A CLASSIFICATION OF LEFT-INVARIANT SYMPLECTIC STRUCTURES ON SOME LIE GROUPS

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1. Introduction

In geometry it is an important problem to study whether a given manifold admits some nice geometric structures. In the setting of Lie groups is very natural to ask about the existence of left-invariant structures. A **symplectic Lie group** is a Lie group G endowed with a left-invariant symplectic form ω (i.e. a nondegenerate closed 2-form). The geometry of symplectic Lie groups is an active field of research. There are many interesting results in the structure of symplectic Lie groups and considerable classification efforts in low dimensions, but the general picture is far from complete. Many conjectures about the existence of Lagrangian normal subgroups are still unsolved. Some nice known results include:

(1) Unimodular symplectic Lie groups are solvable. In particular symplectic Lie groups of dim 4 are solvable.
(2) Classification for the 4 dimensional case.
(3) All nilpotent symplectic Lie algebra of dimension ≤ 6 have a Lagrangian ideal.

In [6] and [4] we can find a novel method to find nice (e.g. Einstein or Ricci soliton) left-invariant pseudo-Riemannian and left-invariant Riemannian metrics respectively. The main idea in these papers is to first study the "moduli space of pseudo-Riemannian (or Riemanian, respectively) metrics" which is a certain orbit space and then search inside this space for nice metrics. In this paper we develop and apply the same ideas to the case of a symplectic Lie group: we first study the " moduli space of left-invariant nondegenerate 2-forms" and then we search inside this moduli space for the 2-forms that are symplectic .

2. A general procedure

2.1. **The moduli space.** Let G be a Lie group ($\dim(G) = 2n$) and $\mathfrak{g}$ its corresponding Lie algebra. We are interested in the space

$$\Omega\left(G\right) := \left\{ \omega(\cdot,\cdot) \in \bigwedge^2 T^*G \mid \omega^n \neq 0, \text{left-invariant} \right\} \text{ (nondegenerate 2-forms of } G).$$

It is well known that this space can be identified with the space

$$\Omega(\mathfrak{g}) := \left\{ \omega\left(\cdot,\cdot\right) \in \bigwedge^2 \mathfrak{g}^* \mid \omega^n \neq 0 \right\} \text{ (nondegenerate 2-forms of } \mathfrak{g}).$$

Nondegenerate 2-forms in a vector space are also called **symplectic forms**.

The general linear group $\mathrm{GL}(2n,\mathbb{R})$ acts transitively on $\Omega\left(\mathfrak{g}\right)$ by

$$g.\omega(\cdot,\cdot) = \omega(g^{-1}(\cdot),g^{-1}(\cdot)) \quad \forall g \in \mathrm{GL}(2n,\mathbb{R}).$$

Recall that the group of linear maps that preserve $\omega \in \Omega(\mathfrak{g})$ is called **symplectic group**. In terms of matrices (for $\dim(\mathfrak{g}) = 2n$) it is denoted by $\mathrm{Sp_n}(\mathbb{R})$ and is defined by

$$\mathrm{Sp_n}(\mathbb{R}) := \left\{ A \in \mathrm{GL}(2n,\mathbb{R}) \mid {}^tAJA = J \right\}$$

where $J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Remark 2.1. The symplectic group $\mathrm{Sp_n}(\mathbb{R})$ is itself a Lie group, but this Lie group is not the definition of "symplectic Lie group".

Then by the theory of homogeneous spaces one has the next identification.

Proposition 2.2.

$$\Omega(\mathfrak{g}) \cong \mathrm{GL(2n,\mathbb{R})/Sp_n(\mathbb{R})} \tag{2.1}$$

as homogeneous spaces.

Consider the automorphism group of $\mathfrak{g}$ defined by

$$\mathrm{Aut}(\mathfrak{g}) := \{\phi \in \mathrm{GL}(2n,\mathbb{R}) \mid \phi[\cdot,\cdot] = [\phi(\cdot),\phi(\cdot)]\}.$$

Also define $\mathbb{R}^\times := \mathbb{R} \setminus 0$. Then we can consider the set

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) := \left\{\phi \in \mathrm{GL}(2n,\mathbb{R}) \mid \phi \in \mathrm{Aut}(\mathfrak{g}), c \in \mathbb{R}^\times\right\}.$$

This last set is a subgroup of $\mathrm{GL}(2n,\mathbb{R})$ so it naturally acts on $\Omega(\mathfrak{g})$. We can then consider the orbit space of this action.

Definition 2.3. The orbit space of the action $\mathbb{R}^\times\mathrm{Aut}(\mathfrak{g}) \curvearrowright \Omega(\mathfrak{g})$ will be called the **moduli space of left-invariant nondegenerate 2-forms** and will be denoted by

$$\mathfrak{P}\Omega(\mathfrak{g}) := \mathbb{R}^\times\mathrm{Aut}(\mathfrak{g}) \setminus \Omega(\mathfrak{g}) := \left\{\mathbb{R}^\times\mathrm{Aut}(\mathfrak{g}).\omega \mid \omega \in \Omega(\mathfrak{g})\right\}.$$

On the other side there is a natural way of defining an equivalence relation between 2-forms.

Definition 2.4. $\omega_1, \omega_2 \in \Omega(\mathfrak{g})$ are **symplectomorphically equivalent up to scale** if there exists $\phi \in \mathrm{Aut}(\mathfrak{g})$ and $c > 0$, such that $c\phi^*\omega_1 = \omega_2$.

So we can consider the quotient space for this equivalence relation. This quotient space will be denoted by

$$\Omega(\mathfrak{g})/\text{"up to symplectomorphism and scale"}.$$

Proposition 2.5. The moduli space of definition 2.3 and the previous quotient space of definition are equivalent, i.e.

$$\mathfrak{P}\Omega(\mathfrak{g}) \cong \Omega(\mathfrak{g})/\text{"up to symplectomorphism and scale"}.$$

2.2. **Milnor frames procedure.** Let ω_0 be the canonical symplectic form on a Lie algebra $\mathfrak{g}$. To simplify notation lets denote the orbit of $\mathbb{R}^\times\mathrm{Aut}(\mathfrak{g})$ through $\omega \in \Omega(\mathfrak{g})$ by

$$[\omega] := \mathbb{R}^\times\mathrm{Aut}(\mathfrak{g}).\omega := \left\{\phi.\omega \mid \phi \in \mathbb{R}^\times\mathrm{Aut}(\mathfrak{g})\right\}.$$

Definition 2.6. A subset $U \subset \mathrm{GL}(2n,\mathbb{R})$ is called a **set of representatives** of $\mathfrak{P}\Omega(\mathfrak{g})$ if

$$\mathfrak{P}\Omega(\mathfrak{g}) = \{[h.\omega_0] \mid h \in U\}.$$

Remark 2.7. Of course the set of representatives is not unique. In practice we want the set of representatives to be as small as possible.

Let $[[g]]$ denote the double space coset of $g \in \mathrm{GL}(2n,\mathbb{R})$ defined by

$$[[g]] := \mathbb{R}^\times\mathrm{Aut}(\mathfrak{g})g\mathrm{Sp}(2n,\mathbb{R}) := \left\{\phi g s \mid \phi \in \mathbb{R}^\times\mathrm{Aut}(\mathfrak{g}), s \in \mathrm{Sp}(2n,\mathbb{R})\right\}.$$

Now we state a criteria for a set U to be a set of representatives. Now we state a theorem for obtaining Milnor type frames in the symplectic case.

Theorem 2.8 (Milnor type theorem)**.** *Let U be a set of representatives of $\mathfrak{P}\Omega(\mathfrak{g})$. Then for every $\omega \in \Omega(g)$ there exists $k > 0$, $\phi \in \mathrm{Aut}(\mathfrak{g})$ and $g \in U$ such that $\{\phi g e_1, \ldots, \phi g e_{2n}\}$ is a symplectic basis with respect to $k\omega$.*

The basis obtained in last theorem will be called **Milnor frames**. Notice that if U has a small number of parameters, the bracket relations of the Milnor frames will also be given in terms of a small set of parameters. When we study a particular Lie algebra we put $x_i := \phi g e_i$ and study the bracket relations between them.

2.3. **Closed condition procedure.** Suppose we are given a 2-form $\omega_{\mathfrak{g}} \in \Omega(\mathfrak{g})$, let $\omega_G \in \Omega(\mathrm{G})$ be the corresponding 2-form on the Lie group. The next is a well known fact.

Proposition 2.9. ω_G is closed iff

$$d\omega_{\mathfrak{g}}(x,y,z) := -\omega_{\mathfrak{g}}([x,y],z) + \omega_{\mathfrak{g}}([x,z],y) - \omega_{\mathfrak{g}}([y,z],x) = 0$$

for all $x, y, z \in \mathfrak{g}$.

An $\omega_{\mathfrak{g}} \in \Omega(\mathfrak{g})$ that satisfies the previous property will be called **closed 2-form or closed symplectic form**. Dropping the distinction between $\omega_{\mathfrak{g}}$ and ω_G, the set $(\mathfrak{g}, \omega)$ where $\mathfrak{g}$ is a lie algebra and ω is a closed symplectic form will be called a **symplectic Lie algebra**.

Remark 2.10. Notice that a symplectic Lie algebra is more than just a symplectic vector space that also has a Lie algebra structure.

In order to check if a given 2-form is closed it is enough to consider any basis of $\mathfrak{g}$ and that if the basis is also symplectic the computations are easier. Suppose we obtained a set of Milnor frames in terms of a nice amount of parameters, then we can use this condition to evaluate which correspond to closed 2-forms.

Denote the set of all symplectic 2-forms in $\mathfrak{g}$ by

$$\Omega^c(\mathfrak{g}) := \left\{ \omega\left(\cdot,\cdot\right) \in \bigwedge^2 \mathfrak{g}^* \mid \omega^n \neq 0, \text{closed} \right\}.$$

2.4. **Lagrangian subspaces.** One of the most important type of subspaces in a symplectic vector space (V, ω) are Lagrangian subspaces. Recall that a subspace $\mathfrak{l} \subset V$ is Lagrangian if $\mathfrak{l}^\perp = \mathfrak{l}$ where

$$l^\perp := \left\{ v \in V \mid \omega(v,w) = 0 \text{ for all } w \in V \right\}.$$

On the other side on a Lie algebra, subalgebras and in particular ideals are the most important subspaces. So if we are given a symplectic Lie algebra it is natural to consider Lagrangian subspaces and in particular Lagrangian ideals. In this paper we will show that for the Lie groups considered we can easily show the existence of Lagrangian ideals using the Milnor frames.

3. Main Results

In this paper we are interested in the following two Lie groups.

(1) $G_{\mathbb{R}\mathrm{H}^{2n}} := \mathbb{R} \ltimes \mathbb{R}^{2n-1}$: the semidirect product of the abelian group $\mathbb{R}$ and $\mathbb{R}^{2n-1}$, where $\mathbb{R}$ acts on $\mathbb{R}^{2n-1}$ by $t.x := e^t x$ $(t \in \mathbb{R}, x \in \mathbb{R}^{2n-1})$. Corresponding Lie algebra: $\mathfrak{g}_{\mathbb{R}\mathrm{H}^{2n}}$.
(2) $H^3 \times \mathbb{R}^{2n-3}$:the direct product of the 3-dimensional Heisenberg Lie group H^3 and the abelian group $\mathbb{R}^{2n-3}$. Corresponding Lie algebra: $\mathfrak{h}^3 \bigoplus \mathbb{R}^{2n-3}$.

As a first result we obtained Milnor-type theorems for the previous Lie groups.

Theorem 3.1 (The Lie group $G_{\mathbb{R}\mathrm{H}^{2n}}$)**.** *For all $\omega \in \Omega\left(\mathfrak{g}_{\mathbb{R}H^{2n}}\right)$, there exists $t > 0$ and a symplectic basis $\{x_1, \ldots, x_{2n}\} \subset \mathfrak{g}_{\mathbb{R}H^{2n}}$ with respect to $t\omega$ such that*

$$[x_1, x_k] = x_k \quad \textit{for } k = 2, \ldots, 2n.$$

Theorem 3.2 (The Lie group $H^3 \times \mathbb{R}^{2n-3}$)**.** *For all $\omega \in \Omega(\mathfrak{h}^3 \bigoplus \mathbb{R}^{2n-3})$, there exists $t > 0$, $k = 0, 1$ and a symplectic basis $\{x_1, \ldots, x_{2n}\} \subset \mathfrak{h}^3 \bigoplus \mathbb{R}^{2n-3}$ with respect to $t\omega$ such that*

$$[x_1, x_2] = x_{n+1} - x_2$$
$$[x_1, x_{n+1}] = x_{n+1} - x_2,$$

or

$$[x_1, x_2] = x_{n+1} - kx_3,$$

or

$$[x_1, x_2] = x_{2n}$$
$$[x_1, x_{n+1}] = kx_{2n}.$$

Using these theorems we can search for 2-forms that are closed. The main results with respect to the existence and classification of left-invariant symplectic structures can be summarized in the next table.

Lie Group	**Number of Left-invariant symplectic structures**	**Existence of Lagrangian ideal**
$G_{\mathbb{R}\mathrm{H}^{2n}}$	1 if $\dim\mathfrak{g} = 2$	Yes
	0 if $\dim\mathfrak{g} > 2$	No
$H \times \mathbb{R}^{2n-3}$	1	Yes

This table is obtained using a new/different approach of that in the existing literature: we study the " moduli space of left-invariant nondegenerate 2-forms" (the orbit space of certain action on $\Omega(\mathfrak{g})$), then we obtain Milnor-type theorems and finally we search for the 2-forms that are symplectic . In certain cases we can use this approach to discuss high dimension examples.

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