ON CLASSIFICATION OF MINIMAL ORBITS OF THE HERMANN ACTION SATISFYING KOIKE'S CONDITIONS (JOINT WORK WITH MINORU YOSHIDA)

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Abstract. Let G be a connected compact Lie group. Let (G, K_1, θ_1) and (G, K_2, θ_2) be two compact Riemannian symmetric pairs. Then the natural left group action of K_2 on a compact Riemannian symmetric space $M = G/K_1$ is called the *Hermann action*. Suppose that G is semi-simple and $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. Assume that $\mathrm{rank}(G/K_1)$ is equal to the cohomogeneity of K_2 on $M = G/K_1$. Naoyuki Koike ([9], [10]) has provided the three conditions on orbits of the Hermann action on M and he proved that if an orbit of the Hermann action satisfies one of the three conditions, then the induced metric on the orbit is proportional to the metric induced from the Killing-Cartan form of G, and in the case when the orbit is a minimal orbit satisfying one of the three conditions, he showed a simplified formula of its Jacobi linear operator in terms of the Casimir operators of K_2 and G/K_1. Moreover he gave some examples of minimal orbits satisfying his conditions. In this note we mention our recent results on the classification of all minimal orbits of the Hermann action satisfying one of Koike's conditions (I), (II), (III) (which were slightly improved). This is a joint work with Mr. Minoru Yoshida.

Introduction

Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$ and let M be a complete Riemannian manifold. Suppose that K acts isometrically on M. For an orbit N of K on M and a point $a \in M$, define an (infinite dimensional) path space

$$\Omega(M, N; a) := \{\gamma : [0, 1] \to M \mid H^1\text{-maps}, a(0) \in N, \gamma(1) = a\}.$$

The concept of *variational completeness* for the Lie group action was introduced by Bott-Samelson ([1]). Assume that the group action of K on M is *variationally complete*. If $a \in M$ is regular, then the energy functional

$$E : \Omega(M, N; a) \ni \gamma \longmapsto \frac{1}{2}\int_0^1 \|\gamma'(t)\|^2 dt \in \mathbb{R}$$

is a perfect Morse function and a homology basis can be constructed explicitly.

Let G be a connected compact Lie group. Suppose that (G, K_1, θ_1) and (G, K_2, θ_2) be two compact Riemannian symmetric pairs. Then the natural left group action of K_2 on a compact Riemannian symmetric space $M = G/K_1$ is shown to be variationally complete first by Robert Hermann ([4]) and further it is known to be hyperpolar ([3]). This group action is called the *Hermann action*. General orbits

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of the Hermann action on a compact symmetric space are compact homogeneous submanifold with nice properties that the mean curvature vector field is parallel with respect to the normal connection and the normal connection coincides with the induced connection from the canonical connection as a reductive homogeneous space ([7]).

Suppose that G is semi-simple and $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. Assume that $\mathrm{rank}(G/K_1)$ is equal to the cohomogeneity of K_2 on $M = G/K_1$. Naoyuki Koike ([9], [10]) has provided the three conditions on an orbit of the Hermann action on M and he proved that if an orbit of the Hermann action satisfies one of the three conditions, then the induced metric on the orbit is proportional to the metric induced from the Killing-Cartan form of G, and thus it is a normal homogeneous metric. In the case when the orbit is a minimal orbit satisfying one of the three conditions, he showed a simplified formula describing its Jacobi linear operator in terms of the Casimir operators of K_2 and G/K_1. Moreover he gave some examples of orbits satisfying his conditions.

In this note we shall mention our recent results on the classification of all minimal orbits of the Hermann action satisfying one of Koike's conditions (I), (II), (III) (which were slightly improved). This is a joint work of the author and Mr. Minoru Yoshida who is my former master student at Osaka City University. The results of this note are contained in his master thesis (March, 2017).

This note is organized as follows: In Section 2, we begin with the definition of the Hermann action on compact symmetric spaces and review some nice properties of the Hermann action and its orbits. In Section 3, we recall the Lie algebraic setting associated to the Hermann action on compact symmetric spaces. In Section 4, we explain the Koike's conditions on orbits of the Hermann action and the Koike's theorems. In the final section we describe our recent results on the classification problem of minimal orbits of the Hermann actions satisfying the Koike's conditions.

We shall discuss this classification problem in detail in the forthcoming paper [15].

1. Hermann actions on compact symmetric spaces

Let G be a connected compact Lie group. Let (G, K_1, θ_1) be a Riemannian symmetric pair and (G, K_2, θ_2) be another Riemannian symmetric pair. The left group action of $K_2 \subset G$ on a compact symmetric space $M = G/K_1$ defined by

$$K_2 \times M \ni (a, bK_1) \longmapsto abK_1 \in M$$

is called the *Hermann action*. In the case when $K_1 = K_2$, the Hermann action is nothing but the *isotropy action* of $K_1 = K_2$ on $M = G/K_2$.

Fundamental propeties of the Hermann action are as follows:

Theorem 1.1 ([4]). *The Hermann action is variationally complete.*

The isometric action of a Lie group on a Riemannian manifold is called *hyperpolar* if there is a closed flat totally geodesic submanifold (*flat section*) to which any orbit meets orthogonally.

Theorem 1.2 ([3]). *The Hermann action is hyperpolar.*

We should mention the following Conlon's results.

Theorem 1.3 ([2]). *The hyperpolar action of a compact Lie group on a complete Riemannian manifold is variationally complete.*

The orbits of the Hermann action have nice properties from the viewpoint of submanifolds in Riemannian geometry.

Theorem 1.4 ([7]). *Any orbit of the Hermann action has parallel mean curvature vector field with respect to the normal connection.*

Theorem 1.5 ([7]). *The normal connection of each orbit of the Hermann action coincides with the induced connection from the canonical connection as a reductive homogeneous space.*

Compare with [8] and [11].

2. Lie algebraic setting

Define an AdG-invariant inner product of $\mathfrak{g}$ as $\langle \cdot, \cdot \rangle := -B_{\mathfrak{g}}(\cdot, \cdot)$. Here $B_{\mathfrak{g}}(\cdot, \cdot)$ denotes the Killing-Cartan form of $\mathfrak{g}$. Denote by $\mathfrak{k}_i$ and $\mathfrak{m}_i$ the eigenspaces of $d\theta_i$ with eigenvalues 1 and -1, respectively. Then we have the canonical decompositions

$$\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{m}_1 = \mathfrak{k}_2 \oplus \mathfrak{m}_2$$

as symmetric Lie algebras. By using the inner product $(\mathfrak{m}_1, \langle \cdot, \cdot \rangle)$, we define a G-invariant Riemannian metric h on $M = G/K_1$ and thus (M, h) is a Riemannian symmetric space. Let $\pi : G \to M = G/K_1$ denote the natural projection. Then the Hermann action of a symmetric group K_2 on (M, h) is isometric.

Suppose that the Hermann action satisfies the commutativity condition

$$\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1.$$

Then we have an orthogonal direct sum decomposition

$$\mathfrak{g} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{k}_2 \cap \mathfrak{m}_1) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2),$$

and its complexification

$$\mathfrak{g}^{\mathbb{C}} = ((\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2))^{\mathbb{C}} \oplus ((\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{k}_2 \cap \mathfrak{m}_1))^{\mathbb{C}},$$

Moreover choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}_1 \cap \mathfrak{m}_2$. Here note that $\mathrm{Exp}(\mathfrak{a})$ is a section of the Hermann action as a hyperpolar action ([3]).

Let

$$\begin{aligned}
&\mathrm{ad} : \mathfrak{a} \to \mathfrak{gl}(\mathfrak{g}^{\mathbb{C}}),\\
&\mathrm{ad} : \mathfrak{a} \to \mathfrak{gl}((\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2))^{\mathbb{C}},\\
&\mathrm{ad} : \mathfrak{a} \to \mathfrak{gl}((\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{k}_2 \cap \mathfrak{m}_1))^{\mathbb{C}}
\end{aligned}$$

be three Lie algebra homomorphisms from $\mathfrak{a}$.

Let

$$V = \mathfrak{g}^{\mathbb{C}}, \ \ ((\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2))^{\mathbb{C}} \ \text{ or } \ ((\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{k}_2 \cap \mathfrak{m}_1))^{\mathbb{C}}.$$

For a real linear function $\beta : \mathfrak{a} \to \mathbb{R}$, we define a complex vector subspace V_β of V by

$$V_\beta := \{X \in V \mid \mathrm{ad}(H)(X) = \sqrt{-1}\beta(H)X \ \text{ for } \forall H \in \mathfrak{a}\}.$$

For $V = \mathfrak{g}^{\mathbb{C}}$, define

$$\widetilde{\Sigma} := \{\beta : \mathfrak{a} \to \mathbb{R} \ \text{ real linear function}, \ \beta \neq 0, \ V_\beta \neq 0\}.$$

For $V = ((\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2))^{\mathbb{C}}$, define

$$\Sigma := \{\beta : \mathfrak{a} \to \mathbb{R} \ \text{ real linear function}, \ \beta \neq 0, \ V_\beta \neq 0\}.$$

For $V = ((\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{k}_2 \cap \mathfrak{m}_1))^{\mathbb{C}}$, define

$$W := \{\beta : \mathfrak{a} \to \mathbb{R} \ \text{ real linear function}, \ \beta \neq 0, \ V_\beta \neq 0\}.$$

Then we have $\widetilde{\Sigma} = \Sigma \cup W$. Moreover, for a simple root system $\widetilde{\Pi}$ of $\widetilde{\Sigma}$, we equip a lexicographic order on $\mathfrak{a}^*$ relative to a basis of $\mathfrak{a}^*$. Let $\widetilde{\Sigma}^+$ denote the set of all positive elements of $\widetilde{\Sigma}$ with respect to this linear order, and set

$$\Sigma^+ := \widetilde{\Sigma}^+ \cap \Sigma,$$
$$W^+ := \widetilde{\Sigma}^+ \cap W.$$

Define

$$\begin{aligned} P_0 := \{H \in \mathfrak{a} \ \mid \ & \beta(H) \in (0, \pi) \text{ for } \forall \beta \in \Sigma^+, \\ & \beta(H) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ for } \forall \lambda \in W^+\}. \end{aligned}$$

Theorem 2.1 (See [5])**.** *For every orbit N of the Hermann action on M, there exists a unique element $Z_0 \in \overline{P_0}$ (up to the Weyl group action) such that $N = K_2(\mathrm{Exp}(Z_0))$.*

3. Kioke's conditions and theorems

Suppose that $Z_0 \in \overline{P_0}$, $g_0 := \exp Z_0$, $M = K_2 g_0 K_1 = K_2(\mathrm{Exp}(Z_0))$. In [9], [10] Naoyuki Koike (Tokyo U. of Sci.) introduced Conditions (I), (II) and (III) on $Z_0 \in \overline{P_0}$ as follows:

Condition (I)

$$\Sigma^+ \cap W^+ = \emptyset$$
$$\{\beta(Z_0) \mid \beta \in \Sigma^+\} \subset \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\},$$
$$\{\beta(Z_0) \mid \beta \in W^+\} \subset \left\{\pm\frac{\pi}{2}, \pm\frac{\pi}{6}\right\}.$$

Condition (II)

$$\Sigma^+ \cap W^+ \subset \left\{\frac{\pi}{4}\right\},$$
$$\{\beta(Z_0) \mid \beta \in \Sigma^+\} \subset \left\{0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi\right\},$$
$$\{\beta(Z_0) \mid \beta \in W^+\} \subset \left\{\pm\frac{\pi}{2}, \pm\frac{\pi}{4}\right\}.$$

Condition (III)

$$\Sigma^+ \cap W^+ = \emptyset,$$
$$\{\beta(Z_0) \mid \beta \in \Sigma^+\} \subset \left\{0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi\right\},$$
$$\{\beta(Z_0) \mid \beta \in W^+\} \subset \left\{\pm\frac{\pi}{2}, \pm\frac{\pi}{3}\right\}.$$

Remark that $\Sigma^+ \cap W^+ \subset \left\{\frac{\pi}{4}\right\}$ is different from [9], [10] in Condition (II) and it was slightly improved by M. Yoshida.

Theorem 3.1 ([9], [10]). *For $Z_0 \in \overline{P_0}$, set $N := K_2 \mathrm{Exp} Z_0$. Suppose that $\mathrm{rk}(G/K_1) = \mathrm{cohom}(K_2 \curvearrowright N)$. Assume that $Z_0 \in \overline{P_0}$ satisfies just one of the above three conditions (I), (II) and (III). Then the induced Riemannian metric g on M coincides with a K_2-invariant Riemannian metric on M obtained from the restriction of a positive definite inner product $c\langle \cdot, \cdot \rangle$ of $\mathfrak{k}_2$ to a vector subspace $\mathfrak{m}_{\mathfrak{k}_2}$. Here c is given as*

$$c = \begin{cases} \frac{3}{4} & \text{(I)}, \\ \frac{1}{2} & \text{(II)}, \\ \frac{1}{4} & \text{(III)}. \end{cases} \tag{3.1}$$

Theorem 3.2 ([9], [10]). *For $Z_0 \in \overline{P_0}$, set $N := K_2 \mathrm{Exp} Z_0$. Suppose that $\mathrm{rk}(G/K_1) = \mathrm{cohom}(K_2 \curvearrowright N)$. Assume that $Z_0 \in \overline{P_0}$ satisfies just one of the three conditions (I), (II) and (III). If $N = K_2 \mathrm{Exp} Z_0$ is a minimal orbit, then the Jacobi differential operator of N is given as*

$$\widetilde{\mathfrak{J}(V)} = -C_{K_2}(\tilde{V}) + C_{G/K_2} \circ \tilde{V}.$$

for each $V \in C^\infty(T^\perp N)$.

Some examples of minimal orbits of the Hermann action on compact symmetric spaces satisfying one of Conditions (I), (II) and (III) were given in [9], [10].

Problem. Classify all minimal orbits of the Hermann action on compact symmetric spaces satisfying one of Conditions (I), (II) and (III)

4. Classification

In our recent work we have determined all $Z_0 \in \overline{P_0}$ which satisfy one of Conditions (I), (II) and (III) and correspond to minimal orbits ([15]).

The Hermann group actions on compact irreducible symmetric spaces with $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ are classified as folllows (O. Ikawa [5], [6], [14]):

1. $K_1 = K_2$, isotropy actions of Type I symmetric spaces.
2. $K_1 = K_2$, isotropy actions of Type II symmetric spaces.
3. $\theta_1 \not\sim \theta_2$ (A),
4. $\theta_1 \not\sim \theta_2$ (B),
5. $\theta_1 \not\sim \theta_2$ (C).

Here $\theta_1 \not\sim \theta_2$ means that θ_1 and θ_2 can be transformed each other by an inner automorphism of $\mathfrak{g}$. It is known that Cases (3), (4) and (5) correspond to symmetric triads (O. Ikawa).

4.1. $K_1 = K_2$, **Type I.**

Theorem 4.1 ([15]). *Type AI: $G = SU(n)$, $K_1 = K_2 = SO(n)$. If $Z_0 \in \overline{P_0}$ satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then $Z_0 \in \overline{P_0}$ satisfies Condition (I) and is given as one of the following*

1. *$n = 3k$ $(k \geq 1)$, $\alpha_k(Z_0) = \alpha_{2k}(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ for $i \neq k, 2k$.*
2. *$n = 3k$ $(k \geq 1)$, $\alpha_l(Z_0) = \alpha_{l+k}(Z_0) = \alpha_{l+2k}(Z_0) = \frac{\pi}{3}$ for each $l \in \mathbb{N}$ $(l + 2k \leq r)$, $\alpha_i(Z_0) = 0$ for $i \neq l, l+k.l+2k$.*

In each case the dimension of the corresponding orbit is equal to $3k^2$.

Theorem 4.2 ([15]). *Type AIII:* $G = SU(p+q)$, $K_1 = K_2 = S(U(p) \times U(q))$. *If* $Z_0 \in \overline{P_0}$ *satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is given as one of the following*

(1) $p = q = 3k$, $\alpha_k(Z_0) = \alpha_{3k}(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq k, 3k$. *The dimension of the corresponding orbit is equal to* $12k^2$.

(2) $p = q = 3k$, $\alpha_{2k}(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq k.3k$. *The dimension of the corresponding orbit is equal to* $12k^2$.

(3) $p + q = 3k$, $\alpha_k(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq k$. *The dimension of the corresponding orbit is equal to* $3k^2$.

Theorem 4.3 ([15]). *Type BI:* $G = SO(p+q)$, $K_1 = K_2 = SO(p) \times SO(q)$, *set* $k = p - q$ *and* $p + q$ *is odd,* $p \geq q$. *If* $Z_0 \in \overline{P_0}$ *satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is given as one of the following*

(1) $p + q = 3l - 1$ $(2 \leq l \leq q - 1)$, $\alpha_l(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 1, q$. *The dimension of the corresponding orbit is equal to* $\frac{3}{2}l(l-1)$.

(2) $p + q = 3k + 2$, $\alpha_1(Z_0) = \alpha_q(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 1, q$. *The dimension of the corresponding orbit is equal to* $\frac{3}{2}k(k+1)$.

(3) $p+q = 3l-1$ $(2 \leq l \leq q-1)$, $\alpha_1(Z_0) = \alpha_l(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 1, l$. *The dimension of the corresponding orbit is equal to* $\frac{3}{2}l(l-1)$.

Theorem 4.4 ([15]). *Type CI:* $G = Sp(n)$, $K_1 = K_2 = U(n)$. *If* $Z_0 \in \overline{P_0}$ *satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is given as one of the following*

(1) $n = 3l + 2$, $\alpha_{2l+1}(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 2l + 1$. *The dimension of the corresponding orbit is equal to* $3(2l+1)(l+1)$.

(2) $n = 3k - 1$, $\alpha_k(Z_0) = \alpha_n(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq k, n$. *The dimension of the corresponding orbit is equal to* $3k(2k-1)$.

Theorem 4.5 ([15]). *Type CII:* $G = Sp(p+q)$, $K_1 = K_2 = Sp(p) \times Sp(q)$, $p \geq q$, *set* $k = p - q$. *There is no* $Z_0 \in \overline{P_0}$ *satisfying one of Conditions (I), (II) and (III) and corresponding to a minimal orbit,*

Theorem 4.6 ([15]). *Type DI:* $G = SO(p+q)$, $K_1 = K_2 = SO(p) \times SO(q)$, $p \geq q$ *and* $p + q$ *is even. If* $Z_0 \in \overline{P_0}$ *satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is given as one of the following*

(1) $p = q = 4$, $\alpha_1(Z_0) = \alpha_3(Z_0) = \frac{\pi}{3}$, $\alpha_1(Z_0) = \alpha_4(Z_0) = \frac{\pi}{3}$, $\alpha_4(Z_0) = \alpha_3(Z_0) = \frac{\pi}{3}$ *or* $\alpha_1(Z_0) = \alpha_3(Z_0) = \alpha_4(Z_0) = \frac{\pi}{3}$. $\alpha_i(Z_0) = 0$ *for* $i \neq 1, q$. *The dimension of the corresponding orbit is equal to* 9.

(2) $p = q = 3l - 1$, $\alpha_l(Z_0) = \alpha_{q-1}(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq l, q - 1$. *The dimension of the corresponding orbit is equal to* $6l^2 + 3l$.

(3) $p = q = 3l - 1$, $\alpha_l(Z_0) = \alpha_{q-1}(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq l, q - 1$. *The dimension of the corresponding orbit is equal to* $6l^2 + 3l$.

(4) $p + q = 3f - 1$ $(2 \leq f \leq q)$, $\alpha_f(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq f$. *The dimension of the corresponding orbit is equal to* $\frac{3}{2}f(f-1)$.

(5) $p + q = 3f - 1$ $(2 \leq f \leq q)$, $\alpha_f(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq f$. *The dimension of the corresponding orbit is equal to* $\frac{3}{2}f(f-1)$.

Theorem 4.7 ([15]). *Type EI: $G = E_6$, $K_1 = K_2 = Sp(4)$. If $Z_0 \in \overline{P_0}$ satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then $Z_0 \in \overline{P_0}$ satisfies Condition (I) and is given as one of the following*

(1) $\alpha_4(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 4$. *The dimension of the corresponding orbit is equal to* 27.

(2) $\alpha_1(Z_0) = \alpha_6(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 1, 6$. *The dimension of the corresponding orbit is equal to* 24.

Theorem 4.8 ([15]). *Type EII: $G = E_6$, $K_1 = K_2 = SU(6) \cdot SU(2)$. If $Z_0 \in \overline{P_0}$ satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then $Z_0 \in \overline{P_0}$ satisfies Condition (I) and is given as one of the following*

(1) $\alpha_2(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 2$. *The dimension of the corresponding orbit is equal to* 27.

(2) $\alpha_2(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 4$. *The dimension of the corresponding orbit is equal to* 24.

Theorem 4.9 ([15]). *Type $EIII$: $G = E_6$, $K_1 = K_2 = Spin(10) \cdot U(1)$. There is no $Z_0 \in \overline{P_0}$ satisfying one of Conditions (I), (II) and (III) and corresponding to a minimal orbit.*

Theorem 4.10 ([15]). *Type EIV: $G = E_6$, $K_1 = K_2 = F_4$. If $Z_0 \in \overline{P_0}$ satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then $Z_0 \in \overline{P_0}$ satisfies Condition (I) and is just given by $\alpha_1(Z_0) = \alpha_2(Z_0) = \frac{\pi}{3}$. The dimension of the corresponding orbit is equal to* 24.

Theorem 4.11 ([15]). *Type EV: EV: $G = E_7$, $K_1 = K_2 = SU(8)$. If $Z_0 \in \overline{P_0}$ satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then $Z_0 \in \overline{P_0}$ satisfies Condition (I) and is given as one of the following*

(1) $\alpha_3(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 3$. *The dimension of the corresponding orbit is equal to* 45.

(2) $\alpha_5(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 5$. *The dimension of the corresponding orbit is equal to* 45.

Theorem 4.12 ([15]). *Type EVI: $G = E_7$, $K_1 = K_2 = SO(12) \cdot SU(2)$. If $Z_0 \in \overline{P_0}$ satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then $Z_0 \in \overline{P_0}$ satisfies Condition (I) and is just given by $\alpha_2(Z_0) = \frac{\pi}{3}$ and $\alpha_i(Z_0) = 0$ for $i \neq 2$. The dimension of the corresponding orbit is equal to* 24.

Theorem 4.13 ([15]). *Type $EVII$: $G = E_7$, $K_1 = K_2 = E_6 \cdot SO(2)$. There is no $Z_0 \in \overline{P_0}$ satisfying one of Conditions (I), (II) and (III) and corresponding to a minimal orbit.*

Theorem 4.14 ([15]). *Type $EVIII$: $G = E_8$, $K_1 = K_2 = Spin(16)$. There is no $Z_0 \in \overline{P_0}$ satisfying one of Conditions (I), (II) and (III) and corresponding to a minimal orbit.*

Theorem 4.15 ([15]). *Type EIX: $G = E_8$, $K_1 = K_2 = E_7 \cdot SU(2)$. If $Z_0 \in \overline{P_0}$ satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then $Z_0 \in \overline{P_0}$ satisfies Condition (I) and is just given by $\alpha_2(Z_0) = \frac{\pi}{3}$ and $\alpha_i(Z_0) = 0$ for $i \neq 2$. The dimension of the corresponding orbit is equal to* 81.

Theorem 4.16 ([15]). *Type FI: $G = F_4$, $K_1 = K_2 = Sp(3) \cdot SU(2)$. If $Z_0 \in \overline{P_0}$ satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit,* *then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is just given by* $\alpha_2(Z_0) = \frac{\pi}{3}$ *and* $\alpha_i(Z_0) = 0$ *for* $i \neq 2$. *The dimension of the corresponding orbit is equal to* 18.

Theorem 4.17 ([15]). *Type* FII*:* $G = F_4$, $K_1 = K_2 = Spin(9)$. *There is no* $Z_0 \in \overline{P_0}$ *satisfying one of Conditions (I), (II) and (III) and corresponding to a minimal orbit.*

Theorem 4.18 ([15]). *Type G:* $G = G_2$, $K_1 = K_2 = SO(4)$. *If* $Z_0 \in \overline{P_0}$ *satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is just given by* $\alpha_1(Z_0) = \frac{\pi}{3}$ *and* $\alpha_2(Z_0) = 0$. *The dimension of the corresponding orbit is equal to* 3.

4.2. $K_1 = K_2$**, Type II.**

Theorem 4.19 ([15]). A_{n-1}*:* $U = SU(n)$. *If* $Z_0 \in \overline{P_0}$ *satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is given as one of the following*

(1) $n = 3k$ $(k \geq 1)$, $\alpha_k(Z_0) = \alpha_{2k}(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq k, 2k$.
(2) $n = 3k$ $(k \geq 1)$, $\alpha_l(Z_0) = \alpha_{l+k}(Z_0) = \alpha_{l+2k}(Z_0) = \frac{\pi}{3}$ *for each* $l \in \mathbb{N}$ $(l + 2k \leq r)$, $\alpha_i(Z_0) = 0$ *for* $i \neq l, l+k.l+2k$.

In each case the dimension of the corresponding orbit is equal to $3k^2$.

Theorem 4.20 ([15]). B_n*:* $U = SO(2n+1)$. *If* $Z_0 \in \overline{P_0}$ *satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is given as one of the following*

(1) $2n = 3l - 2$ $(2 \leq l \leq n, l \in \mathbb{N})$, $\alpha_1(Z_0) = \alpha_l(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 1, l$. *The dimension of the corresponding orbit is equal to* $\frac{3}{2}l(l-1)$.
(2) $2n = 3l - 2$ $(2 \leq l \leq n, l \in \mathbb{N})$, $\alpha_1(Z_0) = \alpha_l(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq$. *The dimension of the corresponding orbit is equal to* $\frac{3}{2}l(l-1)$.

Theorem 4.21 ([15]). *Type* C_n*:* $U = Sp(n)$. *If* $Z_0 \in \overline{P_0}$ *satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is given as one of the following*

(1) $n = 3l + 2$, $\alpha_{2l+1}(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 2l+1$. *The dimension of the corresponding orbit is equal to* $3(2l+1)(l+1)$.
(2) $n = 3k - 1$, $\alpha_k(Z_0) = \alpha_n(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq k, n$. *The dimension of the corresponding orbit is equal to* $3k(2k-1)$.

Theorem 4.22. *Type* $DIII$*:* $U = SO(2n)$. *There is no* $Z_0 \in \overline{P_0}$ *satisfying one of Conditions (I), (II) and (III) and corresponding to a minimal orbit.*

Theorem 4.23 ([15]). *Type* E_6*:* $U = E_6$. *If* $Z_0 \in \overline{P_0}$ *satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is given as one of the following*

(1) $\alpha_4(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 4$. *The dimension of the corresponding orbit is equal to* 27.
(2) $\alpha_1(Z_0) = \alpha_6(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 1, 6$. *The dimension of the corresponding orbit is equal to* 24.

Theorem 4.24 ([15]). *Type* E_7*:* $U = E_7$. *If* $Z_0 \in \overline{P_0}$ *satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then* $Z_0 \in \overline{P_0}$ *satisfies Condition (I) and is given as one of the following*

(1) $\alpha_3(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 3$. *The dimension of the corresponding orbit is equal to* 45.
(2) $\alpha_5(Z_0) = \frac{\pi}{3}$, $\alpha_i(Z_0) = 0$ *for* $i \neq 5$. *The dimension of the corresponding orbit is equal to* 45.

Theorem 4.25 ([15]). *Type E_8: $U = E_8$. There is no $Z_0 \in \overline{P_0}$ satisfying one of Conditions (I), (II) and (III) and corresponding to a minimal orbit.*

Theorem 4.26 ([15]). *Type F_4: $U = F_4$. If $Z_0 \in \overline{P_0}$ satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then $Z_0 \in \overline{P_0}$ satisfies Condition (I) and is just given by $\alpha_2(Z_0) = \frac{\pi}{3}$ and $\alpha_i(Z_0) = 0$ for $i \neq 2$. The dimension of the corresponding orbit is equal to* 18.

Theorem 4.27 ([15]). *Type G_2: $U = G_2$. If $Z_0 \in \overline{P_0}$ satisfies one of Conditions (I), (II) and (III) and corresponds to a minimal orbit, then $Z_0 \in \overline{P_0}$ satisfies Condition (I) and is just given by $\alpha_1(Z_0) = \frac{\pi}{3}$ and $\alpha_2(Z_0) = 0$. The dimension of the corresponding orbit is equal to* 3.

4.3. **Classification $\theta_1 \not\sim \theta_2$ (A).** This case is defined by the assumption that G is simple and θ_1 and θ_2 cannot be transformed each other by an inner involutive automorphism of $\mathfrak{g}$ · E · E In this case there is no orbit of the Hermann action satisfying Condition (I) or Condition (III).

Because

Lemma 4.1 ([5] Matsuki). $\theta_1 \sim \theta_2$ *if and only if* $\Sigma \cap W = \emptyset$.

Theorem 4.28 ([15]). *If $G = SO(r+s+t), K_1 = SO(r) \times SO(s+t), K_2 = SO(r+s) \times SO(t), (1 \leq r < t, s \geq 1)$, then there exists an orbit of the Hermann action satisfying Condition* (II) *only in case $r = 1$. For $s \geq 2$ there exists a minimal orbit of the Hermann action only in case $s = t$ and for $s = 1$ there exists a minimal orbit of the Hermann action only in case $t = 2$ The dimensions of the corresponding orbits are $2t - 2$ and 2, respectively. Moreover the above minimal orbits are austere submanifolds of M.*

Theorem 4.29 ([15]). *If $G = SO(4r), K_1 = U(2r), K_2 = SO(2r) \times SO(2r), (r \geq 1)$, then there exist orbits satisfying Condition* (II) *only in case $r = 1$ and however there is no minimal orbit among them.*

Theorem 4.30 ([15]). *If $G = SU(2r), K_1 = S(U(r) \times U(r)), K_2 = SO(2r), (r \geq 1)$, then there exist orbits satisfying Condition* (II) *only in case $r = 1$ and however there is no minimal orbit among them.*

Theorem 4.31 ([15]). *If $G = SU(r+s), K_1 = S(U(r) \times U(s)), K_2 = SO(r+s), (1 \leq r < s)$, then there exists an orbit of the Hermann action satisfying Condition* (II) *only in case $r = 1$. The dimension of the corresponding orbit is $2s - 2$. Moreover this orbit is an austere submanifolds of M and thus a minimal orbit.*

Theorem 4.32 ([15]). *If $G = SU(4r), K_1 = Sp(2r), K_2 = S(U(2r) \times U(2r)), (r \geq 1)$, then there exists an orbit of the Hermann action satisfying Condition* (II) *only in case $r = 1$. However there is no minimal orbit of the Hermann action satisfying Condition* (II)

Theorem 4.33 ([15]). *If $G = Sp(2r), K_1 = Sp(r) \times Sp(r), K_2 = U(2r)$, then there exists an orbit of the Hermann action satisfying Condition* (II) *only in case $r = 1$. A minimal orbit of the Hermann action satisfying Condition* (II) *is of dimension* 3 *and but it is an austere submanifold of M.*

4.4. **Classification** $\theta_1 \not\sim \theta_2$ **(B).** This case is defined by the assumption that there are a simple connected compact Lie group U and a Riemannian symmetric pair (U, K, τ) such that

$$\begin{aligned}
&G = U \times U,\\
&K_1 = \Delta G = \{(u, u) \mid u \in U\},\\
&\theta_1(u_1.u_2) = (u_2.u_1),\\
&K_2 = K \times K,\\
&\theta_2(u_1.u_2) = (\tau(u_1), \tau(u_2)).
\end{aligned}$$

In this case we have only to treat the case of $V(\mathfrak{m}_1 \cap \mathfrak{k}_2) = 0$. And we observe that if there is an orbit satisfying Condition (II), then the root system of the symmetric triad must be of dimension 1 and thus

(1) $U = SU(n)$, $K = SO(n)$,
(2) $U = SU(p+q)$, $K = S(U(p) \times U(q))$,
(3) $U = SO(p+q)$, $K = SO(p) \times SO(q)$.

Theorem 4.34 ([15])**.** *] There exist minimal orbits of the Hermann action satisfying Condition* (II) *if and only if*

(1) $n = 2$,
(2) $p = q = 1$, *or*
(3) $p = 2$, $q = 1$.

They all are austere submanifolds of M.

4.5. **Classification** $\theta_1 \not\sim \theta_2$ **(C).** This case is defined by the assumption that there are a simple connected compact Lie group U or $U = SO(4)$ and an involutive *outer* automorphism σ of $\mathfrak{u}$ such that

$$\begin{aligned}
&G = U \times U,\\
&K_1 = \Delta G = \{(u, u) \mid u \in U\},\\
&\theta_1(u_1.u_2) = (u_2.u_1),\\
&K_2 = \{(u_1, u_2) \in U \times U \mid (\sigma(u_2), \sigma(u_1)) = (u_1, u_2)\},\\
&\theta_2(u_1.u_2) = (\sigma(u_2), \sigma(u_1)).
\end{aligned}$$

Then we have $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. We have only to treat the case of $V(\mathfrak{m}_1 \cap \mathfrak{k}_2) = 0$. Similarly we observe that if there is an orbit satisfying Condition (II). then the root system of the symmetric triad must be of dimension 1.

Theorem 4.35. *In the case when U is simple, there is no minimal orbit of the Hermann action satisfying Condition* (II)*. In the case when $U = SO(4)$ and $K = SO(2) \times SO(2)$, all orbits satisfying Condition* (II) *are austere submanifolds and thus minimal orbits.*

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