Biharmonic principal G-bundles and vector bundles

Hajime URAKAWA, Tohoku University,

Yuzawa, November 29, 2019.

Table of Contents

§1 (1) Introduction, biharmonic isometric immersions

- B.Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, *Soochow J. Math.*, **17** (1991), 169–188.
- Consider an isometric immersion $f : (M^m, g) \hookrightarrow (\mathbb{R}^k, g_0)$ and $f(x) = (f_1(x), \cdots, f_k(x))$ $(x \in M)$. Then,
- $\Delta f := (\Delta f_1, \cdots, \Delta f_k) = m\,\mathbf{H}$,
- $\mathbf{H} := \frac{1}{m}\sum_{i=1}^m B(e_i, e_i)$, mean curvature vector field,

  $B(X, Y) := D^0_X(f_*Y) - f_*(\nabla_X Y)$,

  the second fundamental form.

§1 (2) Introduction, biharm. isometric immersions

- Def $f : (M^m, g) \hookrightarrow (\mathbb{R}^k, g_0)$ is minimal if $\mathbf{H} \equiv \mathbf{0}$.
- Chen defined that f is to be biharmonic if

$$\Delta\mathbf{H} = \Delta(\Delta f) \equiv \mathbf{0}.$$

- Thm (Chen) If $\dim M = 2$,
  any biharmonic submanifold is minimal.
- B.Y. Chen's Conjecture:

  Any biharmonic isometric immersion into $(\mathbb{R}^k, g_0)$ must be minimal.

§1 (3) Introduction. first variation formula

- For a smooth map $f : (M, g) \to (N, h)$, the energy functional is: $E(f) := \frac{1}{2}\int_M |df|^2\, v_g$.
- The first variation formula is:

$$\frac{d}{dt}\bigg|_{t=0} E(f_t) = -\int_M \langle \tau(f), V\rangle\, v_g.$$

- Here, $V_x = \frac{d}{dt}|_{t=0} f_t(x) \in T_{f(x)}N$, $(x \in M)$, and

$$\tau(f) := \sum_{i=1}^m B(f)(e_i, e_i),$$

$$B(f)(X, Y) := \nabla^N_{df(X)} df(Y) - df(\nabla_X Y),\ X, Y \in \mathfrak{X}(M).$$

- $f : (M, g) \to (N, h)$ is harmonic if $\tau(f) = \mathbf{0}$.

§1 (4) Introduction. second variation formula

- The second variation formula for the energy functional $E(\bullet)$ for a harmonic map $f : (M, g) \to (N, h)$ is:

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(f_t) = \int_M \langle J(V), V\rangle v_g,$$

- where

$$J(V) := \overline{\Delta} V - \mathcal{R}(V),$$

$$\overline{\Delta} V := \overline{\nabla}^* \overline{\nabla} V,\ \ \mathcal{R}(V) := \sum_{i=1}^m R^N(V, df(e_i))df(e_i).$$

§1 (5) Introduction. k - harmonic maps

- The k-energy functional due to Eells-Lemaire is
  $E_k(f) := \frac{1}{2}\int_M |(d+\delta)^k f|^2 v_g \ (k = 1, 2, \cdots),$
- $E_1(f) = \frac{1}{2}\int_M |df|^2\, v_g, \ \ E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2\, v_g.$
- The first variation for $E_2(f)$ (G.Y. Jiang, Chin. Ann. Math. **7A** ('86), Note Mat. **28** ('09), 209–232) is:
  $\frac{d}{dt}\big|_{t=0} E_2(f_t) = -\int_M \langle \tau_2(f), V\rangle v_g,$
  $\tau_2(f) := J(\tau(f)) = \overline{\Delta}\tau(f) - \mathcal{R}(\tau(f)).$
- $f : (M, g) \to (N, h)$ is biharmonic if $\ \tau_2(f) = 0.$
- S.Maeta,Osaka J. Math.**49**('12),1035–1063; S.Maeta,N.Nakauchi&H.Urakawa, Monat. Math. **177**('15),551–567;N.Nakauchi&H.Urakawa,Note Mat. **38**('18),89–100

§2 Biharmonic maps on principal G-bundles

- **Problem** *Let $\pi : (P, g) \to (M, h)$ be a principal G-bundle. If π is biharmonic, is π harmonic ?*
- **Th 1.** (Wang & Ou,'11) Let π, $(M^3(c), g) \to (N^2, h)$, a Riemannian submersion. If π is biharmonic, then π is harmonic, and π is a harmonic morphism.
- **Th 2.** *Let $\pi : (P, g) \to (M, h)$, a compact principal G-bundle & the Ricci tensor of (M, h) is neg. def. Then, if π is biharmonic, then it is harmonic.*
- **Th 3.** *Let $\pi : (P, g) \to (M, h)$, a principal G-bundle & Ricci tensor of (M, h), non-positive. Assume (P, g), complete, π, finite energy & finite bienergy. Then, if π is biharmonic, then it is harmonic.*

§3 (1) Principal G-bundles

- Let $P = P(M, G)$, a principal bundle.
  A compact Lie group G acts on P by
  $(G, P) \ni (a, u) \mapsto u \cdot a \in P.$
- The vertical subsp. $G_u := \{A^*{}_u \,|\, A \in \mathfrak{g}\} \subset T_u P$, $\forall A \in \mathfrak{g}$, the fund. vector field $A^* \in \mathfrak{X}(P)$ def. by
  $A^*{}_u := \frac{d}{dt}\Big|_{t=0} u \exp(t\,A) \in T_u P.$
- Assume a Riemannian metric g on P satisfies $R_a{}^* g = g$ for all $a \in G$. Then, we have
  (a) $T_u P = G_u \oplus H_u$ (ortho. decomp.)
  (b) $G_u = \{A^*{}_u \,|\, A \in \mathfrak{g}\}$, and
  (c) $R_{a*} H_u = H_{u\cdot a}, \quad a \in G,\ u \in P.$
  Here $H_u \subset T_u P$ is the horizontal subspace.

§3 (2) Principal G-bundles

- The (adapted Riemannian metric): We take a Riemannian metric g on the total space P of a principal G-bundle $\pi : P \to M$,
  $$g = \pi^* h + \langle \omega(\cdot), \omega(\cdot)\rangle,$$
  where ω is a $\mathfrak{g}$-valued 1-form on P called a connection form, and $\langle\cdot,\cdot\rangle$ is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ satisfying that
  $$\omega(A^*) = A, \qquad A \in \mathfrak{g},$$
  $$R_a{}^*\omega = \mathrm{Ad}(a^{-1})\,\omega, \qquad a \in G.$$
- Then, $\quad g(X_u, Y_u) = h(\pi_* W_u, \pi_* Z_u) + \langle A, B\rangle,$
  for $X_u = A^*{}_u + W_u, Y_u = B^*{}_u + Z_u,$
  $(A, B \in \mathfrak{g},\ W_u,\ Z_u \in H_u).$

§3 (3) Principal G-bundles

- Assume that the projection $\pi : (P, g) \to (M, h)$ is biharmonic, $J(\tau(\pi)) \equiv 0$, where
-
$$\tau(\pi) := \sum_i \{\nabla^h_{e_i}\pi_* e_i - \pi_*(\nabla_{e_i} e_i)\}, \tag{1}$$
$$JV := \overline{\Delta} V - \mathcal{R}(V) \qquad (V \in \Gamma(\pi^{-1}TN)), \tag{2}$$
$$\overline{\Delta} V := -\sum_i \{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V\}, \tag{3}$$
$$\mathcal{R}(V) := \sum_i R^h(V, \pi_* e_i)\pi_* e_i, \tag{4}$$

$\{e_i\}$ is a local orthonormal frame field on (P, g).

§3 (4) Principal G-bundles

- Since $J(\tau(\pi)) = 0$, we have
$$0 = \int_P \langle J(\tau(\pi)), \tau(\pi)\rangle\, v_g = \int_P \langle \overline{\nabla}^* \overline{\nabla}\, \tau(\pi), \tau(\pi)\rangle\, v_g - \int_P \sum_i \langle R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi)\rangle\, v_g. \tag{5}$$
- Thus, we have
$$\int_P \langle \overline{\nabla}\tau(\pi), \overline{\nabla}\tau(\pi)\rangle\, v_g = \int_P \sum_i R^h(\tau(\pi), e'_i)e'_i, \tau(\pi)\rangle\, v_g = \int_P \langle \rho^h(\tau(\pi)), \tau(\pi)\rangle\, v_g = \int_P \mathrm{Ric}^h(\tau(\pi))\, v_g, \tag{6}$$
- where $\{e'_i\}$ is a local orthonormal frame field, ρ^h is the Ricci tensor, and $\mathrm{Ric}^h(X)$, $X \in TM$, is the Ricci curvature of (M, h).

§3 (5) Principal G-bundles

- By the assumption that the Ricci curvature of (M, h) is negative definite,

$$\mathrm{Ric}^h(\tau(\pi)) \leq 0,$$

so that the right hand side of (6) is non-positive.
- Since the left hand side of (6) is non-negative, so that the both hand sides must vanish.
- Then, we have

$$\mathrm{Ric}^h(\tau(\pi)) \equiv 0 \quad \text{and} \quad \overline{\nabla}\,\tau(\pi) \equiv 0. \qquad (6')$$

§3 (6) Principal G-bundles

- Thus, $\tau(\pi) \equiv 0$, namely, $\pi : (P, g) \to (M, h)$ is harmonic. □
  Therefore, we obtain:
- Theorem 2 Let $\pi : (P, g) \to (M, h)$ be a principal G-bundle over a compact Riemannian manifold (M, h) whose Ricci tensor of (M, h) is negative definite.
- If π is biharmonic, then it is harmonic.
- Theorem 3 Let $\pi : (P, g) \to (M, h)$, a principal G-bundle & Ricci tensor of (M, h), non-positive. Assume (P, g), complete, π, finite energy & finite bienergy. If π is biharmonic, then it is harmonic.

§3 (7) Principal G-bundles

For these works, see

Hajime Urakawa,

Biharmonic maps on principal G-bundles over complete Riemannian manifolds of nonpositive Ricci curvature,

Michigan Math. J., **68** (2019), 19–31.

§4 (0) The warped products

We treat the next, harmonic maps and biharmonic maps on the warped product which is a recent work:

Hajime Urakawa, Harmonic maps and biharmonic maps on principal bundles and warped products,

J. Korean Math. Soc., **55** (2018), no. 3, 553–574, accepted in 2018, January.

§4 (1) The warped products

- (Definition of the warped product) I.e., the product manifold $P = M \times F$ for two Riemannian manifolds (M, h), (F, k), and $f \in C^\infty(M)$ on M, define the Riemannian metric

$$g = \pi^* h + f^2\, k. \qquad (16)$$

- The projection $\pi : P = M \times F \ni (x, y) \mapsto x \in M$ is a Riemannian submersion $\pi : (P, g) \to (M, h)$, called the *warped product* of (M, h), (F, k) and a *warping function* $f \in C^\infty(M)$.

§4 (2) The warped products

- The tension field $\tau(\pi) := \sum_{i=1}^{m+\ell}\{\overline{\nabla}_{e_i}\pi_{*}e_i - \pi_*(\nabla^g{}_{e_i}e_i)\}$ of the warped product $\pi : (P, g) \to (M, h)$ with $g = \pi^* h + f^2\, k$, $f \in C^\infty(M)$, $m = \dim M$, is given:

$$\tau(\pi) = \ell\, \frac{\nabla f}{f}, \quad \ell := \dim F, \quad \nabla f := \mathrm{grad} f,$$

$\{e_i\}$ is a locally defined o.n. frame field on (P, g).
- The bitension field is $\tau_2(\pi) := \overline{\Delta}\tau(\pi) - \mathcal{R}^h(\tau(\pi))$,

$$\overline{\Delta} V = -\sum_i \{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i}\}, \qquad (17)$$

$$\mathcal{R}^h V = \sum_i R^h(V, \pi_* e_i)\pi_* e_i, \qquad V \in \Gamma(\pi^{-1}TM).$$

§4 (3) The warped products

- The bitension field $\tau_2(\pi)$ of the warped product π is

$$\tau_2(\pi) = \overline{\Delta}(\tau(\pi)) - \rho^h(\tau(\pi)) - \ell\, \overline{\nabla}_{\frac{\nabla f}{f}} \tau(\pi), \qquad (18)$$

- $\overline{\Delta}$ is the rough Laplacian acting on $\Gamma(\pi^{-1}TM)$,
  $\overline{\nabla}$ is the induced connection from the Levi-Civita connection ∇^h of (M, h), and
  ρ^h is the Ricci transform of (M, h).
- If the warped product π is *biharmonic*, $\tau_2(\pi) = 0$,

$$J_{\mathrm{id}}(\tau(\pi)) = \ell\, \overline{\nabla}_{\frac{\nabla f}{f}} \tau(\pi), \qquad (19)$$

$J_{\mathrm{id}} := \overline{\Delta} - \rho^h$, Jacobi operator of the id. of (M, h).

§4 (4) The warped product

- If (M, h) is a compact Riemannian manifold whose the Ricci transform ρ^h is non-positive, then the Jacobi operator J_{id} by

$$J_{\mathrm{id}}(V) := \overline{\Delta} V - \rho^h(V), \quad V \in \Gamma(TM) \qquad (20)$$

  is a non-negative operator acting on $\Gamma(TM)$.
- Therefore, if (M, h) is a compact Riemannian manifold of non-positive Ricci curvature ρ^h, then

$$0 \leq \int_M \langle J_{\mathrm{id}}(\tau(\pi)), \tau(\pi)\rangle\, v_h = \ell^3 \int_M \langle \overline{\nabla}_{\frac{\nabla f}{f}} \frac{\nabla f}{f}, \frac{\nabla f}{f}\rangle\, v_h$$

which is a restriction to $f \in C^\infty(M)$. (21)

§4 (5) The warped product

- Let $(P, g) = F \times_f \mathbb{R}$, the warped product with the fiber space (F, k) over the standard line $(\mathbb{R}, dt^2)$,

$$g = \pi^*(dt^2) + f^2\, k. \qquad (22)$$

-

$$\tau_2(\pi) = J_{\mathrm{id}}\Big(\ell\, \frac{\nabla f}{f}\Big) - \ell^2\, \nabla^h{}_{\frac{\nabla f}{f}} \frac{\nabla f}{f} = -\ell\, \Big(\frac{f'}{f}\Big)'' - \ell^2\, \nabla^h{}_{\frac{f'}{f}} \frac{f'}{f}$$

$$= -\ell \Big(\frac{f'}{f}\Big)'' - \ell^2 \Big(\frac{f' f''}{f^2} - \frac{f'^3}{f^3}\Big)$$

$$= -\frac{\ell}{f^3}\Big(f'''\, f^2 + (\ell - 3)\, f''\, f'\, f + (-\ell + 2)\, f'^3\Big). \qquad (23)$$

§4 (6) The warped product

- For $f \in C^\infty(\mathbb{R})$ and (F, k), the warped product $\pi : (P, g) \to (\mathbb{R}, dt^2)$ with $g = \pi^*(dt^2) + f^2\, k$ over $(\mathbb{R}, dt^2)$, is biharmonic, $\tau_2(\pi) = 0$ if and only if

$$f'''\, f^2 + (\ell - 3)\, f''\, f'\, f + (-\ell + 2)\, f'^3 = 0. \qquad (24)$$

- To solve the ODE (24), put

$$u := (\log f)' = \frac{f'}{f}.$$

- Then, (24) is equivalent to

$$u'' + \frac{\ell}{2}\, (u^2)' = 0. \qquad (25)$$

§4 (7) The warped product

- A general solution u of (25) is

$$u(t) = a\ \tanh\Big[a\, \frac{\ell}{2}\, t + b\Big], \qquad (26)$$

  where a and b are arbitrary constants. Then,
- Theorem For a compact Riemannian manifold (F, k) and a C^∞ function f on $\mathbb{R}$ given by

$$f(t) = c\ \exp\Big(\int_{t_0}^{t} a\ \tanh\Big[a\, \frac{\ell}{2}\, r + b\Big]\, dr\Big) \qquad (27)$$

  where $a \neq 0$, b, $c > 0$ are arbitrary constants,
- the warped product $\pi : (\mathbb{R} \times_f F, g) \to (\mathbb{R}, dt^2)$ with $g = \pi^* dt^2 + f^2\, k$, is biharmonic but not harmonic.

§5 (1) Pseudo harmonic maps

- For two Riemannian manifolds (M^{2n+1}, g_θ), (N, h), and for $f \in C^\infty(M, N)$, let the pseudo energy be

$$E_b(f) = \frac{1}{2} \int_M \sum_{i=1}^{2n} (f^* h)(X_i, X_i)\, v_{g_\theta},$$

  where $\{X_i\}$ is an o.n. frame field on $(H(M), g_\theta)$.
- (the first variation formula)

$$\frac{d}{dt}\Big|_{t=0} E_b(f_t) = -\int_M h(\tau_b(f), V)\, v_{g_\theta},$$

  where $\tau_b(f) = \sum_{i=1}^{2n} B_f(X_i, X_i)$, the pseudo tension field, $B_f(X, Y)$, the second fundamental form.

§5 (2) Pseudo harmonic maps

- (the second variation formula)
$$\frac{d^2}{dt^2}\bigg|_{t=0} E_b(f_t) = \int_M h(J_b(V), V)\, v_{g_\theta},$$
where $J_b(V) = \Delta_b V - \mathcal{R}_b(V)$,
$$\Delta_b V = -\sum_{i=1}^{2n} \{\overline{\nabla}_{X_i}(\overline{\nabla}_{X_i} V) - \overline{\nabla}_{\nabla_{X_i} X_i} V\},$$
$$\mathcal{R}_b(V) = \sum_{i=1}^{2n} R^h(V, df(X_i))df(X_i).$$
Here, $\overline{\nabla}$ is the induced connection of ∇^h, ∇ is the Tanaka-Webster connection.
- The pseudo bienergy is
$$E_{b,2}(f) = \frac{1}{2}\int_M h(\tau_b(f), \tau_b(f))\, v_{g_\theta}, \quad v_{g_\theta} = \theta \wedge (d\theta)^n.$$

§5 (3) Pseudo biharmonic maps

- (the first variation formula of $E_{b,2}$)
$$\frac{d}{dt}\bigg|_{t=0} E_{b,2}(f_t) = -\int_M h(\tau_{b,2}(f), V)\, v_{g_\theta},$$
- where $\tau_{b,2}(f)$ is the pseudo bitension field given by
$\tau_{b,2}(f) = \Delta_b(\tau_b(f)) - \sum_{i=1}^{2n} R^h(\tau_b(f), df(X_i))df(X_i)$.
- A C^∞ map $f : (M, g_\theta) \to (N, h)$ is pseudo biharmonic if $\tau_{b,2}(f) = 0$.
- A pseudo harmonic map is pseudo biharmonic.
- (CR analogue of the g. Chen's conjecture):
If (N, h) has non-positive curvature, then every pseudo biharmonic isometric immersion $f : (M, g_\theta) \to (N, h)$ must be pseudo harmonic.

§6 (1) Geometry of foliated maps

- Let φ, a foliated map of $(M, g, \mathcal{F})$ into $(M', g', \mathcal{F}')$, i.e., $\forall$ leaf L of $\mathcal{F}$, $\exists$ a leaf L' of $\mathcal{F}'$, $\varphi(L) \subset L'$.
- $\sigma : Q \to L^\perp$, a bundle map s.th. $\pi \circ \sigma = \mathrm{id}$.
- Let $d_T\varphi := \pi' \circ d\varphi \circ \sigma;\ Q \to Q'$ be a bundle map:
$$Q \xrightarrow{\sigma} L^\perp \subset TM \xrightarrow{d\varphi} TM' \xrightarrow{\pi'} Q'.$$
Here, $Q^* \subset T^*M$, $\pi : TM \to Q = TM/L$,
$\pi' : TM' \to Q' = TM'/L'$.
- Then, $d_T\varphi \in \Gamma(Q^* \otimes \varphi^{-1}Q')$.

§6 (2) First variation, Chiang-Wolak, Jung

- The transversal energy $E_{\mathrm{tr}}(\varphi) := \frac{1}{2}\int_M |d_T\varphi|^2\, v_g$.
- $\forall$ C^∞ foliated variation $\{\varphi_t\}$ with $\varphi_0 = \varphi$ and $\frac{d\varphi_t}{dt}|_{t=0} = V \in \varphi^{-1}Q'$,
$$\frac{d}{dt}\bigg|_{t=0} E_{\mathrm{tr}}(\varphi_t) = -\int_M \langle V, \tau_{\mathrm{tr}}(\varphi)\rangle\, v_g.$$
- Here, $\tau_{\mathrm{tr}}(\varphi)$ is the transversal tension field def. by
$$\tau_{\mathrm{tr}}(\varphi) := \sum_{a=1}^{q} (\widetilde{\nabla}_{E_a} d_T\varphi)(E_a).$$
Here, $\widetilde{\nabla}$ is the induced connection in $Q^* \otimes \varphi^{-1}Q'$ from the Levi-Civita connection of (M', g'), and $\{E_a\}_{a=1}^{q}$ is a local orthonormal frame field on Q.
- A C^∞ foliated map $\varphi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is said to be transversally harmonic if $\tau_{\mathrm{tr}}(\varphi) \equiv 0$.

§6 (3) Second variation formula

- For every transversally harmonic map $\varphi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$, let $\varphi_{s,t} : M \to M'$, any foliated variation of φ with $V = \frac{\partial\varphi_{s,t}}{\partial s}|_{(s,t)=(0,0)}$, $W = \frac{\partial\varphi_{s,t}}{\partial t}|_{(s,t)=(0,0)}$ and $\varphi_{0,0} = \varphi$,
$$\frac{\partial^2}{\partial s\partial t}\bigg|_{(s,t)=(0,0)} E_{\mathrm{tr}}(\varphi_{s,t}) = \int_M \langle J_{\mathrm{tr},\varphi}(V), W\rangle\, v_g,$$
- Here, for $V \in \Gamma(\varphi^{-1}Q')$,
$$J_{\mathrm{tr},\varphi}(V) := \overline{\nabla}^*\overline{\nabla}V - \overline{\nabla}_\tau V - \mathrm{trace}_Q R^{Q'}(V, d_T\varphi)d_T\varphi$$
$$= -\sum_{a=1}^{q} (\overline{\nabla}_{E_a}\overline{\nabla}_{E_a} - \overline{\nabla}_{\nabla_{E_a}E_a})V$$
$$- \sum_{a=1}^{q} R^{Q'}(V, d_T\varphi(E_a))d_T\varphi(E_a).$$
- We want the condition to have $\int_M \langle \overline{\nabla}_\tau V, V\rangle\, v_g = 0$.

§6 (4) Transversal bitension field and transversally biharmonic maps

- The transversal bitension field $\tau_{\mathrm{tr},2}(\varphi)$ of a smooth foliated map φ is defined by
$\tau_{2,\mathrm{tr}}(\varphi) := J_{\mathrm{tr},\varphi}(\tau_{\mathrm{tr}}(\varphi))$.
- The transversal bienergy $E_{2,\mathrm{tr}}$ of a smooth foliated map φ is defined by $E_{2,\mathrm{tr}}(\varphi) := \frac{1}{2}\int_M |\tau_{\mathrm{tr}}(\varphi)|^2\, v_g$.
- A smooth foliated map $\varphi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is said to be transversally biharmonic if $\tau_{2,\mathrm{tr}}(\varphi) \equiv 0$.

§7 (1) Rigidity of pseudo biharmonic maps

- We want to show
  Theorem 1. Let φ be a pseudo biharmonic map of a complete strictly pseudoconvex CR manifold (M, g_θ) into a Riemannian manifold (N, h) of non-positive curvature.
- If $E_{b,2}(\varphi) < \infty$ and $E_b(\varphi) < \infty$,
  then φ is pseudo harmonic.

§7 (2) Refs. on pseudo (bi-)harmonic maps

- (1) (pseudo harmonic): E. Barletta, S. Dragomir & H. Urakawa, *Pseudoharmonic maps from a non-degenerate CR manifolds into a Riemannian manifold*,
  Indiana Univ. Math. J., **50** (2001), 719–746.
- (2) (pseudo biharm.): S. Dragomir, S. Montaldo, *Subelliptic biharmonic maps*,
  J. Geom. Anal., **24** (2014), 223–245.
- (3) (CR rigidity): H. Urakawa, *CR rigidity of pseudo harmonic maps and pseudo biharmonic maps*,
  Hokkaido Math. J., **46** (2017), 141–187.

§8 (1) Rigidity of transversally biharmonic maps

- **The generalized Chen's conjecture for foliated Riemannian manifolds**:

  For every transversally biharmonic map from a foliated Riemannian manifold into another foliated Riemannian manifold whose transversally sectional curvature is non-positive.
- Then, it must be transversally harmonic.

  We want to show

§8 (2) Rigidity of transversally biharmonic maps.

- Theorem 2. Let φ be a C^∞ foliated map of a foliated Riemannian manifold $(M, g, \mathcal{F})$ into a foliated Riemannian manifold $(M', g', \mathcal{F}')$ satisfying conservation law and
  transversally volume preserving.
- Assume that (M, g) is complete and the transversal sectional curvature of $(M', g', \mathcal{F}')$ is non-positive.
- Then, if φ is transversally biharmonic with finite transversal energy and finite transversal 2-energy, then φ is transversally harmonic.

§8 (3) Rigidity of transversally biharmonic maps.

- Let $\varphi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$, a C^∞ fol. map.
- Let $\alpha(X, Y)$ $(X, Y \in \Gamma(L))$, the sec. f. form of $\mathcal{F}$,
  $$\alpha(X, Y) = \pi(\nabla^Q_X Y), \quad (X, Y \in \Gamma(L)),$$
  $\pi : TM \to Q$, $Q = TM/L$, L, tangent bundle of $\mathcal{F}$.
  The tension field τ of $\mathcal{F}$,
  $$\tau = \sum_{i,j=1}^p g^{ij}\alpha(X_i, X_j), \ (\{X_i\}_{i=1}^p \text{ spanns } \Gamma(L)).$$
- $\mathcal{F}$ is trans. volume preserving if $\operatorname{div}(\tau) = 0$.
- φ satisfies conservation law if $\{E_a\}$ $(a = 1, \dots, q)$, a local o.n. frame field of $\Gamma(Q)$,
  $$\operatorname{div}_{\overline{\nabla}} S(\varphi)(\cdot) = \sum(\overline{\nabla}_{E_a} S(\varphi))(E_a, \cdot) = 0,$$
  $S(\varphi) = \frac{1}{2}|d_T\varphi|^2 g_Q - \varphi^* g_{Q'}$, transver. stress-energy.

§8 (4) Rigidity of transversally biharmonic maps

This work is due to

S. Ohno, T. Sakai, and H. Urakawa,

Rigidity of transversally biharmonic maps between foliated Riemannian manifolds,

Hokkaido Mathematical Journal, Vol. 47 (2018), 1-18,

(accepted in Hokkaido Mathematical Journal, October 27, 2016.)

§9 (1) The Riemannian submersions

- Let us recall the Riemannian submersion setting: A C^∞ mapping of (P, g) into (M, h) is a *Riemannian submersion* if **(0)** π, surjective,
  (1) $d\pi = \pi_* : \ T_uP \to T_{\pi(u)}M$, surjective,
  (2) $T_uP = \mathcal{V}_u \oplus \mathcal{H}_u$, orthogonal decomposition,
  (3) $\mathcal{V}_u = \mathrm{Ker}(\pi_{*\,u})$, and
  (4) $\pi_*|_{\mathcal{H}_u} : \ (\mathcal{H}_u, g_u) \to (T_{\pi(u)}M, h_{\pi(u)})$, onto isometry, $\quad (\forall u \in P)$.
- A Riemannian metric g on P is *adapted* if $g = \pi^* h + k$ where k is the Riemannian metric on each fiber $\pi^{-1}(x)$, $(x \in M)$. We call $\mathcal{V}_u$, the *vertical subspace*, $\mathcal{H}_u$, the *horizontal subspace*.

§9 (2) Adapted local o.n. frame fields

- We assume $\mathbf{dim}(\mathcal{V}_u) = \mathbf{1}$ $(u \in P)$, for simplicity.
- Let $\{e_1, \ e_1, \ \ldots, \ e_m\}$, an adapted l.o.n. frame field, being $e_m = e_{n+1}$, vertical, $\{e_1, \ldots, e_n\}$, the basic o.n. frame field on (P, g) corresp. to an o.n. frame field $\{\epsilon_1, \ \epsilon_2, \ \ldots, \ \epsilon_n\}$ on (M, g). Here, $Z = X^* \in \mathfrak{X}(P)$ is *basic* if Z is the horizontal lift of $X \in \mathfrak{X}(M)$.
- $[V, Z]$ is vertical on P if Z is basic and V is vertical (cf. O'Neill, Michigan M.J.**13** (1966), 459–469). So, $[e_i, e_{n+1}] = \kappa_i\, e_{n+1}, \ \kappa_i \in C^\infty(P)$ $(i = 1, \ldots, n)$.
- Being X^*, the horizontal lift on P of $X \in \mathfrak{X}(M)$, $[X^*, Y^*]$ is π-related to $[X, Y] \in \mathfrak{X}(M)$. Let us write $[e_i, e_j] = \sum_{k=1}^{n+1} D_{ij}^k\, e_k$, $D_{ij}^k \in C^\infty(P)$ $(1 \leq i, \ j \leq n)$.

§9 (3) The tension field and bitension field

- $\pi : \ (P, g) \to (M, h)$ is to be *harmonic* if $\tau(\pi) = \mathbf{0}$, and *biharmonic* if $\tau_2(\pi) = J(\tau(\pi)) = \mathbf{0}$.
- The Jacobi operator J for the projection π by $J(V) := \overline{\Delta} V - \mathcal{R}(V), \qquad V \in \Gamma(\pi^{-1}TM)$.
- $\overline{\Delta} V := -\sum_{i=1}^{p} \left\{ \overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V \right\} = \overline{\Delta}_{\mathcal{H}} V + \overline{\Delta}_{\mathcal{V}} V$,
  $\overline{\Delta}_{\mathcal{H}} V := -\sum_{i=1}^{m} \left\{ \overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V \right\}$,
  $\overline{\Delta}_{\mathcal{V}} V := -\sum_{i=1}^{k} \{ \overline{\nabla}_{A^*_{m+i}}(\overline{\nabla}_{A^*_{m+i}} V) - \overline{\nabla}_{\nabla_{A^*_{m+i}} A^*_{m+i}} V \}$.
  Here, $\{e_i\}_{i=1}^{p}$, a local o.n. frame field on (P, g) s. th. $\{e_i\}_{i=1}^{m}$, a local o.n. horizontal field, and $\{e_{m+i}\}_{i=1}^{k}$, the one on the vertical sp. $\mathcal{V}$, $(p = m + k, \ k = 1)$.

§9 (4) The tension field and the bitension field (2)

- We obtain:
  Theorem Let $\pi : \ (P, g) \to (M, h)$ be a Riemannian submersion over (M, h). Then,
  (1) The *tension field* $\tau(\pi)$ of π is given by $\tau(\pi) = -\sum_{i=1}^{n} \kappa_i \epsilon_i$. Here $\kappa_i \in C^\infty(P)$, $(i = 1, \ldots, n)$.
- **(2)** The *bitension field* $\tau_2(\pi)$ of π is given by
  $$\tau_2(\pi) = -\overline{\Delta}^h \left(\sum_{j=1}^{n} \kappa_j \epsilon_j \right) + \nabla^h_{(\sum_{i=1}^{n} \kappa_i \epsilon_i)} \sum_{j=1}^{n} \kappa_j \epsilon_j + \mathrm{Ric}^h\left(\sum_{j=1}^{n} \kappa_j \epsilon_j \right).$$

§9 (5) The Riemannian submersions (1)

We obtain:
Theorem 1 Let $\pi : \ (P, g) \to (M, h)$ be a compact Riemannian submersion over a weakly stable Einstein manifold (M, g) whose Ricci tensor ρ^h satisfies $\rho^h = c\,\mathrm{Id}$ for some constant c.
Assume that π is biharmonic, i.e.,
$$\tau_2(\pi) = -\overline{\Delta}^h X + \nabla^h_X X + \mathrm{Ric}^h(X) = \mathbf{0},$$
where $X = \sum_{i=1}^{n} \kappa_i \epsilon_i$, and assume that $\mathbf{div}(X) = \mathbf{0}$.
Then, we have
$$\overline{\Delta}^h X = cX, \quad \text{and} \quad \nabla^h_X X = \mathbf{0}.$$

§9 (6) The Riemannian submersions (2)

- *Theorem 2.* Let $\pi : \ (P, g) \to (M, h)$, a compact Riemannian submersion over a compact Hermitian symmetric space $(M, h) = (K/H, h)$, K, a compact semi-simple Lie group, H, a closed subgroup of K, and h, an invariant metric on M. Let $X \in \mathfrak{k}$ be an invariant vector field on M. Then, $\mathrm{div}(X) = \mathbf{0}$, and
  $$\overline{\Delta}^h X = cX, \quad \text{and} \quad \nabla^h_X X = \mathbf{0}.$$
- *Corollary.* If $\pi : \ (P, g) \to (M^n, h)$, S^1- bundle over a compact Herm. symm. sp. (M, h), then we have $\tau(\pi) = -\sum_{j=1}^{n} \kappa_j \overline{\epsilon}_j$. Assume $X = \sum_{i=1}^{n} \kappa_j \epsilon_j \not\equiv \mathbf{0}$ is a Killing v. field. Then π is biharm. but not harmonic.

§9 (7) The Riemannnian submersions (3)

- (M. Obata) Let (M, h), a compact Kähler-Einstein, $\lambda_1(h) > 0$, the first eigenvalue. Then, $\lambda_1(h) \geq 2c$. If $\lambda_1(h) = 2c$, f, eigenfun., then ∇f, analytic v. f., $J_{\mathrm{Id}}(\nabla f) := \overline{\Delta}^h (\nabla f) - 2\,\mathrm{Ric}^h(\nabla f) = 0$.
- *Theorem 3* Let $\pi : (P, g) \to (M, h)$, a compact Riemannian submersion. For $X = \tau(\pi)$, assume $X = \nabla f$, where $f \in C^\infty(M)$, with $\Delta^h f = 2c\, f$.
- Let $X = \nabla f = \sum_{i=1}^n \kappa_i \epsilon_i \in \mathfrak{X}(M)$, $\{\epsilon_i\}_{i=1}^n$, o.n. frame, $\{e_i\}_{i=1}^{n+1}$, o.n. on (P, g), with vertical v.f. e_{n+1}. Then,
- X, an analytic vector field on M, $J_{\mathrm{Id}}(X) = 0$, $\overline{\Delta}^h X = cX$, $\nabla^h_X X = 0$, and $\mathrm{div}(X) = \sum_{i=1}^n e_i \kappa_i$.

§9 (8) The Riemannnian submersions (4)

- $(M, h) = (K/T, h)$, Kähler-Einstein flag mfd. with $\mathrm{Ric}^h = c\,\mathrm{Id}$ $(c > 0)$, and $T \subset K = SU(r+1)$, a maximal torus, $\lambda_I : T \to S^1$, a homomorphism,

$$\lambda_I : \begin{bmatrix} e^{2\pi\sqrt{-1}\theta_1} & & O \\ & \ddots & \\ O & & e^{2\pi\sqrt{-1}\theta_{r+1}} \end{bmatrix} \mapsto e^{2\pi\sqrt{-1}(a_1\theta_1 + \cdots + a_{r+1}\theta_{r+1})}.$$

- For $\lambda = \lambda_I$, let $P = \mathcal{S}_\lambda = SU(r+1) \times S^1/\sim$, the S^1 bundle over K/T, where the equivalence relation is given by: $(x', e^{2\pi\sqrt{-1}\theta'}) \sim (x, e^{2\pi\sqrt{-1}\theta})$ iff $x' = xt$ and $e^{2\pi\sqrt{-1}\theta'} = e^{2\pi\sqrt{-1}\theta}\lambda_I(t^{-1})$.

§9 (9) The Riemannnian submersions (5)

- Let

$$K = SU(2) \subset T = \left\{ \begin{bmatrix} e^{2\pi\sqrt{-1}\theta} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\theta} \end{bmatrix} \mid \theta \in \mathbb{R} \right\},$$

$\dim(M) = \dim(K/T) = 2$.

- For a_1, $a_2 \in \mathbb{Z}$ and $\ell = a_1 - a_2$, let

$$\lambda_I : T \ni \begin{bmatrix} e^{2\pi\sqrt{-1}\theta} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\theta} \end{bmatrix} \mapsto e^{2\pi\sqrt{-1}\ell\theta} \in S^1$$

and T acts on $SU(2) \times S^1$ by

$$(x, e^{2\pi\sqrt{-1}\xi}) \cdot a := (xa, e^{2\pi\sqrt{-1}\ell\theta}\, e^{2\pi\sqrt{-1}\xi}),$$

$$a = \begin{bmatrix} e^{2\pi\sqrt{-1}\theta} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\theta} \end{bmatrix} \in T,\ x \in SU(2),\ \xi \in \mathbb{R}.$$

§9 (10) The Riemannnian submersions (6)

- We have a S^1 bundle: $\pi :\ P = \mathcal{S}_{\lambda_I} \to M = K/T = SU(2)/S^1 = P^2(\mathbb{C})$.
- Let $\langle\cdot,\cdot\rangle$, the inner product on $\mathfrak{k}$ defined by

$$\langle X, Y\rangle = -\tfrac{1}{2}\mathrm{Tr}(XY)\ X,\ Y \in \mathfrak{k},$$

$$\mathfrak{k} = \mathfrak{su}(2) = \left\{ X \in \mathfrak{gl}(2,\mathbb{C}) | {}^t\overline{X} + X = 0, \mathrm{Tr}(X) = 0 \right\},$$

and let h, the $SU(2)$-invariant Riemannian metric on $M = K/T = P^2(\mathbb{C})$ induced from $\langle\cdot,\cdot\rangle$, where

$$\mathfrak{t} = \left\{ \begin{bmatrix} \sqrt{-1}\,\theta & 0 \\ 0 & -\sqrt{-1}\,\theta \end{bmatrix} \mid \theta \in \mathbb{R} \right\},$$

$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & -\overline{z} \\ z & 0 \end{bmatrix} \mid z \in \mathbb{C} \right\}, \quad \text{and} \quad \mathfrak{k} = \mathfrak{t} \oplus \mathfrak{m}.$$

§9 (11) The Riemannnian submersions (7)

- Let $\{H_1, X_1, X_2\}$, an o.n. basis of $\mathfrak{k}$ w.r.t. $\langle\cdot,\cdot\rangle$ by

$$H_1 = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix},\ X_1 = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{which satisfy that}$$

$$[H_1, X_1] = 2X_2,\ [X_2, H_1] = 2X_1,\ [X_1, X_2] = 2H_1.$$

- Let us take a local coordinate around $k \in SU(2)$, $SU(2) \ni k\exp(sX_1 + tX_2)\exp(uH_1) \mapsto (s, t, u)$.
- Let us take a locally defined o.n.frame field $\{e_i\}_{i=1}^3$, on $SU(2)$ by

$$e_1 = a\frac{\partial}{\partial s} + b\frac{\partial}{\partial t},\ e_2 = c\frac{\partial}{\partial s} + d\frac{\partial}{\partial t},\ e_3 = e^{C\ell(\ell-1)u(As+Bt)}\frac{\partial}{\partial u},$$

for constants a, b, c, d, A, B, C.

§9 (12) The Riemannnian submersions (8)

- For $X = \tau(\pi) = -(\kappa_1\overline{\epsilon_1} + \kappa_2\overline{\epsilon_2})$, $\{e_1, e_2, e_3\}$ satisfy $\mathcal{V}_p = \mathbb{R}\, e_{3\,p}$, $\mathcal{H}_p = \mathbb{R}\, e_{1\,p} \oplus \mathbb{R}\, e_{2\,p}$ $(p \in P)$, and

$$[e_i, e_3] = \kappa_i\, e_3 \quad (i = 1, 2),$$

where $\kappa_i \in C^\infty(P)$ satisfy

$$\kappa_1 = C\ell(\ell-1)u(aA + bB),$$
$$\kappa_2 = C\ell(\ell-1)u(cA + dB), \quad \text{and}$$
$$\mathrm{div}(X) = e_1\,\kappa_1 + e_2\,\kappa_2 \equiv 0.$$

- $X = \tau(\pi) = -(\kappa_1\overline{\epsilon_1} + \kappa_2\overline{\epsilon_2})$
  $= -C\ell(\ell-1)u\{(aA + bB)\overline{\epsilon_1} + (cA + dB)\overline{\epsilon_2}\}$.
- If $\ell = 0, 1$, $X = \tau(\pi) = 0$, $\pi : \mathcal{S}_{\lambda_I} \to P^1(\mathbb{C})$ is harm.
  If $\ell \geq 2$, $X = \tau(\pi) \not\equiv 0$, $\overline{\Delta}^h X = \frac{1}{2}X$, $\nabla^h_X X = 0$, $\pi : \mathcal{S}_{\lambda_I} \to P^1(\mathbb{C})$ is biharmonic, but not harmonic.

§9 (13) Harmonic maps and biharmonic Riemannian submersions

The above example, Theorem 2, and its Corollary are the first examples of
proper biharmonic compact Riemannian submersions over compact Riemannian symmetric spaces.

This work is due to:

Hajime Urakawa, Harmonic maps and biharmonic Riemannian submersions,
Note di Mat., **39** (1) (2019),1–24;
a preprint, 2018, February.

§10 (1) Biharmonic Hermitian vector bundles over compact Kaehler manifolds and compact Einstein manifolds

- **Thm 1** Let $\pi : (E, g) \to (M, h)$, a vector bundle over a compact Kaehler Einstein Riemannian manifold. If π is biharmonic, then it is harmonic.
- **Thm 2** Let $\pi : (E, g) \to (M, h)$, a biharmonic vector bundle over a compact Einstein manif. with $\mathbf{Ric}^h = c$ $(c > 0)$. Then, either (i) π is harmonic, (ii) $f_0 := \langle \tau(\pi), \tau(\pi) \rangle$, constant, or (iii) $0 < \frac{n}{n-1}\, c \leq \lambda_1(M, h) \leq \frac{2c}{1-X}$, where
$$0 < X := \frac{1}{\mathrm{Vol}(M,h)} \left(\int_M f_0\, v_h \right)^2 / \int_M f_0{}^2\, v_h < 1.$$

§10 (2) Biharmonic Hermitian vector bundles over compact Kaehler manifolds and compact Einstein manifolds

Proposition 1

- Let $\pi : (E, g) \to (M, h)$, a vector bundle over a compact Kaehler Einstein manifold (M, h). Assume that π is biharmonic. Then, we have:
- (i) the tension field $\tau(\pi)$ satisfies that
$$\overline{\nabla}_{X'} \tau(\pi) = 0 \quad (\forall\ X' \in \mathfrak{X}(M)).$$
- (ii) The pointwise norm $|\tau(\pi)|^2$ is constant, say d.
- (iii) The bienergy $E_2(\pi)$ satisfies that
$$E_2(\pi) := \frac{1}{2} \int_M |\tau(\pi)|^2\, v_h = \frac{d}{2}\, \mathrm{Vol}(M, h).$$

§10 (3) Proof of Proposition 1.

- For an Hermitian vector bundle $\pi : (E, g) \to (M, h)$, $\dim E = m$, $\dim M = n$, recall the defs. of $\tau(\pi)$, $\tau_2(\pi)$:
$$\tau(\pi) = \sum_{j=1}^m \left\{ \overline{\nabla}^h_{e_j} \pi_* e_j - \pi_* \left(\nabla^g{}_{e_j} e_j \right) \right\},$$
- $$\begin{aligned} \tau_2(\pi) &= \overline{\Delta} \tau(\pi) - \sum_{j=1}^m R^h(\tau(\pi), \pi_* e_j) \pi_* e_j, \\ &= \overline{\Delta} \tau(\pi) - \sum_{j=1}^n R^h(\tau(\pi), e'_j) e'_j \\ &= \overline{\Delta} \tau(\pi) - \mathbf{Ric}^h(\tau(\pi)), \qquad (1) \end{aligned}$$

where $\{e_i\}_{i=1}^m$, $\{e'_j\}_{j=1}^n$, loc. o.n. on (E, g), (M, h) s.th. $\pi_* e_j = e'_j$ $(1 \leq j \leq n)$, $\pi_*(e_j) = 0$ $(n+1 \leq j \leq m)$.

§10 (4) Proof of Proposition 1.

- Let (M^n, h), compact Kaehler Einstein manifold, $\mathbf{Ric}^h = c\, \mathbf{Id}$. Due to (1), $\pi : (E, g) \to (M, h)$, biharmonic iff
$$\overline{\Delta} \tau(\pi) = c\, \tau(\pi). \qquad (2)$$
Then, since the Laplacian $\Delta^h = -\sum (e'^2_j - \nabla_{e'_j} e'_j)$ on $C^\infty(M)$, we have
- $$\begin{aligned} &\Delta^h \langle \tau(\pi), \tau(\pi) \rangle \\ &= 2 \langle \overline{\Delta} \tau(\pi), \tau(\pi) \rangle - 2 \sum_{j=1}^n \langle \overline{\nabla}_{e'_j} \tau(\pi), \overline{\nabla}_{e'_j} \tau(\pi) \rangle \\ &\leq 2 \langle \overline{\Delta} \tau(\pi), \tau(\pi) \rangle, \qquad (3) \end{aligned}$$

because of $\langle \overline{\nabla}_{e'_j} \tau(\pi), \overline{\nabla}_{e'_j} \tau(\pi) \rangle \geq 0$.

§10 (5) Proof of Propisition 1.

- Assume that $\pi : (E, g) \to (M, h)$, biharmonic. By (2) and (3), we have
$$\Delta^h \langle \tau(\pi), \tau(\pi) \rangle \leq 2c \langle \tau(\pi), \tau(\pi) \rangle. \qquad (4)$$
- Recall the theorem of Obata (cf. Urakawa's book):

Thm Assume that (M, h) is a compact Kaehler manifold (M, h) with Ricci transform ρ^h satisfying $h(\rho^h(u), u) \geq c\, h(u, u)$, $(u \in T_x M)$, (some $c > 0$).

Then, $\qquad \lambda_1(M, h) \geq 2c. \qquad (5)$

If the equality holds in (5), then M admits a non-zero holomorphic vector field.

§10 (6) Proof of Proposition 1.

- Then, we have

$$\lambda_1(M,h) = 2c, \text{ and} \qquad (6)$$

$$\Delta^h \langle \tau(\pi), \tau(\pi)\rangle = 2c\, \langle \tau(\pi), \tau(\pi)\rangle. \qquad (7)$$

- Therefore, we have

$$\sum_{j=1}^n \langle \overline{\nabla}_{e'_j}\tau(\pi), \overline{\nabla}_{e'_j}\tau(\pi)\rangle = 0, \quad \text{i.e.,}$$

$$\overline{\nabla}_{X'}\tau(\pi) = 0, \ (X' \in \mathfrak{X}(M)).$$

- Then, we have

$$X' \langle \tau(\pi), \tau(\pi)\rangle = 2\, \langle \overline{\nabla}_{X'}\tau(\pi), \tau(\pi)\rangle = 0,$$

i.e., $\langle \tau(\pi), \tau(\pi)\rangle$ is constant.

- By (6), (7), $\langle \tau(\pi), \tau(\pi)\rangle \equiv 0$, i.e., $\tau(\pi) \equiv 0$. □

§10 (7) Proof of Theorem 1.

Assume that $\pi : (E,g) \to (M,h)$ is biharmonic.

By Proposition 1, $\overline{\nabla}_{X'}\tau(\pi) = 0 \ (X' \in \mathfrak{X}(M))$. Then,

$$\mathrm{div}(\tau(\pi)) = \sum_{i=1}^n (\overline{\nabla}_{e'_i}\tau(\pi))(e'_i) = 0.$$

For all $f \in C^\infty(M)$,

$$0 = \int_M f\, \mathrm{div}(\tau(\pi))\, v_h = -\int_M \langle \nabla f, \tau(\pi)\rangle\, v_h.$$

We have

$$\tau(\pi) \equiv 0. \quad □$$

§10 (8) Proof of Theorem 2.

- Recall the famous Lichnerowicz-Obata theorem.
- Thm (Lichnerowicz-Obata)

  Let (M,h) be a compact Riemannian manifold (M,h) without boundary.

- Assume that the Ricci curvature of (M,h), Ric^h, is bounded below by a positive constant $c > 0$:

$$\mathrm{Ric}^h \geq c\, \mathrm{Id}.$$

  Then, the first eigenvalue $\lambda_1(M,h)$ satisfies that

- $$\lambda_1(M,h) \geq \frac{n}{n-1}\, c.$$

§10 (9) Proof of Theorem 2.

- Assume that $\pi : (E,g) \to (M,h)$ is biharmonic. Recall that we have

$$\Delta^h \langle \tau(\pi), \tau(\pi)\rangle \leq 2c\, \langle \tau(\pi), \tau(\pi)\rangle. \qquad (4)$$

- I.e., denoting $f_0 := \langle \tau(\pi), \tau(\pi)\rangle \in C^\infty(M)$,

$$\Delta^h f_0 \leq 2c\, f_0.$$

(The first step) Assume that $f_0 \not\equiv 0$, and not a constant. Then, $\int_M {f_0}^2\, v_h > 0$, and

$$2c \geq \frac{\int_M f_0\, (\Delta^h f_0)\, v_h}{\int_M {f_0}^2\, v_h} = \frac{\int_M |\nabla f_0|^2\, v_h}{\int_M {f_0}^2\, v_h}. \qquad (5)$$

§10 (10) Proof of Theorem 2.

- (The second step) Let $f_1 := f_0 - \frac{\int_M f_0\, v_h}{\mathrm{Vol}(M,h)} \in C^\infty(M)$.
- Then, $\int_M f_1\, v_h = 0$, (6)
- $\nabla f_1 = \nabla f_0$, and $|\nabla f_1|^2 = |\nabla f_0|^2$, (7)
- $\int_M {f_1}^2\, v_h = \int_M {f_0}^2\, v_h - \frac{\left(\int_M f_0\, v_h\right)^2}{\mathrm{Vol}(M,h)}$. (8)
- (Schwarz Inequality) For all continuous functions f and g on a compact Riemannian manifold (M,h),

$$\left(\int_M f(x) g(x)\, v_h(x)\right)^2 \leq \int_M f(x)^2\, v_h \int_M g(x)^2\, v_h. \qquad (9)$$

- The equality holds iff there exist constants λ, μ s.th.

$$\lambda f + \mu g = 0. \qquad (10)$$

§10 (11) Proof of Theorem 2.

(The third step) Then, the first eigenvalue $\lambda_1(M,h)$ of (M,h) satisfies that, by (6), we have

- $$\lambda_1(M,h) \leq \frac{\int_M |\nabla f_1|^2\, v_h}{\int_M {f_1}^2\, v_h} = \frac{\int_M |\nabla f_0|^2\, v_h}{\int_M {f_0}^2\, v_h - \frac{\left(f_0{}^2\, v_h\right)^2}{\mathrm{Vol}(M,h)}}. \qquad (11)$$

- By (5), the right hand side of (11) is smaller than or equal to

$$\leq 2c \frac{\int_M {f_0}^2\, v_h}{\int_M {f_0}^2\, v_h - \frac{\left(\int_M f_0\, v_h\right)^2}{\mathrm{Vol}(M,h)}} = 2c \frac{1}{1-X}, \qquad (12)$$

where

$$X := \frac{1}{\mathrm{Vol}(M,h)} \frac{\left(\int_c f_0\, v_h\right)^2}{\int_M {f_0}^2\, v_h}, \quad 0 < X < 1. \qquad (13)$$

§10 (12) Proof of Theorem 2.

- $X < 1$ if and only if
$$\left(\int_M f_0\, v_h\right)^2 < \mathrm{Vol}(M,h)\ \int_M {f_0}^2\, v_h,$$
and
$$0 < X \iff 0 < \int_M f_0\, v_h \iff 0 \not\equiv f_0.$$
- $\lambda_1(M,h) \leq 2c\,\frac{1}{1-X}$ if and only if
$$1 - \frac{2c}{\lambda_1(M,h)} \leq X.$$
- With the Lichinerowicz-Obata, we have $-1 <$
$$1 - 2\frac{n-1}{n} \leq 1 - \frac{2c}{\lambda_1(M,h)} \leq X \leq \frac{1}{\mathrm{Vol}(M,h)}\,\frac{\left(\int_M f_0\, v_h\right)^2}{\int_M {f_0}^2\, v_h} < 1.$$

Therefore, we have Theorem 2. □

Hajime Urakawa,
Biharmonic Hermitian vector bundles over compact Kähler manifolds and compact Einstein Riemannian manifolds,
Note di Matematica,
accepted at November 12, 2019.

§11 (1) Harmonic morphisms from positive curvature spaces onto flag manifolds

- We want to give,
"an infinite family of distinct harmonic morphisms with minimal circle fibers
from the 7-dimensional homogeneous Allof-Wallach spaces of positive curvature
onto the 6-dimensional flag manifolds."
- Fuglede (1978) and Ishihara (1979), independently, initiated harmonic morphism:
- a harmonic morphism $\pi : (P, g) \to (M, h)$ is,
$\forall$ a harmonic function f on $V \subset M$ (open subset),
$f \circ \pi : \pi^{-1}(V) \subset P$ is harmonic.

§11 (2) Harmonic morphisms from positive curvature spaces onto flag manifolds

- **Theorem** Let $(M_{k,\ell}, g_t) = (SU(3)/T_{k,\ell}, g_t)$, $k,\ \ell \in \mathbb{Z},\ (k,\ell) = 1;\ -1 < t < 0$, or $0 < t < \frac{1}{3}$, infinitely many distinct homog. the 7-dim. Allof-Wallach spaces with positive sectional curv.,
- (M, h), the 6-dim. flag manifold $(SU(3)/T, h)$.
- Then, all the Riem. submersions with circle fibers, $\pi : (M_{k,\ell}, g_t) \to (M,h) = (SU(3)/T, h)$ are harmonic morphisms with minimal fibers.
- Here, the subgroups $T_{k,\ell}$ and T of $SU(3)$ and the homogeneous space $M_{k,\ell}$ are given as follows:

§11 (3) Harmonic morphisms from positive curvature spaces onto flag manifolds

- $$T_{k,\ell} := \left\{ \begin{pmatrix} e^{2\pi k i\theta} & 0 & 0 \\ 0 & e^{2\pi i \ell\theta} & 0 \\ 0 & 0 & e^{-2\pi i(k+\ell)\theta} \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\}$$
$$\subset T := \left\{ \begin{pmatrix} e^{2\pi i\theta_1} & 0 & 0 \\ 0 & e^{2\pi i\theta_2} & 0 \\ 0 & 0 & e^{-2\pi i(\theta_1+\theta_2)} \end{pmatrix} \,\middle|\, \theta_1,\ \theta_2 \in \mathbb{R} \right\}$$
$$\subset G := SU(3),$$
- and $M_{k,\ell} := G/T_{k,\ell} = SU(3)/T_{k,\ell}$, where
$H^4(SU(3)/T_{k,\ell}) = \mathbb{Z}/r\,\mathbb{Z}, \quad (r := |k^2 + \ell^2 + k\ell|).$

§11 (4) Harmonic morphisms from positive curvature spaces onto flag manifolds

- Proof of our Theorem is obtained by applying Proposition (Fuglede, Ishihara, see Book, Baird & Wood, p.123) A Riemannian submersion $\pi : (P, g) \to (M, h)$ is a harmonic morphism iff π is harmonic and has minimal fibers.
- Example Let $K \subset H \subset G$ be compact Lie groups. The projection $\pi : (G/K, g) \to (G/H, h)$ is a harmonic Riemannian submersion with totally geodesic fibers if the metrics g and h, induced from the $\mathrm{Ad}(G)$-invariant product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g}$ of G. Our Allof-Wallach's metrics are not.

§11 (5) Harmonic morphisms from positive curvature spaces onto flag manifolds

- (Allof-Wallach's metric g_t on $SU(3)/T_{k,\ell}$) Let $\langle X, Y\rangle_0 := -\mathrm{Re}(\mathrm{Tr}(XY))$, $X, Y \in \mathfrak{g} = \mathfrak{su}(3)$, and $G_1 := \left\{\begin{pmatrix} x & 0 \\ 0 & \det(x^{-1})\end{pmatrix} \mid x \in U(2)\right\} \subset G = SU(3)$.
- Let

$$\mathfrak{m} = \mathfrak{g}_1{}^{\perp} := \left\{\begin{pmatrix} 0 & 0 & z_2 \\ 0 & 0 & z_1 \\ -\overline{z}_2 & -\overline{z}_1 & 0\end{pmatrix} \mid z_1,\ z_2 \in \mathbb{C}\right\},$$

$$\mathfrak{t}_{k,\ell} := \left\{\begin{pmatrix} 2\pi i k\theta & 0 & 0 \\ 0 & 2\pi \ell\theta & 0 \\ 0 & 0 & -2\pi i(k+\ell)\theta\end{pmatrix} \mid \theta \in \mathbb{R}\right\}.$$

§11 (6) Harmonic morphisms from positive curvature spaces onto flag manifolds

- Let

$$V_1 := \mathfrak{t}_{k,\ell}{}^{\perp} \cap \mathfrak{g}_1, \qquad V_2 := \mathfrak{g}_1{}^{\perp} = \mathfrak{m},$$

$$\mathfrak{g} = \mathfrak{su}(3) = \mathfrak{t}_{k,\ell} \oplus V_1 \oplus V_2,$$

be the orthogonal direct decomposition of $\mathfrak{g}$ with respect to the inner product $\langle\ ,\ \rangle_0$.

- For $-1 < t < \infty$, let the new inner product $\langle\ ,\ \rangle_t$ by

$$\langle x_1 + x_2, y_1 + y_2\rangle_t := (1+t)\langle x_1, y_1\rangle_0 + \langle x_2, y_2\rangle_0,$$

$x_i,\ y_i \in V_i\ (i = 1, 2)$. The Allof-Wallach metric g_t is the corresp. G-invariant Riem. metric on $G/T_{k,\ell}$.

§11 (7) Harmonic morphisms from positive curvature spaces onto flag manifolds

- Let $\pi : (P^m, g) \to (M^n, h)$, a Riem. submersion. Assume $\dim(\pi^{-1}(x)) = 1$, $(u \in P,\ \pi(u) = x)$. Let $\{e_1, \ldots, e_n, e_{n+1}\}$, a local o.n. frame field s.th. $e_{n+1} = e_m$, vertical, and $\{e_1, \ldots, e_n\}$, basic o.n. frame field on (P, g) corresp. to an o.n. frame field $\{\epsilon_1, \ldots, \epsilon_n\}$ on (M, g). Here, $Z \in \mathfrak{X}(P)$ is basic if Z is horizontal & π-related to $X \in \mathfrak{X}(M)$.
- $[V, Z]$, vert. if Z, basic & V, vert. ([O'Neill], p. 461).
- So, $[e_i, e_{n+1}]$ $(1 \leq i \leq n)$ is vertical. Then,

$$[e_i, e_{n+1}] = \kappa_i\, e_{n+1}, \quad \kappa_i \in C^{\infty}(P) \quad (1 \leq i \leq n).$$

§11 (8) Harmonic morphisms from positive curvature spaces onto flag manifolds

- $\tau(\pi) = -d\pi\left(\nabla_{e_{n+1}} e_{n+1}\right) = -\sum_{i=1}^{n} \kappa_i\, \epsilon_i.$ $(*)$

$$\begin{aligned} \because \quad \tau(\pi) &= \textstyle\sum_{i=1}^{m} \{\nabla^{\pi}_{e_i} d\pi(e_i) - d\pi(\nabla_{e_i} e_i)\} \\ &= \textstyle\sum_{i=1}^{n} \{\nabla^{\pi}_{e_i} d\pi(e_i) - d\pi\{(\nabla_{e_i} e_i)\} \\ &\quad + \nabla^{\pi}_{e_{n+1}} d\pi(e_{n+1}) - d\pi(\nabla_{e_{n+1}} e_{n+1}) \\ &= -d\pi(\nabla_{e_{n+1}} e_{n+1}) = -\textstyle\sum_{i=1}^{n} \kappa_i \epsilon_i. \end{aligned}$$

- Because, for $i,\ j = 1, \ldots, n$, $d\pi(\nabla_{e_i} e_j) = \nabla^{h}_{\epsilon_i} \epsilon_j$, and $\nabla^{\pi}_{e_i} d\pi(e_i) = \nabla^{h}_{d\pi(e_i)} d\pi(e_i) = \nabla^{h}_{\epsilon_i} \epsilon_i$. Thus, we have
  - $\sum_{i=1}^{n} \left\{\nabla^{\pi}_{e_i} d\pi(e_i) - d\pi\left(\nabla_{e_i} e_i\right)\right\} = 0.$
  - $e_{n+1} = e_m$ is vertical, $d\pi(e_{n+1}) = 0$, $\therefore\ \nabla^{\pi}_{e_{n+1}} d\pi(e_{n+1}) = 0.$

§11 (9) Harmonic morphisms from positive curvature spaces onto flag manifolds

- By definition of the Levi-Civita connection ∇, for $i = 1, \ldots, n$,

$$2g(\nabla_{e_{n+1}e_{n+1}}, e_i) = 2g(e_{n+1}, [e_i, e_{n+1}]) = 2\kappa_i,$$

and

$$2g(\nabla_{e_{n+1}} e_{n+1}, e_{n+1}) = 0.$$

Therefore, we have

$$\nabla_{e_{n+1}} e_{n+1} = \textstyle\sum_{i=1}^{n} \kappa_i e_i,$$

and then,

$$d\pi\left(\nabla_{e_{n+1}} e_{n+1}\right) = \textstyle\sum_{i=1}^{n} \kappa_i \epsilon_i.$$

Thus, we obtain the desired equations $(*)$. □

§11 (10) Harmonic morphisms from positive curvature spaces onto flag manifolds

- $\mathfrak{g} = \mathfrak{su}(3) = \mathfrak{t}_{k,\ell} \oplus \mathfrak{m}$, $\mathfrak{m} = V_1 \oplus V_2$, where $V_1 = \{X_0, X_1, X_2\}_{\mathbb{R}}$, $V_2 = \{X_3, X_4, X_5, X_6\}_{\mathbb{R}}$, and
- $X_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$, $X_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$,

$X_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0\end{pmatrix}$, $X_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0\end{pmatrix}$,

$X_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0\end{pmatrix}$, $X_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0\end{pmatrix}$.

§11 (11) Harmonic morphisms from positive curvature spaces onto flag manifolds

- $$X_0 = \frac{1}{\sqrt{5\Gamma}} \begin{pmatrix} 2\ell + k & 0 & 0 \\ 0 & 2m + \ell & 0 \\ 0 & 0 & 2k + m \end{pmatrix},$$
  where $\Gamma := k^2 + \ell^2 + k\ell$, and $m := -k - \ell$.
- The inner product $\langle\ ,\ \rangle_t$ on $\mathfrak{m}$ for the Allof-Wallach metric g_t on $M_{k,\ell} = SU(3)/T_{k,\ell}$ is the one which takes as an orthonormal basis of $\mathfrak{m}$,

$$\left\{ \frac{1}{\sqrt{1+t}} X_0, \frac{1}{\sqrt{1+t}} X_1, \frac{1}{\sqrt{1+t}} X_2, X_3, X_4, X_5, X_6 \right\},$$

$\{e_0^t, e_1^t, e_2^t, e_3^t, e_4^t, e_5^t, e_6^t\}$, loc. o.n. frame f. on $M_{k,\ell}$.

§11 (12) Harmonic morphisms from positive curvature spaces onto flag manifolds

- We have, for each $X \in \mathfrak{X}(M_{k,\ell})$,
  $$g_t(X, \nabla^{g_t}_{e_0^t} e_0^t) = e_0^t g_t(X, e_0^t) + g_t(e_0^t, [X, e_0^t]).$$
- Here, we have
  $$e_0^t g_t(e_i^t, e_0^t) = 0 \quad (i = 0, 1, \ldots, 6),$$
  $$g_t(e_0^t, [X, e_0^t]) = 0 \quad \forall X = X_i \ (i = 0, 1, \ldots, 6).$$
- Thus, we have
  $g_t(X, \nabla^{g_t}_{e_0^t} e_0^t) = 0$, i.e., $\nabla^{g_t}_{e_0^t} e_0^t = 0$. Therefore,
  $\nabla^{g_t}_{e_0^t} e_0^t = 0$, and $\tau(\pi) = -d\pi(\nabla^{g_t}_{e_0^t} e_0^t) = 0$. □

For further studies for biharmonic isometric immersions, it should be developed the works of W.Y. Hsiang into biharmonic maps. For examples,

- W.Y. Hsiang; *On the compact homogeneous minimal submanifolds*,
  Proc. Nat. Acad. Sci. USA, **56** (1966), 5–6.
- W.Y. Hsiang and H.B. Lawson, *Minimal submanifolds of low cohomogeneity*,
  J. Differential Geometry, **5** (1971), 1–58.

Sasaki manifolds, Kähler cone manifolds

- Theorem 1 Let M^m be an m-dim. submanifold of a Sasakian manifold $(N^{2m+1}, h, J, \xi, \eta)$. Then, M is Legendrian in N if and only if $C(M) \subset C(N)$ is Lagrangian in a Kähler cone manifold $(C(N), \overline{h}, I)$.
- (Proof) M is Legendrian in N iff $h(\xi, X) = 0$ and $h(X, JY) = 0$ for all $X, Y \in \mathfrak{X}(M)$. The Kähler form of $C(N)$ is $\Omega = 2r\, dr \wedge \eta + r^2\, d\eta$ which satifsies

  $$\Omega(f_1\Phi + X, f_2\Phi + Y) = r^2\{h(\xi, f_1 Y - f_2 X) + h(X, JY)\}.$$

  Thus, M is Legendrian
  iff the pullback of Ω to $C(M)$ vanishes, i.e.,
  $C(M) \subset C(N)$ is Lagrangian. □

Main Theorem (1)

- Main Theorem 2 Let $\varphi : (M^m, g) \to N$, a Legendrian submanifold of a Sasakian manifold $(N^{2m+1}, h, J, \xi, \eta)$, and let
  $\overline{\varphi} : C(M) \ni (r, x) \mapsto (r, \varphi(x)) \in C(N)$, the Lagrangian submanifold of a Kähler cone manifold.
  Here, $\overline{g} = dr^2 + r^2 g$, $\overline{h} = dr^2 + r^2 h$. Then,
- (1) $\tau(\overline{\varphi}) = \frac{\tau(\varphi)}{r^2}$, i.e., $\overline{\varphi}$ is harmonic iff φ is harmonic.
- (2)
  $$\tau_2(\overline{\varphi}) := J_{\overline{\varphi}}(\tau(\overline{\varphi})) = \frac{J_\varphi(\tau(\varphi))}{r^4} + \frac{m\,\tau(\varphi)}{r^2} = \frac{\tau_2(\varphi)}{r^4} + \frac{m\,\tau(\varphi)}{r^2}.$$
- I.e., φ is harmonic iff $\overline{\varphi}$ is harmonic.
  φ is biharmonic iff $J_{\overline{\varphi}}(\tau(\overline{\varphi})) = m\,\tau(\overline{\varphi})$.

Main Theoem (2)

Corollary 3 Let $\varphi : (M^m, g) \to N$ be a Legendrian submanifold of a Sasakian manifold $(N^{2m+1}, h, J, \xi, \eta)$, $\overline{\varphi} : C(M) \to C(N)$, the Lagrangian submanifold of a Kähler cone manifold. Then,
$\varphi : (M, g) \to N$ is proper biharmonic if and only is $\tau(\overline{\varphi})$ is an eigensection of $J_{\overline{\varphi}}$ with the eigenvalue m.

Here, $J_{\overline{\varphi}}$ is an elliptic operator of the form:
$$J_{\overline{\varphi}} W := \Delta_{\overline{\varphi}} W - \sum_{i=1}^{m+1} R^{C(N)}(W, \overline{\varphi}_* \overline{e}_i)\overline{\varphi}_* \overline{e}_i,$$
$(W \in \Gamma(\overline{\varphi}^{-1} TC(N)))$, and
$R^{C(N)}$ is the curvature tensor of $(C(N), \overline{h})$.

Main Theorem (3)

- Remarks. (1) Recall Takahashi's theorem (1966):
- Theorem Let (M^m, g) be a compact Riemannian manifold, and $\varphi : (M, g) \to (S^n, ds_0^2)$, an isometric immersion: $\varphi = (\varphi_1, \cdots, \varphi_{n+1})$, $\varphi_i \in C^\infty(M)$. Then,

  φ is minimal iff $\Delta_g \varphi_i = m\, \varphi_i \ (i = 1, \cdots, n+1)$.

  Here Δ_g is the non-negative Laplacian of (M, g).
- (2) Recall the work of T. Sasahara: Let $\varphi(u, v) = (e^{iu},\, i\, e^{-iu} \sin(\sqrt{2}\, v),\, i\, e^{-iu} \cos(\sqrt{2}\, v))/\sqrt{2}$.

  Then, φ is a proper biharmonic Legendrian immersion into (S^5, ds_0^2) (cf. T. Sasahara, 2005).

Symplectic Setting for Biharmonic Maps

- Let (N, J, h) be a complex m-dimensional Kähler manifold, and consider a symplectic form on N by $\omega(X, Y) := h(X, JY)$, $X, Y \in \mathfrak{X}(N)$.
- A real submanifold M in N of dimension m is called to be Lagrangian if the immersion $\varphi : M \to N$ satisfies that $\varphi^*\omega \equiv 0$, i.e.,

$$h_x(T_xM, J(T_xM)) = 0 \ (\forall\, x \in M).$$

- Problem

  When is $\varphi : (M, g) \to (N, J, h)$ biharmonic?
  Here, $g := \varphi^* h$.

This work is due to

H. Urakawa, *Sasaki manifolds, Kähler cone manifolds and biharmonic submanifolds*,
arXiv: 1306.6123v2.

Illinois Journal of Mathematics, Vol. 58, No. 2 (2014), 521–535.

Biharmonic Lagrangian submanifolds (1)

Then, we have

- Thm 2 (Maeta & Urakawa) Let (N, J, h), a Kähler manifold, and (M, g), a Lagrangian submanifold.
- Then, it is biharmonic iff $\qquad (m = \dim M)$

$$\mathrm{Tr}_g(\nabla A_{\mathrm{H}}) + \mathrm{Tr}_g(A_{\nabla^\perp_\bullet \mathrm{H}}(\bullet)) - \sum \langle \mathrm{Tr}_g(\nabla^\perp_{e_i} B) - \mathrm{Tr}_g(\nabla^\perp_\bullet B)(e_i, \bullet), \mathrm{H}\rangle\, e_i = 0,$$

$$\Delta^\perp \mathrm{H} + \mathrm{Tr}_g B(A_{\mathrm{H}}(\bullet), \bullet) + \sum \mathrm{Ric}^N(J\mathrm{H}, e_i) J e_i - \sum \mathrm{Ric}(J\mathrm{H}, e_i) J e_i - J\, \mathrm{Tr}_g A_{B(J\mathrm{H}, \bullet)}(\bullet) + m\, J A_{\mathrm{H}}(J\mathrm{H}) = 0.$$

- Ric, Ric^N are the Ricci tensors of (M, g), (N, h).

Biharmonic maps and symplectic geometry

- What is a relation between biharmonic maps and symplectic geometry?
- One can ask: "When are Lagrangian submanifolds biharmonic immersions into a symplectic manifold? "
- Take as a symplectic manifold, a Kähler manifold: "When is its Lagrangian submanifold biharmonic immersion? "

Biharmonic Lagrangian submanifolds (2)

In particular, we have

Thm 3 (Maeta & Urakawa)

- If $(N, J, h) = N^m(4c)$, the complex space form of complex dim m, with constant holomorphic curvature $4c(< 0, = 0, > 0)$, and, (M, g), a Lagrangian submanifold.
- Then it is biharmonic iff

$$\mathrm{Tr}_g(\nabla A_{\mathrm{H}}) + \mathrm{Tr}_g(A_{\nabla^\perp_\bullet \mathrm{H}}(\bullet)) = 0, \qquad (7)$$

$$\Delta^\perp \mathrm{H} + \mathrm{Tr}_g B(A_{\mathrm{H}}(\bullet), \bullet) - (m+3)c\mathrm{H} = 0. \qquad (8)$$

Biharmonic Lagrangian submanifolds (3)

- B.Y. Chen introduced the following two notions on Lagrangian submanif. M in a Kähler manifold N:
- H-umbilic: M is called H-umbilic if M has a local orthonormal frame field $\{e_i\}$ satisfying that

$$B(e_1, e_1) = \lambda\, J e_1, \quad B(e_1, e_i) = \mu\, J e_i,$$
$$B(e_i, e_i) = \mu\, J e_1, \quad B(e_i, e_j) = 0 \ (i \neq j),$$

where $2 \leq i, j \leq m = \dim M$, B is the second f.f. of $M \hookrightarrow N$, and λ, μ are local functions on M.

- PNMC: M has a parallel normalized mean curvature vector field if $\nabla^{\perp}(\frac{\mathbf{H}}{|\mathbf{H}|}) = 0$.

Bubbling phenomena of harmonic maps and biharmonic maps

- For any $C > 0$, let $\mathcal{F} := \{\varphi : (M^m, g) \to (N^n, h)$ smooth harmonic $\mid \int_M |d\varphi|^m v_g \leq C\}$.
- For any $C > 0$, let $\mathcal{F} := \{\varphi : (M^m, g) \to (N^n, h)$ smooth biharmonic $\mid \int_M |d\varphi|^m v_g \leq C$ & $\int_M |\tau(\varphi)|^2 v_g \leq C\}$.
- Question: Are both $\mathcal{F}$ small or big?
- Our answer: a rather surprising.
- Both $\mathcal{F}$ are small! I.e., both $\mathcal{F}$ cause bubblings, kinds of compactness. More precisely,

Our Main Theorem (Maeta & Urakawa)

- Thm Let $\varphi : M \to (N^m(4c), J, h)$ be a Lagrangian H-umbilic PNMC submanifold.
- Then, it is biharmonic iff $c = 1$ and $\varphi(M)$ is congruent to a submanifold of $P^m(4)$ given by

$$\pi\left(\sqrt{\frac{\mu^2}{1+\mu^2}}\, e^{-\frac{i}{\mu}x}, \sqrt{\frac{1}{1+\mu^2}}\, e^{i\mu x} y_1, \cdots, \sqrt{\frac{1}{1+\mu^2}}\, e^{i\mu x} y_m\right)$$

where $x, y_i \in \mathbb{R}$ with $\sum_{i=1}^m {y_i}^2 = 1$.

- Here, $\pi : S^{2m+1} \to P^m(4)$ is the Hopf fibering, and $\mu = \pm\sqrt{\frac{m+5\pm\sqrt{m^2+6m+25}}{2m}}$, $\quad (\lambda = (\mu^2 - 1)/\mu)$.

Previous bubbling result of harmonic maps

- Thm Let (M, g), (N, h) be compact Riem. mfds. $\dim M \geq 3$. For any $C > 0$, let $\mathcal{F} := \{\varphi : (M^m, g) \to (N^n, h)$ smooth harmonic $\mid \int_M |d\varphi|^m v_g \leq C\}$.
- Then, $\forall \{\varphi_i\} \in \mathcal{F}$, $\exists\, \mathcal{S} = \{x_1, \cdots, x_\ell\} \subset M$, and $\exists$ a harmonic map $\varphi_\infty : (M \backslash \mathcal{S}, g) \to (N, h)$ s.th.
- (1) $\varphi_{i_j} \to \varphi_\infty$ in the C^∞-topology on $M \backslash \mathcal{S}$ $(j \to \infty)$,
- (2) The Radon measures $|d\varphi_{i_j}|^m v_g$ converges to a measure given by

$$|d\varphi_\infty|^m v_g + \sum_{k=1}^{\ell} a_k\, \delta_{x_k} \quad (j \to \infty).$$

The above work is due to:

S. Maeta and H. Urakawa,

Biharmonic Lagrangian submanifolds in Kähler manifolds,
Glasgow Math. J., Vol. 55 (2013), 465-480.

arXiv: 1203.4092v2 [math.DG].

Bubbling of biharmonic maps (with N. Nakauchi)

- Thm (Bubbling) Let (M, g), (N, h) be compact Riem. mfds. $\dim M \geq 3$. For any $C > 0$, let $\mathcal{F} := \{\varphi : (M^m, g) \to (N^n, h)$ smooth biharmonic $\mid \int_M |d\varphi|^m v_g \leq C$ & $\int_M |\tau(\varphi)|^2 v_g \leq C\}$.
- Then, $\forall \{\varphi_i\} \in \mathcal{F}$, $\exists\, \mathcal{S} = \{x_1, \cdots, x_\ell\} \subset M$, and $\exists$ a biharmonic map $\varphi_\infty : (M \backslash \mathcal{S}, g) \to (N, h)$ s.th.
- (1) $\varphi_{i_j} \to \varphi_\infty$ in the C^∞-topology on $M \backslash \mathcal{S}$ $(j \to \infty)$,
- (2) Radon meas. $|d\varphi_{i_j}|^m v_g$ converges to a meas.

$$|d\varphi_\infty|^m v_g + \sum_{1 \leq k \leq \ell} a_k\, \delta_{x_k} \quad (j \to \infty).$$

This is based on the following work:

N. Nakauchi and H. Urakawa,

Bubbling phenomena of biharmonic maps,

arXiv: 0912.4086v4 [Math.DG],
Journal of Geometry and Physics, Vol. 98 (2015), 355–375.

Joint works with Norihito Koiso (1)

- B.Y. Chen's Conjecture: Any biharmonic isometric immersion into $(\mathbb{R}^k, g_0)$ must be minimal.
- Let $\varphi : M^m \looparrowright (\mathbb{R}^{m+1}, g_0)$, a biharm. hypersurface,
- λ_i, the principal curvature, $(i = 1, \cdots, m)$,
  v_i, the unit principal curvature vector fields.
  Let $\tau := \sum \lambda_i$. Then,
  $-\frac{\tau}{2}$ is a simple principal curvature, say $\lambda_m = -\frac{\tau}{2}$.
- Thm (Koiso-Urakawa) Let $\varphi : M^m \looparrowright (\mathbb{R}^{m+1}, g_0)$, a biharmonic hypersurf., with $\lambda_i \neq \lambda_j$ $(i \neq j)$, and
  $g(\nabla_{v_i} v_j, v_k) \neq 0 \quad (\forall i, j, k = 1, \cdots, m-1)$,
  ∇, the induced connect. w.r.t the induced metric g.
- Then, M is minimal.

Joint works with Norihito Koiso (2)

- Thm (Koiso-Urakawa) Every Riemannian manifold (M, g) can be embedded as a biharmonic but not minimal hypersurface in a Riemannian manif., $(M \times \mathbb{R}, \overline{g}(t) := g(t) + dt^2)$ with $g(0) = g$.
- Here $g(t)$ is a solution of the system of ODE's:

$$\alpha = -\frac{1}{2} g'(t), \ \beta = -\frac{1}{2} g''(t) + \frac{1}{4} C_{g(t)} (g'(t) \otimes g'(t)),$$

- $g'(t)(X, Y) = \partial g(t)(X, Y)/\partial t$, $C_{g(t)}(\cdot)$, contraction,
  $\alpha(X, Y) = \overline{g}(\overline{\nabla}_X Y, N)$ $(X, Y \in \mathfrak{X}(M))$, $N = \partial/\partial t$,
  (the unit normal vector field along M at $t = 0$),
  and $\beta(X, Y) := \overline{g}(0)(\overline{R}(N, X)Y, N)$.

Joint works with Norihito Koiso (3)

See the above works in:

Norihito Koiso and Hajime Urakawa:

Biharmonic submanifolds in a Riemannian manifold,
Osaka J. Math., Vol. 55 (2018), 325–346,

(arXiv: 14089.5494v1 [math.DG] 23 Aug 2014,
accepted in Osaka J. Math., January 10, 2017.)

Classif. of all biharm. homog. hypersurfaces in compact symmetric spaces (with S. Ohno, T. Sakai)

- Thm Let (G, K_1, K_2), any commut. symmetric triad, i.e., G, a compact simple Lie gr., G/K_i $(i = 1, 2)$, compact symm. sp., two involutions θ_i, $\theta_1\theta_2 = \theta_2\theta_1$, K_2, K_1 act on G/K_1, G/K_2, of cohom. one, resp.
- K_2-orbit, proper biharm $\Leftrightarrow$ K_1-orbit, proper biharm.
- Case 1: 3 cases.
  $(SO(1 + b + c), SO(1 + b) \times SO(c), SO(b + c))$,
  $(SU(4), S(U(2) \times U(2)), Sp(2))$,
  $(Sp(2), U(2), Sp(1) \times Sp(1))$. In each case,
- $\exists_1$ proper biharm. hypersurfaces K_2-orbit in G/K_1.

Classif. of all biharm. homog. hypersurfaces in compact symmetric spaces (2)

- Case 2: 7 cases.
  $(\mathrm{SO}(2 + 2q),\ \mathrm{SO}(2) \times \mathrm{SO}(2q),\ \mathrm{U}(1 + q)) \quad (q > 1)$,
  $(\mathrm{SU}(1 + b + c),\ \mathrm{S}(\mathrm{U}(1 + b) \times \mathrm{U}(c)),$
  $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(b + c)) \quad (b \geq 0,\ c > 1)$,
  $(\mathrm{Sp}(1 + b + c),\ \mathrm{Sp}(1 + b) \times \mathrm{Sp}(c),$
  $\mathrm{Sp}(1) \times \mathrm{Sp}(b + c)) \quad (b \geq 0,\ c > 1)$,
  $(\mathrm{SO}(8),\ \mathrm{U}(4),\ \mathrm{U}(4)')$,
  $(\mathrm{E}_6,\ \mathrm{SO}(10) \cdot \mathrm{U}(1),\ \mathrm{F}_4)$,
  $(\mathrm{SO}(1 + q),\ \mathrm{SO}(q),\ \mathrm{SO}(q)) \quad (q > 1)$,
  $(\mathrm{F}_4,\ \mathrm{Spin}(9),\ \mathrm{Spin}(9))$.
- $\exists_2$ proper biharm. hyp. orb. of K_2-action on G/K_1.

Classif. of all biharm. homog. hypersurfaces in compact symmetric spaces (3)

- Case 3: 8 cases.
  $(\mathrm{SO}(2c),\ \mathrm{SO}(c)\times\mathrm{SO}(c),\ \mathrm{SO}(2c-1))\quad (c>1)$,
  $(\mathrm{SU}(4),\ \mathrm{Sp}(2),\ \mathrm{SO}(4))$,
  $(\mathrm{SO}(6),\ \mathrm{U}(3),\ \mathrm{SO}(3)\times\mathrm{SO}(3))$,
  $(\mathrm{SU}(1+q),\ \mathrm{SO}(1+q),\ \mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(q)))\quad (q>1)$,
  $(\mathrm{SU}(2+2q),\ \mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2q)),\ \mathrm{Sp}(1+q))\quad (q>1)$,
  $(\mathrm{Sp}(1+q),\ \mathrm{U}(1+q),\ \mathrm{Sp}(1)\times\mathrm{Sp}(q))\quad (q>1)$,
  $(\mathrm{E}_6,\ \mathrm{SU}(6)\cdot\mathrm{SU}(2),\ \mathrm{F}_4)$,
  $(\mathrm{F}_4,\ \mathrm{Sp}(3)\cdot\mathrm{Sp}(1),\ \mathrm{Spin}(9))$.
- In this case, $\forall$ biharm. reg. orbits of K_2-action on G/K_1 (same as, K_1-action on G/K_2) is minimal.

Classif. of all biharm. homog. hypersurfaces in compact symmetric spaces

This work is due to:

Shinji Ohno, Takashi Sakai and Hajime Urakawa,
Biharmonic homogeneous hypersurfaces in compact symmetric spaces,
arXiv: 1507.01738v1 [math.DG] 7 Jul 2015,
Differential Geometry and Its Applications,
Vol. 43 (2015), 155–179.

Classif. all biharmonic homog. hypersurfaces in compact Lie groups (1)

- Thm Let (G, K_1, K_2) be a commutative compact symmetric triad with $\dim\mathfrak{a}=1$. Then, all biharmonic regular orbits for $(K_2\times K_1)$-actions on G are classified as follows:
- (1): All cases admitting regular orbits of the $(K_2\times K_1)$-action on G which
  "$\exists_2$ distinct proper biharmonic hypersurfaces",
  are one of the 15 cases in the next page.
- (2): All cases which "all biharmonic regular orbits of the $(K_2\times K_1)$-action on G must be harmonic", are one of the 4 cases in the page after the next.

(1) (G, K_1, K_2), $\exists_2$ proper biharm. hypersurf.

- $(SO(1+b+c), SO(1+b)\times SO(c), SO(b+c))$
- $(SU(4), Sp(2), SO(4))$ · $(SU(4), S(U(2)\times U(2)), Sp(2))$
- $(Sp(2), U(2), Sp(1)\times Sp(1))$
- $(SO(2+2q), SO(2)\times SO(2q), U(1+q))\quad (q>1)$
- $(SU(1+b+c), S(U(1+b)\times U(c)), S(U(1)\times U(b+c))$
- $(Sp(1+b+c), Sp(1+b)\times Sp(c), Sp(1)\times Sp(b+c))$
- $(SO(1+q), SO(q), SO(q))\quad (q>1)$
- $(SU(1+q), SO(1+q), S(U(1)\times U(q)))\quad (q>52)$
- $(SU(2+2q), S(U(2)\times U(2q)), Sp(1+q))\quad (q>1)$
- $(Sp(1+q), U(1+q), Sp(1)\times Sp(q))\ (q=2, q>45)$
- $(E_6, SO(10)\cdot U(1), F_4)$
- $(F_4, Spin(9), Spin(9))$ · $(F_4, Sp(3)\cdot Sp(1), Spin(9))$
- $(SO(8), U(4), U(4)')$

(2) (G, K_1, K_2), any biharmnic regular orbit of the $(K_2\times K_1)$-action on G is harmonic

Recall the action of $K_2\times K_1$ on G is

$$(k_2, k_1)\cdot x := k_2 x k_1^{-1}\qquad (k_2\in K_2,\ k_1\in K_1,\ x\in G).$$

(2-1) $(SO(6), U(3), SO(3)\times SO(3))$,

(2-2) $(SU(1+q), SO(1+q), S(U(1)\times U(q))\quad (52\geq q>1)$,

(2-3) $(Sp(1+q), U(1+q), Sp(1)\times Sp(q))\quad (45\geq q>2)$,

(2-4) $(E_6, SU(6)\cdot SU(2), F_4)$.

compact symmetric triads (G, K_1, K_2), the K_2-action on G/K_1 is cohomogeneity two

Thm 1 Let (G, K_1, K_2), a compact symmetric triad whose the K_2-action on G/K_1 is of cohomogeneity two.
Then, all singular orbit types are divided into one of the following three cases:
(the codimension of all such orbits of K_2 in $G/K_1\geq 2$).

(i) $\exists_1$ a unique proper biharmonic orbit,
(ii) $\exists_2$ proper biharmonic orbits,
(iii) any biharmonic orbit is harmonic.

Thm 2 The classification is given as follows:

(2) compact symm. triads (G, K_1, K_2), the K_2-action on G/K_1 is cohomogeneity two

(1) $\mathbf{A}_2$: **12** cases (ii), (2) $\mathbf{B}_2$: **6** cases (ii),
(3) $\mathbf{C}_2$: **15** cases (ii), (4) $\mathbf{BC}_2$: **12** cases (ii),
(5) $\mathbf{G}_2$: **4** cases (ii) and **2** cases (iii),
(6) $\mathbf{I\text{-}B}_2$: **2** cases (i), **4** cases in (ii),
(7) $\mathbf{I\text{-}C}_2$: **4** cases (i) and **8** cases (ii),
(8) $\mathbf{I\text{-}C}_2$: **4** cases (i) and **8** cases in (ii),
(9) $\mathbf{I\text{-}BC}_2\text{-}\mathbf{A}_1^2$: **9** cases (ii), (10) $\mathbf{II\text{-}BC}_2$: **9** cases (iii),
(11) $\mathbf{I\text{-}BC}_2\text{-}\mathbf{B}_2$: **4** cases (ii) and **5** cases in (iii),
(12) $\mathbf{III\text{-}A}_2$: **9** cases (iii), (13) $\mathbf{III\text{-}B}_2$: **3** cases (iii),
(14) $\mathbf{III\text{-}C}_2$: **2** cases (i) and **7** cases in (iii),
(15) $\mathbf{III\text{-}BC}_2$: **9** cases (iii), (16) $\mathbf{III\text{-}G}$: **2** cases (iii).

Biharmonic homogeneous submanifolds in compact symmetric spaces and compact Lie groups

This work is due to:

Shinji Ohno, Takashi Sakai and Hajime Urakawa, *Biharmonic homogeneous submanifolds in compact symmetric spaces and compact Lie groups,*

arXiv: 1612.01063v1 [math.DG] 4 Dec 2016,
Hiroshima Math. J., **49** (2019), 47–115,
(accepted in 2018, January).

Thank you very much
for your attentions !