Proof. (1) Put $T^2 := \{A = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) \mid A \in SU(3)\}$, which is a maximal torus of $SU(3)$. Then $F(s_e, T^2) = \{\mathrm{diag}(\pm 1, \pm 1, \pm 1) \in T^2\} =$

$\{e\} \cup \{A_i \mathrm{diag}(1,-1,-1)A_i^{-1} \mid i=1,2,3\}$ for some $A_i \in SU(3)$ $(i=1,2,3)$. By virtue of Proposition 1 (2), $\gamma \in F(s_e, G) \backslash \{e\} = \cup_{g \in G}\, gf(F(s_e, T^2))g^{-1} \backslash \{e\} = \cup_{g\in G}\{g\gamma g^{-1} \mid g \in G\} \cong G/G^\gamma$, which is connected since G is connected. Hence, $G_2/SO(4) \cong F(s_e, G)\backslash\{e\} = M_1^+$.

(2) $F(s_\gamma, M_1^+)\backslash\{\gamma\} = M_1^+ \cap G^\gamma \backslash \{\gamma\} = \{\psi(p,q) \mid (p^2,q^2) = \pm(1,1)\}\backslash\{e,\gamma\} = \{\psi(p,q) \mid (p^2,q^2) = -(1,1)\} \cong (S^2 \times S^2)/\boldsymbol{Z}_2$, because of $e = \psi(1,1)$, $\gamma = \psi(1,-1)$ and $\{p \in Sp(1) \mid p^2 = -1\} = \{p \in Sp(1) \mid p = -\bar{p}\} = \{p = p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k} \mid \sum_{i=1}^3 p_i^2 = 1\}$.

(3) follows from Lemma 2 because of (2). (4) (resp. (5)) follows from Lemma 1 with $a = 0$ because of (3) (resp. (4)) and (1). $\square$

Remark. (1) The result $\sharp_2(S^2 \cdot S^2) = 6$, $\sharp_2 G_2/SO(4) = 7$ *and* $\sharp_2 G_2 = 8$ of Chen-Nagano [1, Examples 3.13] is refined by Theorem 3 since B (resp. B' or B'') is a great antipodal set in $S^2 \cdot S^2$ (resp. $G_2/SO(4)$ or G_2) as unique maximal antipodal set up to congruence.

(2) Lemma 1 provides a priori or clear-sighted geometric method to Theorems 2 and 3. Posteriorly or arithmetically, Theorem 3 (5) is verified by calculations of weights of B'' on $\boldsymbol{O} = \boldsymbol{R}^8$.

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