Asymptotic Expansions and New Numerical Algorithm of the Algebraic Riccati Equation for Multiparameter Singularly Perturbed Systems

Hiroaki Mukaidani*, Tetsu Shimomura** and Koichi Mizukami***

* *Faculty of Information Sciences, Hiroshima City University, 3–4–1, Ozuka–Higashi, Asaminami–ku, Hiroshima, 731–3194 Japan.*
** *Graduate School of Education, Hiroshima University, 1–1–1 Kagamiyama, Higashi–Hiroshima, 739–8524 Japan.*
*** *Faculty of Engineering, Hiroshima Kokusai Gakuin University, 6–20–1, Nakano Aki–ku, Hiroshima, 739–0321 Japan.*
E-mail: mukaida@im.hiroshima-cu.ac.jp

In this paper we study a continuous–time multiparameter algebraic Riccati equation (MARE) with indefinite sign quadratic term. The existence of a unique and bounded solution of the MARE is newly established. We show that the Kleinman algorithm can be used well to solve the sign indefinite MARE. The proof of the convergence and the existence of the unique solution of the Kleinman algorithm is done by using the Newton–Kantorovich theorem. Furthermore, we present new algorithms for solving the generalized multiparameter algebraic Lyapunov equation (GMALE) by means of the fixed point algorithm.

1. INTRODUCTION

The deterministic and the stochastic multimodeling control and the filtering problems have been investigated extensively by several researchers (see e.g., [1, 2, 3, 4, 5, 6]). The multimodeling problems arise in large scale dynamic systems. For example, these multimodel situations in practice are illustrated by the multiarea power system [1] and the passenger car model [6]. In order to obtain the optimal solution to the multimodeling problems, we must solve the multiparameter algebraic Riccati equation (MARE), which are parameterized by two small positive same order parameters ε_1 and ε_2. Various reliable approaches to the theory of the ordinary algebraic Riccati equation (ARE) have been well documented in many literatures (see e.g., [7, 8]). One of the approaches is the invariant subspace approach which is based on the Hamiltonian matrix. However, there is no guarantee of symmetry for the solution of the ARE when the ARE is known to be ill–conditioned [7]. Note that it is very hard to solve directly the singularly perturbed ARE and the MARE due to the presence of the small parameters [5, 6, 15, 18].

A popular approach to deal with the multiparameter singularly perturbed systems (MSPS) is the two–time–scale design method [1]. However, it is known from [5] that an $O(\|\mu\|)$ (where $\mu = (\varepsilon_1,\ \varepsilon_2)$) accuracy is very often not sufficient because the reduced–order controller which is based on the two–time–scale design method might not produce satisfactory results for the desired performance. More recently, the exact slow–fast decomposition method for solving the MARE has been proposed in [5, 6]. The solutions are obtained by solving the Sylvester equations of lower dimensions which are non–symmetric equations by means of the Newton method or the fixed point algorithm. However, the results of [5, 6] need the assumption that the sign of the quadratic term of the MARE corresponding to the optimal control and the filtering problem is positive semidefinite and that Hamiltonian matrices for the fast subsystems have no eigenvalues in common (Assumption 5 of [6]).

In this paper, we investigate the asymptotic expansions for the MARE with indefinite sign quadratic term and propose the iterative technique for solving such MARE. Firstly, we relax the condition for the existence of the solution compared with [3] in the sense that some of the assumptions for the MARE are weakened. It is worth pointing out that existence of a unique and bounded solution of the MARE with indefinite sign quadratic term has not been established so far in the previous literature [3]. Furthermore, note that the MSPS with either standard or nonstandard singular perturbations [4] is considered. Secondly, we propose a new iterative algorithm for solving the sign indefinite MARE. The method studied here is based on the Kleinman algorithm [9]. Therefore, the algorithm achieves the quadratic convergence property. Note that the difference between the results in [9] and the present paper is that the successive approximation technique is used to prove the convergence in [9], while the approach adopted here is composed of Newton–Kantorovich theorem [10, 11]. Thus, we do not assume here that the sign of the quadratic term for the MARE is positive semidefinite. The Newton–Kantorovich theorem plays an also important role in the proof of the existence of the unique solution. The main objective in this paper is to provide a new algorithm for solving the generalized multiparameter algebraic Lyapunov equation (GMALE). The method presented in this paper is based on the fixed point algorithm [14]. Consequently, our proposed algorithm is extremely useful since we have only to solve an algebraic Lyapunov equation (ALE) of lower dimension. In particular, it is important note that so far the algorithm for solving the GMALE has not been established. Finally, a numerical example is given to complement the theoretical results. The resulting algorithms are implemented for the multiparameter H_∞ optimal control problem.

2. PROBLEM FORMULATION AND PRIMARY RESULT

We consider the following MARE

$$A_{\mathcal{E}}^T P_{\mathcal{E}} + P_{\mathcal{E}} A_{\mathcal{E}} - P_{\mathcal{E}} S_{\mathcal{E}} P_{\mathcal{E}} + Q = 0, \tag{1}$$

where

$$P_{\mathcal{E}} = \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ \varepsilon_1 P_{10} & \varepsilon_1 P_{11} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21}^T \\ \varepsilon_2 P_{20} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21} & \varepsilon_2 P_{22} \end{bmatrix} \in \mathbf{R}^{N \times N},$$

$$P_{00} = P_{00}^T, \quad P_{11} = P_{11}^T, \quad P_{22} = P_{22}^T,$$

$$A_{\mathcal{E}} = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ \varepsilon_1^{-1} A_{10} & \varepsilon_1^{-1} A_{11} & 0 \\ \varepsilon_2^{-1} A_{20} & 0 & \varepsilon_2^{-1} A_{22} \end{bmatrix} \in \mathbf{R}^{N \times N},$$

$$S_{\mathcal{E}} = S_{\mathcal{E}}^T = \begin{bmatrix} S_{00} & \varepsilon_1^{-1} S_{01} & \varepsilon_2^{-1} S_{02} \\ \varepsilon_1^{-1} S_{01}^T & \varepsilon_1^{-2} S_{11} & 0 \\ \varepsilon_2^{-1} S_{02}^T & 0 & \varepsilon_2^{-2} S_{22} \end{bmatrix} \in \mathbf{R}^{N \times N},$$

$$S_{00} = S_{00}^T, \quad S_{11} = S_{11}^T, \quad S_{22} = S_{22}^T,$$

$$Q = Q^T = \begin{bmatrix} Q_{00} & Q_{01} & Q_{02} \\ Q_{01}^T & Q_{11} & 0 \\ Q_{02}^T & 0 & Q_{22} \end{bmatrix} \in \mathbf{R}^{N \times N},$$

$$Q_{00} = Q_{00}^T, \quad Q_{11} = Q_{11}^T, \quad Q_{22} = Q_{22}^T,$$

$$P_{00},\ A_{00},\ S_{00},\ Q_{00} \in \mathbf{R}^{n_0 \times n_0},\ P_{11},\ A_{11},\ S_{11},\ Q_{11} \in \mathbf{R}^{n_1 \times n_1},$$

$$P_{22},\ A_{22},\ S_{22},\ Q_{22} \in \mathbf{R}^{n_2 \times n_2},\ \varepsilon_1 > 0,\ \varepsilon_2 > 0,\ N = n_0 + n_1 + n_2.$$

If the sign of the MARE (1) is positive semidefinite, then the equation (1) is known as a regulator ARE, appearing in the multimodeling [3]. However, we do not assume in this paper that the sign of the MARE (1) is positive semidefinite. That is, no assumption is made on the definiteness of $S_{\mathcal{E}}$. In addition, we do not assume here that A_{11} and A_{22} are nonsingular compared with [1, 3].

In order to avoid the ill–conditioned due to the large parameter ε_j^{-1} which is included in the MARE (1), we introduce the following useful lemma.

Lemma 2.1. *The MARE (1) is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE) (2a)*

$$A^T P + P^T A - P^T S P + Q = 0, \tag{2a}$$

$$P_{\mathcal{E}} = \Phi_{\mathcal{E}} P = P^T \Phi_{\mathcal{E}}, \tag{2b}$$

where

$$\Phi_{\mathcal{E}} = \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & \varepsilon_1 I_{n_1} & 0 \\ 0 & 0 & \varepsilon_2 I_{n_2} \end{bmatrix}, \ A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix},$$

$$S = \begin{bmatrix} S_{00} & S_{01} & S_{02} \\ S_{01}^T & S_{11} & 0 \\ S_{02}^T & 0 & S_{22} \end{bmatrix}, \ P = \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ P_{10} & P_{11} & \dfrac{1}{\sqrt{\alpha}} P_{21}^T \\ P_{20} & \sqrt{\alpha} P_{21} & P_{22} \end{bmatrix}.$$

Proof. Firstly, by direct calculation we verify that $P_{\mathcal{E}} = \Phi_{\mathcal{E}} P$. Secondly, it is easy to verify that $A = \Phi_{\mathcal{E}} A_{\mathcal{E}}$, $S = \Phi_{\mathcal{E}} S_{\mathcal{E}} \Phi_{\mathcal{E}}$. Hence,

$$A^T P = A_{\mathcal{E}}^T \Phi_{\mathcal{E}} \Phi_{\mathcal{E}}^{-1} P_{\mathcal{E}} = A_{\mathcal{E}}^T P_{\mathcal{E}}.$$

By using the similar calculation, we can immediately rewrite (1) as (2a). ▮

Before investing the structural properties of the GMARE (2a), let us define a parameter α

$$0 < k_1 \leq \alpha \equiv \frac{\varepsilon_1}{\varepsilon_2} \leq k_2 < \infty. \tag{3}$$

It is assumed that the limit of α exists as ε_1 and ε_2 tend to zero, that is

$$\bar{\alpha} = \lim_{\substack{\varepsilon_1 \to +0 \\ \varepsilon_2 \to +0}} \alpha.$$

The GMARE (2a) can be partitioned into

$$\begin{aligned} f_1 &= A_{00}^T P_{00} + P_{00} A_{00} + A_{10}^T P_{10} + P_{10}^T A_{10} + A_{20}^T P_{20} + P_{20}^T A_{20} \\ &\quad - P_{00} S_{00} P_{00} - P_{10}^T S_{01}^T P_{00} - P_{00} S_{01} P_{10} \\ &\quad - P_{20}^T S_{02}^T P_{00} - P_{00} S_{02} P_{20} - P_{10}^T S_{11} P_{10} - P_{20}^T S_{22} P_{20} + Q_{00} = 0, \quad \text{(4a)} \\ f_2 &= P_{00} A_{01} + P_{10}^T A_{11} + \varepsilon_1 A_{00}^T P_{10}^T + A_{10}^T P_{11} + \sqrt{\alpha} A_{20}^T P_{21} \\ &\quad - \varepsilon_1 (P_{00} S_{00} P_{10}^T + P_{10}^T S_{01}^T P_{10}^T + P_{20}^T S_{02}^T P_{10}^T) \\ &\quad - P_{00} S_{01} P_{11} - P_{10}^T S_{11} P_{11} \\ &\quad - \sqrt{\alpha} (P_{00} S_{02} P_{21} + P_{20}^T S_{22} P_{21}) + Q_{01} = 0, \quad \text{(4b)} \\ f_3 &= P_{00} A_{02} + P_{20}^T A_{22} + \varepsilon_2 A_{00}^T P_{20}^T + A_{20}^T P_{22} + \frac{1}{\sqrt{\alpha}} A_{10}^T P_{21}^T \end{aligned}$$

$$
\begin{aligned}
&-\varepsilon_2(P_{00}S_{00}P_{20}^T + P_{10}^T S_{01}^T P_{20}^T + P_{20}^T S_{02}^T P_{20}^T) \\
&-P_{00}S_{02}P_{22} - P_{20}^T S_{22} P_{22} \\
&-\frac{1}{\sqrt{\alpha}}(P_{00}S_{01}P_{21}^T + P_{10}^T S_{11} P_{21}^T) + Q_{02} = 0, \qquad (4c)\\
f_4 &= A_{11}^T P_{11} + P_{11}A_{11} + \varepsilon_1(A_{01}^T P_{10}^T + P_{10}A_{01}) \\
&-\varepsilon_1(\varepsilon_1 P_{10}S_{00}P_{10}^T + P_{11}S_{01}^T P_{10}^T + \sqrt{\alpha}P_{21}^T S_{02}^T P_{10}^T) \\
&-\varepsilon_1(P_{10}S_{01}P_{11} + \sqrt{\alpha}P_{10}S_{02}P_{21}) \\
&-P_{11}S_{11}P_{11} - \alpha P_{21}^T S_{22} P_{21} + Q_{11} = 0, \qquad (4d)\\
f_5 &= \varepsilon_1 P_{10}A_{02} + \varepsilon_2 A_{01}^T P_{20}^T - \varepsilon_1\varepsilon_2 P_{10}S_{00}P_{20}^T \\
&-\varepsilon_2(P_{11}S_{01}^T P_{20}^T + \sqrt{\alpha}P_{21}^T S_{02}^T P_{20}^T) - \varepsilon_1(P_{10}S_{02}P_{22} + \frac{1}{\sqrt{\alpha}}P_{10}S_{01}P_{21}^T) \\
&+\sqrt{\alpha}P_{21}^T(A_{22} - S_{22}P_{22}) + \frac{1}{\sqrt{\alpha}}(A_{11} - S_{11}P_{11})^T P_{21}^T = 0, \qquad (4e)\\
f_6 &= A_{22}^T P_{22} + P_{22}A_{22} + \varepsilon_2(A_{02}^T P_{20}^T + P_{20}A_{02}) \\
&-\varepsilon_2(\varepsilon_2 P_{20}S_{00}P_{20}^T + P_{22}S_{02}^T P_{20}^T + \frac{1}{\sqrt{\alpha}}P_{21}S_{01}^T P_{20}^T) \\
&-\varepsilon_2(P_{20}S_{02}P_{22} + \frac{1}{\sqrt{\alpha}}P_{20}S_{01}P_{21}^T) \\
&-P_{22}S_{22}P_{22} - \frac{1}{\alpha}P_{21}S_{11}P_{21}^T + Q_{22} = 0. \qquad (4f)
\end{aligned}
$$

By limiting solutions of the GMARE (2a) or (4) as $\varepsilon_1 \to +0$ and $\varepsilon_2 \to +0$, then we obtain the following equations

$$
\begin{aligned}
&A_{00}^T\bar{P}_{00} + \bar{P}_{00}A_{00} + A_{10}^T\bar{P}_{10} + \bar{P}_{10}^T A_{10} + A_{20}^T\bar{P}_{20} + \bar{P}_{20}^T A_{20} - \bar{P}_{00}S_{00}\bar{P}_{00} \\
&\quad -\bar{P}_{10}^T S_{01}^T\bar{P}_{00} - \bar{P}_{00}S_{01}\bar{P}_{10} - \bar{P}_{20}^T S_{02}^T\bar{P}_{00} - \bar{P}_{00}S_{02}\bar{P}_{20} \\
&\quad -\bar{P}_{10}^T S_{11}\bar{P}_{10} - \bar{P}_{20}^T S_{22}\bar{P}_{20} + Q_{00} = 0, \qquad (5a)\\
&\bar{P}_{00}A_{01} + \bar{P}_{10}^T A_{11} + A_{10}^T\bar{P}_{11} + \sqrt{\bar{\alpha}}A_{20}^T\bar{P}_{21} - \bar{P}_{00}S_{01}\bar{P}_{11} \\
&\quad -\bar{P}_{10}^T S_{11}\bar{P}_{11} - \sqrt{\bar{\alpha}}(\bar{P}_{00}S_{02}\bar{P}_{21} + \bar{P}_{20}^T S_{22}\bar{P}_{21}) + Q_{01} = 0, \qquad (5b)\\
&\bar{P}_{00}A_{02} + \bar{P}_{20}^T A_{22} + A_{20}^T\bar{P}_{22} + \frac{1}{\sqrt{\bar{\alpha}}}A_{10}^T\bar{P}_{21}^T - \bar{P}_{00}S_{02}\bar{P}_{22} \\
&\quad -\bar{P}_{20}^T S_{22}\bar{P}_{22} - \frac{1}{\sqrt{\bar{\alpha}}}(\bar{P}_{00}S_{01}\bar{P}_{21}^T + \bar{P}_{10}^T S_{11}\bar{P}_{21}^T) + Q_{02} = 0, \qquad (5c)\\
&A_{11}^T\bar{P}_{11} + \bar{P}_{11}A_{11} - \bar{P}_{11}S_{11}\bar{P}_{11} - \bar{\alpha}\bar{P}_{21}^T S_{22}\bar{P}_{21} + Q_{11} = 0, \qquad (5d)\\
&\sqrt{\bar{\alpha}}\bar{P}_{21}^T(A_{22} - S_{22}\bar{P}_{22}) + \frac{1}{\sqrt{\bar{\alpha}}}(A_{11} - S_{11}\bar{P}_{11})^T\bar{P}_{21}^T = 0, \qquad (5e)\\
&A_{22}^T\bar{P}_{22} + \bar{P}_{22}A_{22} - \bar{P}_{22}S_{22}\bar{P}_{22} - \frac{1}{\bar{\alpha}}\bar{P}_{21}S_{11}\bar{P}_{21}^T + Q_{22} = 0, \qquad (5f)
\end{aligned}
$$

where $\bar{P}_{00},\ \bar{P}_{10},\ \bar{P}_{20},\ \bar{P}_{11},\ \bar{P}_{21}$ and $\bar{P}_{22}$ are the 0–order solutions of the GMARE (2a).

We shall make the following basic condition without loss of generality [15].

(H1) The AREs $A_{jj}^T\tilde{P}_{jj}+\tilde{P}_{jj}A_{jj}-\tilde{P}_{jj}S_{jj}\tilde{P}_{jj}+Q_{jj}=0,\ j=1,2$ have the positive semidefinite stabilizing solutions.

If condition (H1) holds, there exist the matrices $\tilde{P}_{jj},\ j=1,\ 2$ such that the matrices $A_{jj}-S_{jj}\tilde{P}_{jj},\ j=1,\ 2$ are stable. Therefore, we chose the solutions $\bar{P}_{jj},\ j=1,\ 2$ as $\tilde{P}_{jj},\ j=1,\ 2$. Then, the unique solution of (5e) is given by $\bar{P}_{21}=0$ because the matrices $A_{jj}-S_{jj}\bar{P}_{jj}=A_{jj}-S_{jj}\tilde{P}_{jj}$ are stable. As a consequence, the parameter $\bar{\alpha}$ does not appear in (5) automatically, that is, it does not affect the equation (5) in the limit when ε_1 and ε_2 tend to zero. Thus the AREs (5d) and (5f) will produce the unique positive semidefinite stabilizing solution under the conditions (H1).

We now obtain the following 0–order equations

$$A_s^T\bar{P}_{00}+\bar{P}_{00}A_s-\bar{P}_{00}S_s\bar{P}_{00}+Q_s=0, \tag{6a}$$
$$\bar{P}_{j0}^T=\bar{P}_{00}N_{0j}-M_{0j},\ j=1,\ 2, \tag{6b}$$
$$A_{jj}^T\bar{P}_{jj}+\bar{P}_{jj}A_{jj}-\bar{P}_{jj}S_{jj}\bar{P}_{jj}+Q_{jj}=0,\ j=1,\ 2, \tag{6c}$$

where

$$\begin{aligned}
A_s &= A_{00}+N_{01}A_{10}+N_{02}A_{20}+S_{01}M_{01}^T+S_{02}M_{02}^T\\
&\quad +N_{01}S_{11}M_{01}^T+N_{02}S_{22}M_{02}^T,\\
S_s &= S_{00}+N_{01}S_{01}^T+S_{01}N_{01}^T+N_{02}S_{02}^T+S_{02}N_{02}^T\\
&\quad +N_{01}S_{11}N_{01}^T+N_{02}S_{22}N_{02}^T,\\
Q_s &= Q_{00}-M_{01}A_{10}-A_{10}^TM_{01}^T-M_{02}A_{20}-A_{20}^TM_{02}^T\\
&\quad -M_{01}S_{11}M_{01}^T-M_{02}S_{22}M_{02}^T,\\
N_{0j} &= -D_{0j}D_{jj}^{-1},\ M_{0j}=\bar{Q}_{0j}D_{jj}^{-1},\ \bar{Q}_{0j}=A_{j0}^T\bar{P}_{jj}+Q_{0j},\\
D_{00} &= A_{00}-S_{00}\bar{P}_{00}-S_{01}\bar{P}_{10}-S_{02}\bar{P}_{20},\ D_{0j}=A_{0j}-S_{0j}\bar{P}_{jj},\\
D_{j0} &= A_{j0}-S_{0j}^T\bar{P}_{00}-S_{jj}\bar{P}_{j0},\ D_{jj}=A_{jj}-S_{jj}\bar{P}_{jj},\ j=1,\ 2.
\end{aligned}$$

The matrices $A_s,\ S_s$ and Q_s do not depend on $\bar{P}_{jj},\ j=1,\ 2$ because their matrices can be computed by using $T_{pq},\ p,\ q=0,\ 1,\ 2$ which is independent of $\bar{P}_{jj},\ j=1,\ 2$ [5, 6], that is,

$$T_s=T_{00}-T_{01}T_{11}^{-1}T_{10}-T_{02}T_{22}^{-1}T_{20}=\begin{bmatrix} A_s & -S_s\\ -Q_s & -A_s^T\end{bmatrix},$$

$$T_{00} = \begin{bmatrix} A_{00} & -S_{00} \\ -Q_{00} & -A_{00}^T \end{bmatrix},\ T_{0j} = \begin{bmatrix} A_{0j} & -S_{0j} \\ -Q_{0j} & -A_{j0}^T \end{bmatrix},$$
$$T_{j0} = \begin{bmatrix} A_{j0} & -S_{0j}^T \\ -Q_{0j}^T & -A_{0j}^T \end{bmatrix},\ T_{jj} = \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix},\ j = 1,\ 2.$$

Note that the Hamiltonian matrices $T_{jj} := \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix},\ j = 1,\ 2$ are nonsingular under the condition (H1) because of

$$T_{jj} = \begin{bmatrix} I_{n_j} & 0 \\ \bar{P}_{jj}^T & I_{n_j} \end{bmatrix} \begin{bmatrix} D_{jj} & -S_{jj} \\ 0 & -D_{jj}^T \end{bmatrix} \begin{bmatrix} I_{n_j} & 0 \\ -\bar{P}_{jj} & I_{n_j} \end{bmatrix}$$
$$\Leftrightarrow\ T_{jj}^{-1} = \begin{bmatrix} I_{n_j} & 0 \\ \bar{P}_{jj} & I_{n_j} \end{bmatrix} \begin{bmatrix} D_{jj}^{-1} & -D_{jj}^{-1} S_{jj} D_{jj}^{-T} \\ 0 & -D_{jj}^{-T} \end{bmatrix} \begin{bmatrix} I_{n_j} & 0 \\ -\bar{P}_{jj}^T & I_{n_j} \end{bmatrix}.$$

The required solution of the ARE (6a) exists under the following condition [15].
(H2) The ARE (6a) has the positive semidefinite stabilizing solutions.

It should be remarked that the solution $P_{\mathcal{E}}$ of (1) is a function of the multiparameters ε_1 and ε_2. But, the solutions $\bar{P}_{00}$ and $\bar{P}_{jj},\ j = 1,\ 2$ of (6a) and (6c) are independent of the multiparameters ε_1 and ε_2, respectively. The following theorem will establish the relation between $P_{\mathcal{E}}$ and the reduced–order solutions (6) (see [3]).

THEOREM 2.1. *Under the conditions (H1) and (H2), there exist small ε_1^* and ε_2^* such that for all $\varepsilon_1 \in (0,\ \varepsilon_1^*)$ and $\varepsilon_2 \in (0,\ \varepsilon_2^*)$, the MARE (1) admits a symmetric positive semidefinite stabilizing solution $P_{\mathcal{E}}$ which can be written as*

$$P_{\mathcal{E}} = \begin{bmatrix} \bar{P}_{00} + \mathcal{F}_{00} & \varepsilon_1(\bar{P}_{10} + \mathcal{F}_{10})^T & \varepsilon_2(\bar{P}_{20} + \mathcal{F}_{20})^T \\ \varepsilon_1(\bar{P}_{10} + \mathcal{F}_{10}) & \varepsilon_1(\bar{P}_{11} + \mathcal{F}_{11}) & \sqrt{\varepsilon_1\varepsilon_2}\mathcal{F}_{21}^T \\ \varepsilon_2(\bar{P}_{20} + \mathcal{F}_{20}) & \sqrt{\varepsilon_1\varepsilon_2}\mathcal{F}_{21} & \varepsilon_2(\bar{P}_{22} + \mathcal{F}_{22}) \end{bmatrix}, \qquad (7)$$

where

$$\mathcal{F}_{pq} = O(\|\mu\|),\ \|\mathcal{F}_{pq}\| = c_{pq} < \infty,\ pq = 00,\ 10,\ 20,\ 11,\ 21,\ 22.$$

In order to prove Theorem 2.1, we need the following lemma [1].

LEMMA 2.2. *Consider the system*

$$\begin{aligned} \dot{x}_0(t) &= A_{00}x_0(t) + A_{01}x_1(t) + A_{02}x_2(t),\ x_0(t_0) = x_0^0, \\ \varepsilon_1\dot{x}_1(t) &= A_{10}x_0(t) + A_{11}x_1(t) + \varepsilon_3 A_{12}x_2(t),\ x_1(t_0) = x_1^0, \\ \varepsilon_2\dot{x}_2(t) &= A_{20}x_0(t) + \varepsilon_4 A_{21}x_1(t) + A_{22}x_2(t),\ x_2(t_0) = x_2^0, \end{aligned}$$

where $x_0 \in \mathrm{R}^{n_0}$, $x_1 \in \mathrm{R}^{n_1}$ *and* $x_2 \in \mathrm{R}^{n_2}$ *are the state vector.* ε_3 *is a small weak coupling parameter,* ε_1 *and* ε_2 *are small positive singular perturbation parameters of the same order of magnitude with (3). If* A_{jj}^{-1}, $j = 1,\ 2$ *exist, and if* $A_0 \equiv A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20}$, A_{jj}, $j = 1,\ 2$ *are stable matrices, then there exist small* $\hat{\varepsilon}_1$ *and* $\hat{\varepsilon}_2$ *such that for all* $\varepsilon_1 \in (0,\ \hat{\varepsilon}_1)$ *and* $\varepsilon_2 \in (0,\ \hat{\varepsilon}_2)$*, the system is asymptotically stable.*

Now, let us prove Theorem 2.1.

Proof. Since the MARE (1) is equivalent to the GMARE (2a) from Lemma 2.1, we apply the implicit function theorem [3] to (2a). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$. It can be shown, after some algebra, that the Jacobian of (2a) in the limit is given by

$$\begin{aligned} J &= \nabla \mathbf{F} = \left. \frac{\partial \mathrm{vec}(f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6)}{\partial \mathrm{vec}(P_{00},\ P_{10},\ P_{20},\ P_{11},\ P_{21},\ P_{22})^T} \right|_{(\mu,\ \mathcal{P}) = (\mu_0,\ \mathcal{P}_0)} \\ &= \begin{bmatrix} J_{00} & J_{01} & J_{02} & 0 & 0 & 0 \\ J_{10} & J_{11} & 0 & J_{13} & J_{14} & 0 \\ J_{20} & 0 & J_{22} & 0 & J_{24} & J_{25} \\ 0 & 0 & 0 & J_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{55} \end{bmatrix}, \end{aligned} \tag{8}$$

where vec denotes an ordered stack of the columns of its matrix [12] and

$$\begin{aligned}
&\mu = (\varepsilon_1,\ \varepsilon_2),\ \mu_0 = (0,\ 0),\ \mathcal{P} = (P_{00},\ P_{10},\ P_{20},\ P_{11},\ P_{21},\ P_{22}),\\
&\mathcal{P}_0 = (\bar{P}_{00},\ \bar{P}_{10},\ \bar{P}_{20},\ \bar{P}_{11},\ 0,\ \bar{P}_{22}),\\
&J_{00} = (I_{n_0} \otimes D_{00}^T)U_{n_0 n_0} + D_{00}^T \otimes I_{n_0},\\
&J_{0j} = (I_{n_0} \otimes D_{j0}^T)U_{n_0 n_j} + D_{j0}^T \otimes I_{n_0},\\
&J_{j0} = D_{0j}^T \otimes I_{n_0},\ J_{jj} = D_{jj}^T \otimes I_{n_0},\ j = 1,\ 2,\\
&J_{13} = I_{n_1} \otimes D_{10},\ J_{14} = \sqrt{\bar{\alpha}}(I_{n_1} \otimes D_{20})U_{n_1 n_2},\\
&J_{24} = \frac{1}{\sqrt{\bar{\alpha}}} I_{n_2} \otimes D_{10},\ J_{25} = I_{n_2} \otimes D_{20},\\
&J_{33} = (I_{n_1} \otimes D_{11}^T)U_{n_1 n_1} + D_{11}^T \otimes I_{n_1},\\
&J_{44} = \sqrt{\bar{\alpha}} D_{22}^T \otimes I_{n_1} + \frac{1}{\sqrt{\bar{\alpha}}} I_{n_2} \otimes D_{11}^T,\\
&J_{55} = (I_{n_2} \otimes D_{22}^T)U_{n_2 n_2} + D_{22}^T \otimes I_{n_2},
\end{aligned}$$

where $\otimes$ denotes Kronecker products and $U_{n_j n_j},\ j = 0,\ 1,\ 2$ is the permutation matrix in Kronecker matrix sense [12].

The Jacobian (8) can be expressed as

$$\begin{aligned}
\det J &= \det J_{33} \cdot \det J_{44} \cdot \det J_{55} \cdot \det \begin{bmatrix} J_{00} & J_{01} & J_{02} \\ J_{10} & J_{11} & 0 \\ J_{20} & 0 & J_{22} \end{bmatrix} \\
&= \det J_{33} \cdot \det J_{44} \cdot \det J_{55} \cdot \det J_{11} \cdot \det J_{22} \\
&\qquad \cdot \det(J_{00} - J_{01} J_{11}^{-1} J_{10} - J_{02} J_{22}^{-1} J_{20}) \\
&= \det J_{11} \cdot \det J_{22} \cdot \det J_{33} \cdot \det J_{44} \cdot \det J_{55} \\
&\qquad \cdot \det[I_{n_0} \otimes D_0^T U_{n_0 n_0} + D_0^T \otimes I_{n_0}], \qquad (9)
\end{aligned}$$

where $D_0 \equiv D_{00} - D_{01} D_{11}^{-1} D_{10} - D_{02} D_{22}^{-1} D_{20}$. Obviously, $J_{jj},\ j = 1, \cdots, 5$ are nonsingular because the matrices $D_{jj} = A_{jj} - S_{jj} \bar{P}_{jj},\ j = 1,\ 2$ are nonsingular under the condition (H1). After some straightforward algebra but tedious, we see that the $A_s - S_s \bar{P}_{00} = D_{00} - D_{01} D_{11}^{-1} D_{10} - D_{02} D_{22}^{-1} D_{20} = D_0$. Therefore, the matrix D_0 is nonsingular if the condition (H2) holds. Thus, $\det J \neq 0$, i.e., J is nonsingular at $(\mu,\ \mathcal{P}) = (\mu_0,\ \mathcal{P}_0)$. The conclusion of the first part of Theorem 2.1 is obtained directly by using the implicit function theorem. The second part of the proof of Theorem 2.1 is performed by direct calculation. By using (7), we obtain

$$\Phi_{\mathcal{E}}^{-1}(A - SP) = \Phi_{\mathcal{E}}^{-1} \left(\begin{bmatrix} D_{00} & D_{01} & D_{02} \\ D_{10} & D_{11} & 0 \\ D_{20} & 0 & D_{22} \end{bmatrix} + O(\|\mu\|) \right).$$

We know from Lemma 2.2 that for sufficiently small $\|\mu\|$ the matrix $\Phi_{\mathcal{E}}^{-1}(A - SP)$ will be stable. On the other hand, since $\bar{P}_{00} \geq 0$, $\bar{P}_{11} \geq 0$ and $\bar{P}_{22} \geq 0$, $P_{\mathcal{E}}$ is positive semidefinite as long as $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ by using the Schur complement [13]. Therefore, the proof on Theorem 2.1 ends. ∎

3. ITERATIVE ALGORITHM

We now develop an algorithm which converges quadratically to the required solution of the MARE (1). So far, the exact decomposition method for solving the MARE with positive semidefinite sign quadratic term has been proposed in [5, 6]. However, the result of [5, 6] needs the assumption that Hamiltonian matrices for the fast subsystems have no eigenvalues in common.

In this paper we develop an elegant and simple algorithm which converges globally to the positive semidefinite solution of the MARE (1). Taking into account the fact that the MARE (1) is equivalent to the GMARE (2a) from Lemma 2.1, the algorithm is given in term of the GMALE [16], which have to be solved iteratively. We present the iterative algorithm based on the Kleinman algorithm [9]. Here we note that the Kleinman algorithm is based on the Newton type algorithm. In general, the stabilizable–detectable conditions will guarantee the convergence of the Kleinman algorithm for the standard linear–quadratic regulator type GMARE to the required solutions. However, it is difficult to apply the Kleinman algorithm to the equation (2a) presented in this paper because the matrix S is in general indefinite.

In this paper, we show that by using the Newton–Kantorovich theorem, the Kleinman algorithm guarantees the quadratic convergence under the appropriate initial conditions.

We propose the following algorithm for solving the GMARE (2a)

$$(A - SP^{(i)})^T P^{(i+1)} + P^{(i+1)T}(A - SP^{(i)}) + P^{(i)T} SP^{(i)} + Q = 0, \tag{10a}$$
$$P_{\mathcal{E}}^{(i)} = \Phi_{\mathcal{E}} P^{(i)} = P^{(i)T}\Phi_{\mathcal{E}}, \tag{10b}$$

$i = 0,\ 1,\ 2,\ 3,\ \cdots$, with the initial condition obtained from

$$P^{(0)} = \begin{bmatrix} \bar{P}_{00} & 0 & 0 \\ \bar{P}_{10} & \bar{P}_{11} & 0 \\ \bar{P}_{20} & 0 & \bar{P}_{22} \end{bmatrix}, \tag{11}$$

where

$$P^{(i)} = \begin{bmatrix} P_{00}^{(i)} & \varepsilon_1 P_{10}^{(i)T} & \varepsilon_2 P_{20}^{(i)T} \\ P_{10}^{(i)} & P_{11}^{(i)} & \dfrac{1}{\sqrt{\alpha}} P_{21}^{(i)T} \\ P_{20}^{(i)} & \sqrt{\alpha} P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix},$$
$$P_{00}^{(i)} = P_{00}^{(i)T},\ P_{11}^{(i)} = P_{11}^{(i)T},\ P_{22}^{(i)} = P_{22}^{(i)T},$$

and $\bar{P}_{pq},\ pq = 00,\ 10,\ 20,\ 11,\ 22$ are defined by (6).

According to the Newton–Kantorovich theorem [10, 11], it is well known that if the initial condition is very close to the exact solution of the considered equation, the Newton method has the quadratic convergence property. Therefore, we can choose the initial conditions as (11).

Although the sign of the matrix S is in general indefinite, we can prove the quadratic convergence for the resulting algorithm (10) by using the Newton–Kantorovich theorem because the initial condition is very close to the exact solution of the GMARE (2a) for sufficiently small $\|\mu\|$. This idea is derived from the following fact:

$$\begin{aligned}\|P - P^{(0)}\| &= \left\| \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ P_{10} & P_{11} & \dfrac{1}{\sqrt{\alpha}} P_{21}^T \\ P_{20} & \sqrt{\alpha} P_{21} & P_{22} \end{bmatrix} - \begin{bmatrix} \bar{P}_{00} & 0 & 0 \\ \bar{P}_{10} & \bar{P}_{11} & 0 \\ \bar{P}_{20} & 0 & \bar{P}_{22} \end{bmatrix} \right\| \\ &= O(\|\mu\|).\end{aligned}$$

The algorithm (10) has the feature given in the following lemma.

Lemma 3.1. *Under the conditions (H1) and (H2), there exists an $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ such that for all $0 < \varepsilon_1 \le \bar{\varepsilon}_1 \le \varepsilon_1^*$ and $0 < \varepsilon_2 \le \bar{\varepsilon}_2 \le \varepsilon_2^*$ respectively, the iterative algorithm (10) converges to the exact solution of P^* with the rate of quadratic convergence. Then, $P_{\mathcal{E}}^{(i)} = \Phi_{\mathcal{E}} P^{(i)} = P^{(i)T} \Phi_{\mathcal{E}}$ is positive semidefinite. Moreover, there exists unique solution of the GMARE (2a) with the indefinite sign quadratic term in neighborhood of the required solution P^*. That is, the following conditions are satisfied.*

$$\|P^{(i)} - P^*\| \le \frac{O(\|\mu\|^{2^i})}{2^i \beta\gamma} = O(\|\mu\|^{2^i}),\ i = 0,\ 1,\ 2,\ \cdots, \tag{12a}$$

$$P_{\mathcal{E}}^{(i)} = \Phi_{\mathcal{E}} P^{(i)} = P^{(i)T} \Phi_{\mathcal{E}} \ge 0,\ i = 1,\ 2,\ 3,\ \cdots, \tag{12b}$$

$$\|P^{(0)} - P^*\| \le \frac{1}{\beta\gamma}[1 - \sqrt{1 - 2\theta}], \tag{12c}$$

where

$$\mathcal{G}(P) = A^T P + P^T A - P^T S P + Q, \tag{13}$$

$$\gamma = 2\|S\| < \infty,\ \beta = \|[\nabla \mathcal{G}(P^{(0)})]^{-1}\|,\ \eta = \beta \cdot \|\mathcal{G}(P^{(0)})\|,\ \theta = \beta\eta\gamma,$$

$$\nabla \mathcal{G}(P) = \frac{\partial \mathrm{vec}\mathcal{G}(P)}{\partial (\mathrm{vec}P)^T},\ P^* = \begin{bmatrix} P_{00}^* & \varepsilon_1 P_{10}^{*T} & \varepsilon_2\ P_{20}^{*T} \\ P_{10}^* & P_{11}^* & \dfrac{1}{\sqrt{\alpha}} P_{21}^{*T} \\ P_{20}^* & \sqrt{\alpha} P_{21}^* & P_{22}^* \end{bmatrix}.$$

Proof. This proof is equivalent to the proof of existence of the unique solution for the GMARE (2a) [16, 17, 18]. Thus, the proof follows directly by applying the Newton–Kantorovich theorem [10, 11] for the GMARE (2a). We now verify that function $\mathcal{G}(P)$ is differentiable on a convex set $\mathcal{D}$.

Using the fact that

$$\nabla\mathcal{G}(P) = (A - SP)^T \otimes I_N + I_N \otimes (A - SP)^T, \tag{14}$$

we have

$$\|\nabla\mathcal{G}(P_1) - \nabla\mathcal{G}(P_2)\| \le \gamma\|P_1 - P_2\|, \tag{15}$$

where $\gamma = 2\|S\|$. Moreover, using the fact that

$$\nabla\mathcal{G}(P^{(0)}) = \begin{bmatrix} D_{00} & D_{01} & D_{02} \\ D_{10} & D_{11} & 0 \\ D_{20} & 0 & D_{22} \end{bmatrix}^T \otimes I_N + I_N \otimes \begin{bmatrix} D_{00} & D_{01} & D_{02} \\ D_{10} & D_{11} & 0 \\ D_{20} & 0 & D_{22} \end{bmatrix}^T, \tag{16}$$

it follows that $\nabla\mathcal{G}(P^{(0)})$ is nonsingular because D_0 and $D_{jj},\ j = 1,\ 2$ are stable under the conditions (H1) and (H2). Therefore, there exists β such that $\|[\nabla\mathcal{G}(P^{(0)})]^{-1}\| \equiv \beta$. On the other hand, since $\mathcal{G}(P^{(0)}) < O(\|\mu\|)$, there exists η such that $\|[\nabla\mathcal{G}(P^{(0)})]^{-1}\| \cdot \|\mathcal{G}(P^{(0)})\| \equiv \eta = O(\|\mu\|)$. Thus, there exists θ such that $\theta \equiv \beta\gamma\eta < 2^{-1}$ because of $\eta = O(\|\mu\|)$. Using the Newton–Kantorovich theorem, the strict error estimate is given by (12a). Now, let us define

$$t^* \equiv \frac{1}{\gamma\beta}[1 - \sqrt{1 - 2\theta}] = \frac{1}{2\|S\| \cdot \|[\nabla\mathcal{G}(P^{(0)})]^{-1}\|}[1 - \sqrt{1 - 2\theta}]. \tag{17}$$

Clearly, $\mathcal{S} \equiv \{\ P\ : \|P - P^{(0)}\| \le t^*\ \}$ is in the convex set $\mathcal{D}$. In the sequel, since $\|P^* - P^{(0)}\| = O(\|\mu\|)$ holds for small ε_1 and ε_2, we show that P^* is the unique solution in $\mathcal{S}$.

On the other hand, using (12a), we have

$$P_{\mathcal{E}}^{(i)} = \begin{bmatrix} \bar{P}_{00} + O(\|\mu\|) & \varepsilon_1(\bar{P}_{10} + O(\|\mu\|))^T & \varepsilon_2(\bar{P}_{20} + O(\|\mu\|))^T \\ \varepsilon_1(\bar{P}_{10} + O(\|\mu\|)) & \varepsilon_1(\bar{P}_{11} + O(\|\mu\|)) & \sqrt{\varepsilon_1\varepsilon_2}O(\|\mu\|)^T \\ \varepsilon_2(\bar{P}_{20} + O(\|\mu\|)) & \sqrt{\varepsilon_1\varepsilon_2}O(\|\mu\|) & \varepsilon_2(\bar{P}_{22} + O(\|\mu\|)) \end{bmatrix}.$$

Since $\bar{P}_{00} \ge 0$, $\bar{P}_{11} \ge 0$ and $\bar{P}_{22} \ge 0$, $P_{\mathcal{E}}^{(i)}$ is positive semidefinite by using the Schur complement [13]. Therefore, the proof is completed. ▮

4. MAIN RESULTS

Now, we consider a method for solving the pair of GMALE (10a). So far, there is little argument as to the numerical method for solving the GMALE.

Therefore, in order to obtain the solution of the pair of GMALE (10a), we present new algorithm by applying the fixed point algorithm [5, 6, 14]. Let us consider the following GMALE in general form.

$$\Lambda^T Y + Y^T \Lambda + U = 0, \tag{18}$$

where Y is the solution of the GMALE (18) and Λ and U are known matrices defined by

$$\begin{aligned}
&Y = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ Y_{10} & Y_{11} & \dfrac{1}{\sqrt{\alpha}} Y_{21}^T \\ Y_{20} & \sqrt{\alpha} Y_{21} & Y_{22} \end{bmatrix} \in \mathbf{R}^{N \times N}, \\
&Y_{00} = Y_{00}^T, \quad Y_{11} = Y_{11}^T, \quad Y_{22} = Y_{22}^T, \\
&\Lambda = \begin{bmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} \\ \Lambda_{10} & \Lambda_{11} & \mathcal{E}\Lambda_{12} \\ \Lambda_{20} & \mathcal{E}\Lambda_{21} & \Lambda_{22} \end{bmatrix} \in \mathbf{R}^{N \times N}, \\
&U = U^T = \begin{bmatrix} U_{00} & U_{01} & U_{02} \\ U_{01}^T & U_{11} & \mathcal{E}U_{12} \\ U_{02}^T & \mathcal{E}U_{12}^T & U_{22} \end{bmatrix} \in \mathbf{R}^{N \times N}, \\
&U_{00} = U_{00}^T, \quad U_{11} = U_{11}^T, \quad U_{22} = U_{22}^T, \\
&Y_{00},\ \Lambda_{00},\ U_{00} \in \mathbf{R}^{n_0 \times n_0},\ Y_{11},\ \Lambda_{11},\ U_{11} \in \mathbf{R}^{n_1 \times n_1}, \\
&Y_{22},\ \Lambda_{22},\ U_{22} \in \mathbf{R}^{n_2 \times n_2}, \\
&\varepsilon_1 > 0,\ \varepsilon_2 > 0,\ \|\mu\| = \mathcal{E} = \sqrt{\varepsilon_1 \varepsilon_2},\ N = n_0 + n_1 + n_2.
\end{aligned}$$

The required solution of the GMALE (18) exists under the standard condition [1].

(H3) The matrices $\Lambda_{jj},\ j = 1,\ 2$ are nonsingular and $\Lambda_0 \equiv \Lambda_{00} - \Lambda_{01}\Lambda_{11}^{-1}\Lambda_{10} - \Lambda_{02}\Lambda_{22}^{-1}\Lambda_{20}$, $\Lambda_{jj},\ j = 1,\ 2$ are stable.

The GMALE (18) can be partitioned into

$$\begin{aligned}
&\Lambda_{00}^T Y_{00} + Y_{00}\Lambda_{00} + \Lambda_{10}^T Y_{10} + Y_{10}^T \Lambda_{10} \\
&\quad + \Lambda_{20}^T Y_{20} + Y_{20}^T \Lambda_{20} + U_{00} = 0, && (19a) \\
&Y_{00}\Lambda_{01} + Y_{10}^T \Lambda_{11} + \mathcal{E} Y_{20}^T \Lambda_{21} + \varepsilon_1 \Lambda_{00}^T Y_{10}^T + \Lambda_{10}^T Y_{11} \\
&\quad + \sqrt{\alpha}\Lambda_{20}^T Y_{21} + U_{01} = 0, && (19b) \\
&Y_{00}\Lambda_{02} + Y_{20}^T \Lambda_{22} + \mathcal{E} Y_{10}^T \Lambda_{12} + \varepsilon_2 \Lambda_{00}^T Y_{20}^T + \Lambda_{20}^T Y_{22} \\
&\quad + \frac{1}{\sqrt{\alpha}}\Lambda_{10}^T Y_{21}^T + U_{02} = 0, && (19c) \\
&\Lambda_{11}^T Y_{11} + Y_{11}\Lambda_{11} + \varepsilon_1(\Lambda_{01}^T Y_{10}^T + Y_{10}\Lambda_{01})
\end{aligned}$$

$$+\sqrt{\alpha}\mathcal{E}(\Lambda_{21}^T Y_{21} + Y_{21}^T \Lambda_{21}) + U_{11} = 0, \tag{19d}$$

$$\varepsilon_1 Y_{10}\Lambda_{02} + \varepsilon_2 \Lambda_{01}^T Y_{20}^T + \sqrt{\alpha} Y_{21}^T \Lambda_{22} + \frac{1}{\sqrt{\alpha}}\Lambda_{11}^T Y_{21}^T$$

$$+\mathcal{E}(Y_{11}\Lambda_{12} + \Lambda_{21}^T Y_{22}) + \mathcal{E} U_{12} = 0, \tag{19e}$$

$$\Lambda_{22}^T Y_{22} + Y_{22}\Lambda_{22} + \varepsilon_2(\Lambda_{02}^T Y_{20}^T + Y_{20}\Lambda_{02})$$

$$+\frac{1}{\sqrt{\alpha}}\mathcal{E}(\Lambda_{12}^T Y_{21}^T + Y_{21}\Lambda_{12}) + U_{22} = 0. \tag{19f}$$

For the equations (19) above, in the limit, as $\varepsilon_1 \to +0$ and $\varepsilon_2 \to +0$, we obtain the following equations

$$\Lambda_{00}^T \bar{Y}_{00} + \bar{Y}_{00}\Lambda_{00} + \Lambda_{10}^T \bar{Y}_{10} + \bar{Y}_{10}^T \Lambda_{10}$$

$$+\Lambda_{20}^T \bar{Y}_{20} + \bar{Y}_{20}^T \Lambda_{20} + U_{00} = 0, \tag{20a}$$

$$\bar{Y}_{00}\Lambda_{01} + \bar{Y}_{10}^T \Lambda_{11} + \Lambda_{10}^T \bar{Y}_{11} + \sqrt{\bar{\alpha}}\Lambda_{20}^T \bar{Y}_{21} + U_{01} = 0, \tag{20b}$$

$$\bar{Y}_{00}\Lambda_{02} + \bar{Y}_{20}^T \Lambda_{22} + \Lambda_{20}^T \bar{Y}_{22} + \frac{1}{\sqrt{\bar{\alpha}}}\Lambda_{10}^T \bar{Y}_{21}^T + U_{02} = 0, \tag{20c}$$

$$\Lambda_{11}^T \bar{Y}_{11} + \bar{Y}_{11}\Lambda_{11} + U_{11} = 0, \tag{20d}$$

$$\sqrt{\bar{\alpha}}\bar{Y}_{21}^T \Lambda_{22} + \frac{1}{\sqrt{\bar{\alpha}}}\Lambda_{11}^T \bar{Y}_{21}^T = 0, \tag{20e}$$

$$\Lambda_{22}^T \bar{Y}_{22} + \bar{Y}_{22}\Lambda_{22} + U_{22} = 0. \tag{20f}$$

Note that the unique solution of (20e) is given by $\bar{Y}_{21} = 0$ since the matrices $\Lambda_{jj},\ j = 1,\ 2$ are nonsingular under the condition (H3). Thus the parameter $\bar{\alpha}$ does not appear in (20). Consequently, we obtain the following 0–order equations

$$\Lambda_0^T \bar{Y}_{00} + \bar{Y}_{00}\Lambda_0 + U_{00} - U_{01}\Lambda_{11}^{-1}\Lambda_{10} - \Lambda_{10}^T \Lambda_{11}^{-T} U_{01}^T$$

$$-U_{02}\Lambda_{22}^{-1}\Lambda_{20} - \Lambda_{20}^T \Lambda_{22}^{-T} U_{02}^T$$

$$+\Lambda_{10}^T \Lambda_{11}^{-T} U_{11}\Lambda_{11}^{-1}\Lambda_{10} + \Lambda_{20}^T \Lambda_{22}^{-T} U_{22}\Lambda_{22}^{-1}\Lambda_{20} = 0, \tag{21a}$$

$$\bar{Y}_{j0}^T = -(\bar{Y}_{00}\Lambda_{0j} + \Lambda_{j0}^T \bar{Y}_{jj} + U_{0j})\Lambda_{jj}^{-1},\ j = 1,\ 2, \tag{21b}$$

$$\Lambda_{jj}^T \bar{Y}_{jj} + \bar{Y}_{jj}\Lambda_{jj} + U_{jj} = 0,\ j = 1,\ 2. \tag{21c}$$

Now, let us introduce

$$Y = \begin{bmatrix} \bar{Y}_{00} + \mathcal{E}\Xi_{00} & \varepsilon_1(\bar{Y}_{10} + \mathcal{E}\Xi_{10})^T & \varepsilon_2(\bar{Y}_{20} + \mathcal{E}\Xi_{20})^T \\ \bar{Y}_{10} + \mathcal{E}\Xi_{10} & \bar{Y}_{11} + \mathcal{E}\Xi_{11} & \frac{\mathcal{E}}{\sqrt{\alpha}}\Xi_{21}^T \\ \bar{Y}_{20} + \mathcal{E}\Xi_{20} & \sqrt{\alpha}\mathcal{E}\Xi_{21} & \bar{Y}_{22} + \mathcal{E}\Xi_{22} \end{bmatrix}. \tag{22}$$

The approximation of the error terms $\Xi_{pq},\ pq = 00,\ 10,\ 20,\ 11,\ 21,\ 22$ will result in approximation of the required matrix Y_{pq}. That is why we are interested in finding equations of the error terms and a convenient algorithm to find their solutions. Substituting (22) into (19) and subtracting (20) from (19), we arrive at the error equations.

$$\Lambda_{00}^T\Xi_{00} + \Xi_{00}\Lambda_{00} + \Lambda_{10}^T\Xi_{10} + \Xi_{10}^T\Lambda_{10} + \Lambda_{20}^T\Xi_{20} + \Xi_{20}^T\Lambda_{20} = 0, \tag{23a}$$

$$\Xi_{00}\Lambda_{01} + \Xi_{10}^T\Lambda_{11} + \Lambda_{10}^T\Xi_{11} + \sqrt{\alpha}\Lambda_{20}^T\Xi_{21} = -\bar{Y}_{20}^T\Lambda_{21} - \frac{\varepsilon_1}{\mathcal{E}}\Lambda_{00}^T\bar{Y}_{10}^T$$
$$-\varepsilon_1\Lambda_{00}^T\Xi_{10}^T - \mathcal{E}\Xi_{20}^T\Lambda_{21}, \tag{23b}$$

$$\Xi_{00}\Lambda_{02} + \Xi_{20}^T\Lambda_{22} + \Lambda_{20}^T\Xi_{22} + \frac{1}{\sqrt{\alpha}}\Lambda_{10}^T\Xi_{21}^T = -\bar{Y}_{10}^T\Lambda_{12} - \frac{\varepsilon_2}{\mathcal{E}}\Lambda_{00}^T\bar{Y}_{20}^T$$
$$-\varepsilon_2\Lambda_{00}^T\Xi_{20}^T - \mathcal{E}\Xi_{10}^T\Lambda_{12}, \tag{23c}$$

$$\Lambda_{11}^T\Xi_{11} + \Xi_{11}\Lambda_{11} = -\frac{\varepsilon_1}{\mathcal{E}}(\Lambda_{01}^T\bar{Y}_{10}^T + \bar{Y}_{10}\Lambda_{01}) - \varepsilon_1(\Lambda_{01}^T\Xi_{10}^T + \Xi_{10}\Lambda_{01})$$
$$-\mathcal{E}\sqrt{\alpha}(\Lambda_{21}^T\Xi_{21} + \Xi_{21}^T\Lambda_{21}), \tag{23d}$$

$$\Lambda_{22}^T\Xi_{22} + \Xi_{22}\Lambda_{22} = -\frac{\varepsilon_2}{\mathcal{E}}(\Lambda_{02}^T\bar{Y}_{20}^T + \bar{Y}_{20}\Lambda_{02}) - \varepsilon_2(\Lambda_{02}^T\Xi_{20}^T + \Xi_{20}\Lambda_{02})$$
$$-\frac{\mathcal{E}}{\sqrt{\alpha}}(\Lambda_{12}^T\Xi_{21}^T + \Xi_{21}\Lambda_{12}), \tag{23e}$$

$$\sqrt{\alpha}\Xi_{21}^T\Lambda_{22} + \frac{1}{\sqrt{\alpha}}\Lambda_{11}^T\Xi_{21}^T = -\frac{\varepsilon_1}{\mathcal{E}}\bar{Y}_{10}\Lambda_{02} - \frac{\varepsilon_2}{\mathcal{E}}\Lambda_{01}^T\bar{Y}_{20}^T - \bar{Y}_{11}\Lambda_{12} - \Lambda_{21}^T\bar{Y}_{22}$$
$$-U_{12} - \varepsilon_1\Xi_{10}\Lambda_{02} - \varepsilon_2\Lambda_{01}^T\Xi_{20}^T - \mathcal{E}(\Lambda_{21}^T\Xi_{22} + \Xi_{11}\Lambda_{12}). \tag{23f}$$

These equations (23) have very nice form since the unknown quantities Ξ_{pq} in right hand side are multiplied by small parameters ε_1, ε_2 and $\mathcal{E}$. This fact suggests that a fixed point algorithm can be efficient for their solutions. Hence, we propose the following algorithm (24).

$$\Lambda_{jj}^T\Xi_{jj}^{(i+1)} + \Xi_{jj}^{(i+1)}\Lambda_{jj} + \mathcal{G}_{jj}(i) = 0,\ j = 1,\ 2 \tag{24a}$$

$$\sqrt{\alpha}\Xi_{21}^{(i+1)T}\Lambda_{22} + \frac{1}{\sqrt{\alpha}}\Lambda_{11}^T\Xi_{21}^{(i+1)T} + \mathcal{G}_{21}(i) = 0, \tag{24b}$$

$$\Lambda_0^T\Xi_{00}^{(i+1)} + \Xi_{00}^{(i+1)}\Lambda_0 + \mathcal{G}_{00}(i) = 0, \tag{24c}$$

$$\Xi_{j0}^{(i+1)T} = -[\Xi_{00}^{(i+1)}\Lambda_{0j} + \mathcal{G}_{0j}(i)]\Lambda_{jj}^{-1},\ j = 1,\ 2, \tag{24d}$$
$$i = 0,\ 1,\ 2,\ \cdots$$

where

$$\mathcal{G}_{11}(i) = \frac{\varepsilon_1}{\mathcal{E}}(\Lambda_{01}^T\bar{Y}_{10}^T + \bar{Y}_{10}\Lambda_{01}) + \varepsilon_1(\Lambda_{01}^T\Xi_{10}^{(i)T} + \Xi_{10}^{(i)}\Lambda_{01})$$
$$+\mathcal{E}\sqrt{\alpha}(\Lambda_{21}^T\Xi_{21}^{(i)} + \Xi_{21}^{(i)T}\Lambda_{21}),$$
$$\mathcal{G}_{22}(i) = \frac{\varepsilon_2}{\mathcal{E}}(\Lambda_{02}^T\bar{Y}_{20}^T + \bar{Y}_{20}\Lambda_{02}) + \varepsilon_2(\Lambda_{02}^T\Xi_{20}^{(i)T} + \Xi_{20}^{(i)}\Lambda_{02})$$

$$
\begin{aligned}
&\quad +\frac{\mathcal{E}}{\sqrt{\alpha}}(\Lambda_{12}^T\Xi_{21}^{(i)T}+\Xi_{21}^{(i)}\Lambda_{12}),\\
\mathcal{G}_{21}(i) &= \frac{\varepsilon_1}{\mathcal{E}}\bar{Y}_{10}\Lambda_{02}+\frac{\varepsilon_2}{\mathcal{E}}\Lambda_{01}^T\bar{Y}_{20}^T+\bar{Y}_{11}\Lambda_{12}+\Lambda_{21}^T\bar{Y}_{22}+U_{12}\\
&\quad +\varepsilon_1\Xi_{10}^{(i)}\Lambda_{02}+\varepsilon_2\Lambda_{01}^T\Xi_{20}^{(i)T}+\mathcal{E}(\Lambda_{21}^T\Xi_{22}^{(i)}+\Xi_{11}^{(i)}\Lambda_{12}),\\
\mathcal{G}_{01}(i) &= \Lambda_{10}^T\Xi_{11}^{(i+1)}+\sqrt{\alpha}\Lambda_{20}^T\Xi_{21}^{(i+1)}+\bar{Y}_{20}^T\Lambda_{21}\\
&\quad +\frac{\varepsilon_1}{\mathcal{E}}\Lambda_{00}^T\bar{Y}_{10}^T+\varepsilon_1\Lambda_{00}^T\Xi_{10}^{(i)T}+\mathcal{E}\Xi_{20}^{(i)T}\Lambda_{21},\\
\mathcal{G}_{02}(i) &= \Lambda_{20}^T\Xi_{22}^{(i+1)}+\frac{1}{\sqrt{\alpha}}\Lambda_{10}^T\Xi_{21}^{(i+1)T}+\bar{Y}_{10}^T\Lambda_{12}\\
&\quad +\frac{\varepsilon_2}{\mathcal{E}}\Lambda_{00}^T\bar{Y}_{20}^T+\varepsilon_2\Lambda_{00}^T\Xi_{20}^{(i)T}+\mathcal{E}\Xi_{10}^{(i)T}\Lambda_{12},\\
\mathcal{G}_{00}(i) &= -[\Lambda_{10}^T\Lambda_{11}^{-T}\mathcal{G}_{01}(i)^T+\mathcal{G}_{01}(i)\Lambda_{11}^{-1}\Lambda_{10}\\
&\quad +\Lambda_{20}^T\Lambda_{22}^{-T}\mathcal{G}_{02}(i)^T+\mathcal{G}_{02}(i)\Lambda_{22}^{-1}\Lambda_{20}],
\end{aligned}
$$

$$\Lambda_{11}^T\Xi_{11}^{(0)}+\Xi_{11}^{(0)}\Lambda_{11}+\frac{\varepsilon_1}{\mathcal{E}}(\Lambda_{01}^T\bar{Y}_{10}^T+\bar{Y}_{10}\Lambda_{01})=0, \tag{25a}$$

$$\Lambda_{22}^T\Xi_{22}^{(0)}+\Xi_{22}^{(0)}\Lambda_{22}+\frac{\varepsilon_2}{\mathcal{E}}(\Lambda_{02}^T\bar{Y}_{20}^T+\bar{Y}_{20}\Lambda_{02})=0, \tag{25b}$$

$$
\begin{aligned}
&\sqrt{\alpha}\Xi_{21}^{(0)T}\Lambda_{22}+\frac{1}{\sqrt{\alpha}}\Lambda_{11}^T\Xi_{21}^{(0)T}+\frac{\varepsilon_1}{\mathcal{E}}\bar{Y}_{10}\Lambda_{02}+\frac{\varepsilon_2}{\mathcal{E}}\Lambda_{01}^T\bar{Y}_{20}^T\\
&\qquad +\bar{Y}_{11}\Lambda_{12}+\Lambda_{21}^T\bar{Y}_{22}+U_{12}=0,
\end{aligned} \tag{25c}
$$

$$
\begin{aligned}
&\Lambda_0^T\Xi_{00}^{(0)}+\Xi_{00}^{(0)}\Lambda_0-\Lambda_{10}^T\Lambda_{11}^{-T}\Theta_{01}^T-\Theta_{01}\Lambda_{11}^{-1}\Lambda_{10}\\
&\quad -\Lambda_{20}^T\Lambda_{22}^{-T}\Theta_{02}^T-\Theta_{02}\Lambda_{22}^{-1}\Lambda_{20}=0,
\end{aligned} \tag{25d}
$$

$$\Xi_{10}^{(0)T}=-(\Xi_{00}^{(0)}\Lambda_{01}+\Theta_{01})\Lambda_{11}^{-1}, \tag{25e}$$

$$\Xi_{20}^{(0)T}=-(\Xi_{00}^{(0)}\Lambda_{02}+\Theta_{02})\Lambda_{22}^{-1}, \tag{25f}$$

$$
\begin{aligned}
\Theta_{01} &= \Lambda_{10}^T\Xi_{11}^{(0)}+\sqrt{\alpha}\Lambda_{20}^T\Xi_{21}^{(0)}+\bar{Y}_{20}^T\Lambda_{21}+\frac{\varepsilon_1}{\mathcal{E}}\Lambda_{00}^T\bar{Y}_{10}^T,\\
\Theta_{02} &= \Lambda_{20}^T\Xi_{22}^{(0)}+\frac{1}{\sqrt{\alpha}}\Lambda_{10}^T\Xi_{21}^{(0)T}+\bar{Y}_{10}^T\Lambda_{12}+\frac{\varepsilon_2}{\mathcal{E}}\Lambda_{00}^T\bar{Y}_{20}^T.
\end{aligned}
$$

The following theorem indicates the convergence of the algorithm (24).

THEOREM 4.1. *The fixed point algorithm (24) converges to the exact solution of Ξ_{pq} with the rate of convergence of $O(\|\mu\|^{i+1})$, that is*

$$
\begin{aligned}
&\|\Xi_{pq}-\Xi_{pq}^{(i)}\|=O(\|\mu\|^{i+1}),\\
&i=0,\ 1,\ 2,\ \cdots,\quad pq=00,\ 10,\ 20,\ 11,\ 21,\ 22.
\end{aligned} \tag{26}
$$

Proof. The proof is done by using mathematical induction. When $i = 0$ for the equations (24), the first order approximations Ξ_{pq} corresponding to the small parameters ε_1, ε_2 and $\mathcal{E}$ satisfy the equations (25). It follows from these equations that

$$\|\Xi_{pq} - \Xi_{pq}^{(0)}\| = O(\|\mu\|), \quad pq = 00,\ 10,\ 20,\ 11,\ 21,\ 22.$$

When $i = k$ ($k \geq 1$), we assume that $\|\Xi_{pq} - \Xi_{pq}^{(k)}\| = O(\|\mu\|^{k+1})$. Subtracting (24) from (23), we arrive at the following equations.

$$\begin{aligned}
&\Lambda_{00}^T(\Xi_{00} - \Xi_{00}^{(k+1)}) + (\Xi_{00} - \Xi_{00}^{(k+1)})\Lambda_{00} \\
&\quad + \Lambda_{10}^T(\Xi_{10} - \Xi_{10}^{(k+1)}) + (\Xi_{10} - \Xi_{10}^{(k+1)})^T\Lambda_{10} \\
&\quad + \Lambda_{20}^T(\Xi_{20} - \Xi_{20}^{(k+1)}) + (\Xi_{20} - \Xi_{20}^{(k+1)})^T\Lambda_{20} = 0, \\
&(\Xi_{00} - \Xi_{00}^{(k+1)})\Lambda_{01} + (\Xi_{10} - \Xi_{10}^{(k+1)})^T\Lambda_{11} + \Lambda_{10}^T(\Xi_{11} - \Xi_{11}^{(k+1)}) \\
&\quad + \sqrt{\alpha}\Lambda_{20}^T(\Xi_{21} - \Xi_{21}^{(k+1)}) = -\varepsilon_1\Lambda_{00}^T(\Xi_{10} - \Xi_{10}^{(k)})^T - \mathcal{E}(\Xi_{20} - \Xi_{20}^{(k)})^T\Lambda_{21}, \\
&(\Xi_{00} - \Xi_{00}^{(k+1)})\Lambda_{02} + (\Xi_{20} - \Xi_{20}^{(k+1)})^T\Lambda_{22} + \Lambda_{20}^T(\Xi_{22} - \Xi_{22}^{(k+1)}) \\
&\quad + \frac{1}{\sqrt{\alpha}}\Lambda_{10}^T(\Xi_{21} - \Xi_{21}^{(k+1)})^T = -\varepsilon_2\Lambda_{00}^T(\Xi_{20} - \Xi_{20}^{(k)})^T - \mathcal{E}(\Xi_{10} - \Xi_{10}^{(k)})^T\Lambda_{12}, \\
&\Lambda_{11}^T(\Xi_{11} - \Xi_{11}^{(k+1)}) + (\Xi_{11} - \Xi_{11}^{(k+1)})\Lambda_{11} \\
&\quad = -\varepsilon_1[\Lambda_{01}^T(\Xi_{10} - \Xi_{10}^{(k)})^T + (\Xi_{10} - \Xi_{10}^{(k)})\Lambda_{01}] \\
&\qquad - \mathcal{E}\sqrt{\alpha}[\Lambda_{21}^T(\Xi_{21} - \Xi_{21}^{(k)}) + (\Xi_{21} - \Xi_{21}^{(k)})^T\Lambda_{21}], \\
&\Lambda_{22}^T(\Xi_{22} - \Xi_{22}^{(k+1)}) + (\Xi_{22} - \Xi_{22}^{(k+1)})\Lambda_{22} \\
&\quad = -\varepsilon_2[\Lambda_{02}^T(\Xi_{20} - \Xi_{20}^{(k)})^T \\
&\qquad + (\Xi_{20} - \Xi_{20}^{(k)})\Lambda_{02}] - \frac{\mathcal{E}}{\sqrt{\alpha}}[\Lambda_{12}^T(\Xi_{21} - \Xi_{21}^{(k)})^T + (\Xi_{21} - \Xi_{21}^{(k)})\Lambda_{12}], \\
&\sqrt{\alpha}(\Xi_{21} - \Xi_{21}^{(k+1)})^T\Lambda_{22} + \frac{1}{\sqrt{\alpha}}\Lambda_{11}^T(\Xi_{21} - \Xi_{21}^{(k+1)})^T \\
&\quad = -\varepsilon_1(\Xi_{10} - \Xi_{10}^{(k)})\Lambda_{02} - \varepsilon_2\Lambda_{01}^T(\Xi_{20} - \Xi_{20}^{(k)})^T \\
&\qquad - \mathcal{E}[\Lambda_{21}^T(\Xi_{22} - \Xi_{22}^{(k)}) + (\Xi_{11} - \Xi_{11}^{(k)})\Lambda_{12}].
\end{aligned}$$

Using the assumption $\|\Xi_{pq} - \Xi_{pq}^{(k)}\| = O(\|\mu\|^{k+1})$, we have

$$\begin{aligned}
&\Lambda_{00}^T(\Xi_{00} - \Xi_{00}^{(k+1)}) + (\Xi_{00} - \Xi_{00}^{(k+1)})\Lambda_{00} \\
&\quad + \Lambda_{10}^T(\Xi_{10} - \Xi_{10}^{(k+1)}) + (\Xi_{10} - \Xi_{10}^{(k+1)})^T\Lambda_{10} \\
&\quad + \Lambda_{20}^T(\Xi_{20} - \Xi_{20}^{(k+1)}) + (\Xi_{20} - \Xi_{20}^{(k+1)})\Lambda_{20} = 0, \\
&(\Xi_{00} - \Xi_{00}^{(k+1)})\Lambda_{01} + (\Xi_{10} - \Xi_{10}^{(k+1)})^T\Lambda_{11} + \Lambda_{10}^T(\Xi_{11} - \Xi_{11}^{(k+1)})
\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{\alpha}\Lambda_{20}^T(\Xi_{21}-\Xi_{21}^{(k+1)})=O(\|\mu\|^{k+2}),\\
&(\Xi_{00}-\Xi_{00}^{(k+1)})\Lambda_{02}+(\Xi_{20}-\Xi_{20}^{(k+1)})^T\Lambda_{22}+\Lambda_{20}^T(\Xi_{22}-\Xi_{22}^{(k+1)})\\
&\quad+\frac{1}{\sqrt{\alpha}}\Lambda_{10}^T(\Xi_{21}-\Xi_{21}^{(k+1)})^T=O(\|\mu\|^{k+2}),\\
&\Lambda_{11}^T(\Xi_{11}-\Xi_{11}^{(k+1)})+(\Xi_{11}-\Xi_{11}^{(k+1)})\Lambda_{11}=O(\|\mu\|^{k+2}),\\
&\Lambda_{22}^T(\Xi_{22}-\Xi_{22}^{(k+1)})+(\Xi_{22}-\Xi_{22}^{(k+1)})\Lambda_{22}=O(\|\mu\|^{k+2}),\\
&\sqrt{\alpha}(\Xi_{21}-\Xi_{21}^{(k+1)})^T\Lambda_{22}+\frac{1}{\sqrt{\alpha}}\Lambda_{11}^T(\Xi_{21}-\Xi_{21}^{(k+1)})^T=O(\|\mu\|^{k+2}).
\end{aligned}
$$

After the cancellation takes place, since $\Lambda_0,\ \Lambda_{jj},\ j=1,\ 2$ are stable from the condition (H3), we get

$$
\begin{aligned}
&\Lambda_0^T(\Xi_{00}-\Xi_{00}^{(k+1)})+(\Xi_{00}-\Xi_{00}^{(k+1)})\Lambda_0=O(\|\mu\|^{k+2}),\\
&(\Xi_{j0}-\Xi_{j0}^{(k+1)})^T=-(\Xi_{00}-\Xi_{00}^{(k+1)})\Lambda_{0j}\Lambda_{jj}^{-1}+O(\|\mu\|^{k+2}),\ j=1,\ 2,\\
&\Xi_{jj}-\Xi_{jj}^{(k+1)}=O(\|\mu\|^{k+2}),\ j=1,\ 2,\\
&\Xi_{21}-\Xi_{21}^{(k+1)}=O(\|\mu\|^{k+2}).
\end{aligned}
$$

Therefore, we have

$$
\|\Xi_{pq}-\Xi_{pq}^{(k+1)}\|=O(\|\mu\|^{k+2}),\quad pq=00,\ 10,\ 20,\ 11,\ 21,\ 22.
$$

Consequently, the equation (26) holds for all $i\in\mathbf{N}$. This completes the proof of Theorem 4.1 concerned with the fixed point algorithm. ▮

5. MULTIPARAMETER H_∞ OPTIMAL CONTROL PROBLEM

5.1. The Design Problem and Preliminaries

In this section, we study the H_∞ control problem by using the state feedback control law for the MSPS

$$
\begin{aligned}
\dot{x}_0 &= A_{00}x_0+A_{01}x_1+A_{02}x_2+B_{01}u_1+B_{02}u_2\\
&\quad +F_{01}w_1+F_{02}w_2,\ x_0^0=0, &\text{(27a)}\\
\varepsilon_1\dot{x}_1 &= A_{10}x_0+A_{11}x_1+\varepsilon_3A_{12}x_2\\
&\quad +B_{11}u_1+\varepsilon_3B_{12}u_2+F_{11}w_1+\varepsilon_3F_{12}w_2,\ x_1^0=0, &\text{(27b)}\\
\varepsilon_2\dot{x}_2 &= A_{20}x_0+\varepsilon_3A_{21}x_1+A_{22}x_2\\
&\quad +\varepsilon_3B_{21}u_1+B_{22}u_2+\varepsilon_3F_{21}w_1+F_{22}w_2,\ x_2^0=0, &\text{(27c)}
\end{aligned}
$$

$$z = \begin{bmatrix} C_{00} & C_{01} & 0 \\ C_{10} & 0 & C_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{27d}$$

where $x_0 \in \mathrm{R}^{n_0}$, $x_1 \in \mathrm{R}^{n_1}$ and $x_2 \in \mathrm{R}^{n_2}$ are the state vector, $u_j \in \mathrm{R}^{m_j}$, $j = 1,\ 2$ is the control input, $w_j \in \mathrm{R}^{l_j}$, $j = 1,\ 2$ is the disturbance, $z \in \mathrm{R}^n$ is the controlled output. In order to simplify derivations, without loss of generality, we assume that the fast state variables are not connected among themselves, i.e., $\varepsilon_3 \equiv 0$, [3, 4, 5, 6].

We discuss the H_∞ optimal control problem that the closed–loop system is internally stable and $\|G_\mathcal{E}\|_\infty < \gamma$, where

$$G_\mathcal{E} = (C + HK_\mathcal{E})(sI_N - A_\mathcal{E} - B_\mathcal{E}K_\mathcal{E})^{-1}F_\mathcal{E}, \tag{28}$$

$$B_\mathcal{E} = \begin{bmatrix} B_{01} & B_{02} \\ \varepsilon_1^{-1}B_{11} & 0 \\ 0 & \varepsilon_2^{-1}B_{22} \end{bmatrix},\ F_\mathcal{E} = \begin{bmatrix} F_{01} & F_{02} \\ \varepsilon_1^{-1}F_{11} & 0 \\ 0 & \varepsilon_2^{-1}F_{22} \end{bmatrix},$$

$$C = \begin{bmatrix} C_{00} & C_{01} & 0 \\ C_{10} & 0 & C_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ H_1 & 0 \\ 0 & H_2 \end{bmatrix},\ H^TH > 0,$$

by using the following state feedback controller (29)

$$u = K_\mathcal{E} \begin{bmatrix} x_0^T & x_1^T & x_2^T \end{bmatrix}^T = K_\mathcal{E}x. \tag{29}$$

The next result was shown by Doyle *et al.* [19].

LEMMA 5.1. *The following are equivalent:*

i) *$A_\mathcal{E} + B_\mathcal{E}K_\mathcal{E}$ is stable and the transfer matrix $G_\mathcal{E}$ satisfies the inequality $\|G_\mathcal{E}\|_\infty < \gamma$.*

ii) *The MARE (30) has the positive semidefinite stabilizing solution.*

$$\begin{aligned} &A_\mathcal{E}^TX_\mathcal{E} + X_\mathcal{E}A_\mathcal{E} + \gamma^{-2}X_\mathcal{E}F_\mathcal{E}F_\mathcal{E}^TX_\mathcal{E} \\ &\quad -X_\mathcal{E}B_\mathcal{E}(H^TH)^{-1}B_\mathcal{E}^TX_\mathcal{E} + C^TC = 0. \end{aligned} \tag{30}$$

Moreover, one such optimal controller that guarantees the γ level of optimality is given by

$$u = K_\mathcal{E}x = -(H^TH)^{-1}B_\mathcal{E}^TX_\mathcal{E}x. \tag{31}$$

Note that the MARE (30) is not a convex function with respect to $P_{\mathcal{E}}$ because the matrix $\gamma^{-2}F_{\mathcal{E}}F_{\mathcal{E}}^T - B_{\mathcal{E}}(H^TH)^{-1}B_{\mathcal{E}}^T$ is in general indefinite.

5.2. Solvability Condition

The H_∞ control problem for the MSPS defined in (27) will be solved by using the algorithm (10). In that respect, we set

$$X_{\mathcal{E}} \ \Rightarrow \ P_{\mathcal{E}}, \ B_{\mathcal{E}}(H^TH)^{-1}B_{\mathcal{E}}^T - \gamma^{-2}F_{\mathcal{E}}F_{\mathcal{E}}^T \ \Rightarrow \ S_{\mathcal{E}}, \ C^TC \ \Rightarrow \ Q \quad (32)$$

where $\Rightarrow$ stands for the replacement.

The AREs (6c) will produce the unique positive semidefinite stabilizing solution under the condition (H1) if γ is large enough. Therefore, let us define the sets as [17, 18]

$\Gamma_{jf} := \{\gamma > 0|$ the pair of AREs (6c) have the positive semidefinite stabilizing solutions$\}$,

$\gamma_{jf} := \inf\{\gamma | \gamma \in \Gamma_{jf}\}$.

Moreover, let us define the set as

$\Gamma_{1s} := \{\gamma > 0|$ the ARE (6a) has a positive semidefinite stabilizing solution$\}$,

$\gamma_{1s} := \inf\{\gamma | \gamma \in \Gamma_{1s}\}$.

As the results, for every $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \ \gamma_{jf}\}$, the MARE (30) has the positive semidefinite stabilizing solutions if $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are small enough. Then, we have the following result.

Corollary 5.1. *If we select a parameter* $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \ \gamma_{jf}\}$, *then there exist small* $\tilde{\varepsilon}_1$ *and* $\tilde{\varepsilon}_2$ *such that for all* $\varepsilon_1 \in (0, \ \tilde{\varepsilon}_1)$ *and* $\varepsilon_2 \in (0, \ \tilde{\varepsilon}_2)$, *the MARE (30) admits a solution such that* $P_{\mathcal{E}}$ *is the symmetric positive semidefinite stabilizing solution, which can be written as (7).*

Proof. Since the proof is similar to Theorem 2.1, it is omitted. ∎

Remark 5. 1. Note that the condition such as $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \ \gamma_{jf}\}$ corresponding to the parameter γ is equivalent to the conditions that the AREs (6c) have the positive semidefinite stabilizing solutions under the conditions (H1) and (H2).

5.3. Numerical Example

In the rest of this section, in order to demonstrate the efficiency of our proposed algorithm, we have run a numerical example. The system matrix is given as a modification of an Appendix A in [1].

$$A_{00} = \begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 \\ 0 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & -0.05 & 0 & -0.1 \\ 0 & 0 & 0 & -0.05 & 0.1 \\ 0 & 0 & 32.7 & -32.7 & 0 \end{bmatrix},$$

$$A_{01} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ A_{02} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0 \end{bmatrix}, \ A_{20} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 \end{bmatrix},$$

$$A_{11} = A_{22} = \begin{bmatrix} -0.05 & 0.05 \\ 0 & -0.1 \end{bmatrix}, \ F_{11} = F_{22} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix},$$

$$F_{01} = F_{02} = B_{01} = B_{02} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ B_{11} = B_{22} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

$$C^T C = \mathrm{diag}(1,\ 1,\ 1,\ 1,\ 1,\ 0.5,\ 0.5,\ 0.5,\ 0.5),\ H^T H = \mathrm{diag}(20,\ 20).$$

Firstly, the numerical results are obtained for small parameter $\varepsilon_1 = \varepsilon_2 = 10^{-3}$. The simulation results for the different parameter ε_j will be discussed later. Note that we can not apply the technique proposed in [5, 6] to the MARE (30) since the Hamiltonian matrices $T_{jj},\ j = 1,\ 2$ have eigenvalues in common. The two basic quantities for the system are $\gamma_{jf} = 9.7590 \times 10^{-2}$, $\gamma_{1s} = 4.4721 \times 10^{-1}$. Thus, for every boundary value $\gamma > \bar{\gamma} = \max\{\gamma_{1s},\ \gamma_{jf}\} = 4.472 \times 10^{-1}$, the AREs (6c) and (6a) have the positive semidefinite stabilizing solutions. On the other hand, by using MATLAB, the minimum value $\hat{\gamma}$ such that there exists the feedback controller is $\hat{\gamma} = 4.472 \times 10^{-1}$.

Now, we choose $\gamma = 1.0$ ($> \bar{\gamma}$) to solve the MARE (30). We give a solution of the MARE (30).

$$P_{\mathcal{E}} = \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ \varepsilon_1 P_{10} & \varepsilon_1 P_{11} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21}^T \\ \varepsilon_2 P_{20} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21} & \varepsilon_2 P_{22} \end{bmatrix}$$

$$P_{00} = \begin{bmatrix} 6.0730e+000 & 8.4607e-001 & 5.1386e+001 & -1.3695e-001 & 2.5846e-001 \\ 8.4607e-001 & 6.0730e+000 & -1.3695e-001 & 5.1386e+001 & -2.5846e-001 \\ 5.1386e+001 & -1.3695e-001 & 6.9744e+002 & -2.3924e+002 & 5.0568e+000 \\ -1.3695e-001 & 5.1386e+001 & -2.3924e+002 & 6.9744e+002 & -5.0568e+000 \\ 2.5846e-001 & -2.5846e-001 & 5.0568e+000 & -5.0568e+000 & 1.3473e+000 \end{bmatrix}$$

$$\varepsilon_1 P_{10} = \begin{bmatrix} 1.0158e-001 & -9.0348e-005 & 1.3678e+000 & -4.7187e-001 & 8.3654e-003 \\ 5.0000e-002 & 7.2948e-015 & 6.6053e-001 & -2.3187e-001 & 3.7279e-003 \end{bmatrix}$$

$$\varepsilon_2 P_{20} = \begin{bmatrix} -9.0348e-005 & 1.0158e-001 & -4.7187e-001 & 1.3678e+000 & -8.3654e-003 \\ 6.0176e-015 & 5.0000e-002 & -2.3187e-001 & 6.6053e-001 & -3.7279e-003 \end{bmatrix}$$

$$\varepsilon_1 P_{11} = \begin{bmatrix} 7.6993e-003 & 2.9751e-003 \\ 2.9751e-003 & 3.9561e-003 \end{bmatrix},$$

$$\varepsilon_2 P_{22} = \begin{bmatrix} 7.6993e-003 & 2.9751e-003 \\ 2.9751e-003 & 3.9561e-003 \end{bmatrix}$$

$$\sqrt{\varepsilon_1 \varepsilon_2} P_{21} = \begin{bmatrix} -9.3283e-004 & -4.5889e-004 \\ -4.5889e-004 & -2.2587e-004 \end{bmatrix}$$

We find that the solution of the MARE (30) converges to the exact solution with accuracy of $\|\mathcal{G}(P_{\mathcal{E}}^{(i)})\| < 10^{-10}$ after 3 iterative iterations. In order to verify the exactitude of the solution, we calculate the remainder per iteration by substituting $P_{\mathcal{E}}^{(i)}$ into the MARE (30). In Table 1 we present results for the error $\|\mathcal{G}(P_{\mathcal{E}}^{(i)})\|$. It can be seen that the initial guess (11) for the algorithm (10) is quite good.

In order to verify the exactitude of the solution, when we substitute the obtained reference solution $P_{\mathcal{E}}^{\mathrm{sch}}$ by using the function `are` of MATLAB into the MARE (30), the remainder is $\|\mathcal{G}(P_{\mathcal{E}}^{\mathrm{sch}})\| = 1.7864e-009$. For different values of ε_1 and ε_2, the remainder of the algorithm (10) versus MATLAB are given by Table 2.

TABLE 1.

Errors per Iteration

i	$\|\mathcal{G}(P_{\mathcal{E}}^{(i)})\|$
0	$3.2505e-010$
1	$1.0193e-002$
2	$5.0362e-005$
3	$4.2618e-012$

TABLE 2.

Error $\|\mathcal{G}(P_{\mathcal{E}})\|$

$\varepsilon_1 = \varepsilon_2$	Revised Kleinman algorithm	MATLAB
10^{-2}	$8.8142e-011$	$3.3465e-010$
10^{-3}	$5.9038e-012$	$1.7864e-009$
10^{-4}	$3.4592e-011$	$2.2509e-008$
10^{-5}	$4.1606e-012$	$1.3073e-005$
10^{-6}	$8.7978e-012$	$5.2618e-004$
10^{-7}	$6.1600e-012$	$1.4103e-003$
10^{-8}	$1.5099e-011$	$3.0732e-002$

TABLE 3.

CPU Times [sec]

$\varepsilon_1 = \varepsilon_2$	Revised Kleinman algorithm	MATLAB
10^{-2}	$5.44e-001$	$2.80e-002$
10^{-3}	$1.32e-001$	$2.70e-002$
10^{-4}	$8.00e-002$	$2.60e-002$
10^{-5}	$8.00e-002$	$2.70e-002$
10^{-6}	$4.10e-002$	$2.60e-002$
10^{-7}	$4.30e-002$	$2.70e-002$
10^{-8}	$2.50e-002$	$2.70e-002$

From Table 2, it should be noted that although the dimensionality of the MARE (30) is small, when the parameter ε_j is quite small, the loss of accuracy corresponding to the error $\|\mathcal{G}(P_{\mathcal{E}})\|$ for MATLAB is obvious for this numerical example. On the other hand, the resulting algorithm which combine the Kleinman algorithm (10) and the fixed point algorithm (24) computes the solution to full accuracy for all ε_j. Hence, the resulting algorithm of this paper is very useful at least in this example. In Table 3, we give the results of the CPU times when we have run the new method versus MATLAB. From Table 3, even if the iterative algorithm (10) takes a lot of CPU times in the case of not very small value of the singular perturbation parameter, our algorithm can obtain the exact solution.

6. CONCLUSION

In this paper, we have investigated the MARE with an indefinite quadratic term in general associated with the MSPS. We have shown that there exists a unique and bounded solution for the MARE. Furthermore, we have presented the iterative method for solving the sign indefinite GMARE. Finally, based on the fixed point algorithm, we have presented the new numerical methods for solving the GMALE appearing in the Kleinman algorithm. It should be noted that so far the algorithm for solving the GMALE with multiparameter has not been established.

The algorithms for solving the GMARE and GMALE are applied to a wide class of control law synthesis involving a solution of the MARE such as the robust stabilizing control problem and the guaranteed cost control problem.

REFERENCES

1. H. K. Khalil and P. V. Kokotovic, Control strategies for decision makers using different models of the same system, *IEEE Trans. Automat. Control*, **AC–23**, No. 2 (1978), 289–298.
2. Z. Gajic and H. K. Khalil, Multimodel strategies under random disturbances and imperfect partial observations, *Automatica*, **22**, No. 1 (1986), 121–125.
3. Z. Gajic, The existence of a unique and bounded solution of the algebraic Riccati equation of multimodel estimation and control problems, *Systems and Control Lett.*, **10**, (1988), 185–190.
4. Y.-Y. Wang, P. M. Frank and N. E. Wu, Near–optimal control of nonstandard singularly perturbed systems, *Automatica*, **30**, No. 2 (1994), 277–292.
5. C. Coumarbatch and Z. Gajic, Exact decomposition of the algebraic Riccati equation of deterministic multimodeling optimal control problems, *IEEE Trans. Automat. Control*, **AC–45**, No. 4 (2000), 790–794.
6. C. Coumarbatch and Z. Gajic, Parallel optimal Kalman filtering for stochastic systems in multimodeling form, *Trans. ASME, J. Dynamic Systems, Measurement, and Control*, **122**, No. 12 (2000), 542–550.
7. A. J. Laub, A Schur method for solving algebraic Riccati equations, *IEEE Trans. Automat. Control*, **AC–24**, No. 6 (1979), 913–921.
8. V. Ionescu, C. Oară and M. Weiss, Generalized matrix pencil techniques for the solution of algebraic Riccati equations: A unified approach, *IEEE Trans. Automat. Control*, **AC–42**, No. 8 (1997), 1085–1097.

9. D. L. Kleinman, On an iterative technique for Riccati equation computations, *IEEE Trans. Automat. Control*, **AC–13**, No. 2 (1968), 114–115.

10. J. M. Ortega and W. C. Rheinboldt, "Iterative Solution of Nonlinear Equations in Several Variables," Academic Press, New York, 1970.

11. T. Yamamoto, A method for finding sharp error bounds for Newton's method under the Kantorvich assumptions, *Numerische Mathematik*, **49**, (1986), 203–220.

12. J. R. Magnus and H. Neudecker, "Matrix Differential Calculus with Applications in Statistics and Econometrics," John Wiley and Sons, New York, 1999.

13. K. Zhou, "Essentials of Robust Control," Prentice–Hall, New Jersey, 1998.

14. Z. Gajic, D. Petkovski and X. Shen, "Singularly Perturbed and Weakly Coupled Linear System–a Recursive Approach: Lecture Notes in Control and Information Sciences, 140," Springer–Verlag, Berlin, 1990.

15. V. Dragan, Asymptotic expansions for game–theoretic Riccati equations and stabilization with disturbance attenuation for singularly perturbed systems, *Systems and Control Lett.*, **20**, (1993), 455–463.

16. H. Mukaidani and K. Mizukami, The guaranteed cost control problem of uncertain singularly perturbed systems, *J. Math. Anal. Appl*, **251**, (2000), 716–735.

17. H. Mukaidani, H. Xu and K. Mizukami, Recursive approach of H_∞ control problems for singularly perturbed systems under perfect and imperfect state measurements, *Int. J. Systems Science*, **30**, No.5 (1999), 467–477.

18. H. Mukaidani, H. Xu and K. Mizukami, An iterative technique for solving singularly perturbed algebraic Riccati equation with indefinite sign related to H_∞ control, *IEEE Trans. Automat. Control*, **AC–46**, No.11 (2001), (to appear).

19. J. C. Doyle, K. Glover, P. P. Khargonekar and B. A. Francis, State space solution to standard H_2 and H_∞ control problems, *IEEE Trans. Automat. Control*, **AC–34**, No. (1989), 831–847.