Transition temperature and Equation of State from RBC-Bielefeld Collaboration

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Aim of this talk

QCD Thermodynamics from Lattice QCD

Lattice QCD enables us to perform
the first principle calculations of QCD

What are necessary to control some systematic uncertainties ?

What were not sufficient in previous studies ?

What are improved in our calculation ?

This talk (of first part) is based on our recent publication
M.Chen et al., Phys. Rev. D74 (2006) 054507.
→ T_c = 192(7)(4) MeV

There is no section for comparison with other group's results.
You should check the Frithjof's talk in QM2006 !!

Contents of this talk

[1] Aim of this talk

[2] For the first principle calculation

- sources of systematic uncertainties
- problem in previous studies

[3] Our approach

- strategy
- choice of lattice action

[4] Transition temperature *M.Chen et al. Phys.Rev.D74 054507 (2006)*

[5] Equation of State *on going calculation*

[6] Conclusion

These may be important for non-lattice people…

Source of uncertainties

We have to control many sources of uncertainties.

- Dynamical quark effects

- Finite volume effects

- Continuum extrapolation (a→0 limit)

- Statistical error

Sufficient calculation needs huge computational cost !!

but one can compromise some of them depending on aim of study.

Source of uncertainties

We have to control many sources of uncertainties.

- Dynamical quark effects
  - Num. of flavors: N_f=2+1 increases parameter space
  - quark masses : lighter quark masses increase comp. cost

    chiral extrapolation with heavier quarks → source of sys. error
- Finite volume effects
  - $L^3 = (N_s/a)^3$: larger volume increases comp. cost

    lighter quark mass needs larger volume at T=0 e.g. $Lm_\pi \geq$ 3–4
- Continuum extrapolation (a→0 limit)
  - the extrapolation has to be performed in scaling region
  - $L^3 = (N_s/a)^3$: smaller "a" increase comp. cost

  Improved actions allow some observables to scale with larger "a"
- Statistical error

Sufficient calculation needs huge computational cost !!

but one can compromise some of them depending on aim of study.

Case of QCD thermodynamics

One can compromise some of them
depending on aim of study

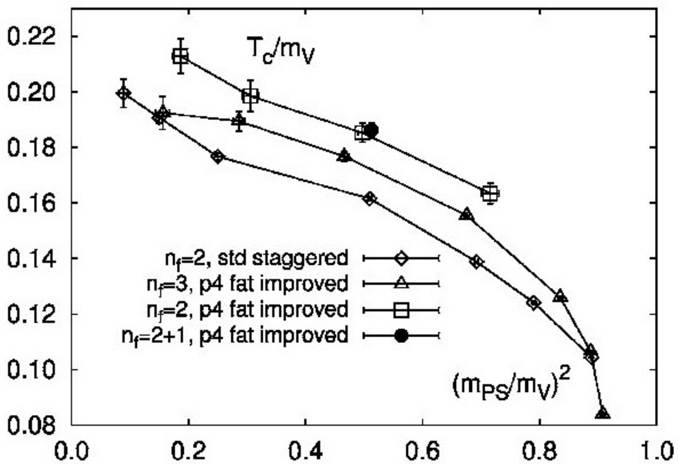
For example, qualitative study of hadron spectroscopy
→ (quenched/small V/no a→0 limit) are not so bad

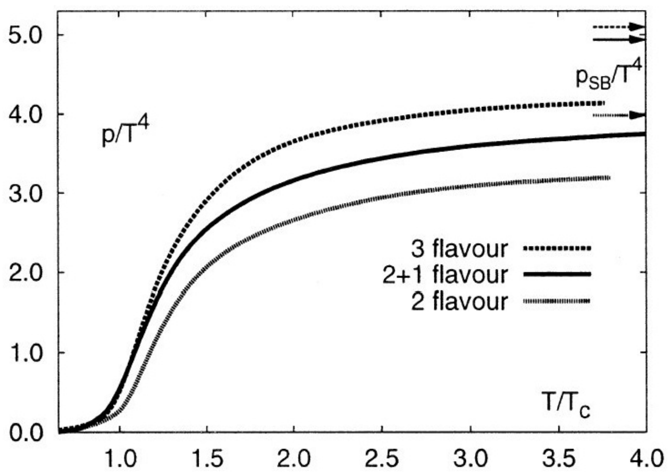
However, our aim is
"Quantitative study of QCD thermodynamics in Lattice QCD"

- Some observables are sensitive to N_f & m_q
- Study of critical phenomena requires large volume (volume dep.)
- Determination of transition point requires (very) large statistics

Although basic methodology has been established,
there is no calculation in which systematic uncertainties
are controlled in a few % level

Previous studies





F.Karsch et al., Nucl. Phys. B605 (2001) 579.

F.Karsch et al., Phys. Lett. B478 (2000) 447.

T_c = 173(8) MeV at chiral limit

- $N_f = 2$ （ $N_f = 3$: 154(8) MeV ）
- only $N_t = 4$, no continuum limit
- pion mass ≃ 400 – 1000 MeV
- scale setting by rho meson mass
- $N_s/N_t = 2 - 4$
- Improved Staggered quark
  p4fat3 action

These were the first systematic studies for QCD thermodynamics in full QCD !!

Our approach

Quantitative study of QCD thermodynamics

T_c, EoS, phase diagram, small μ, etc…

from recent studies, we know

these quantities strongly depend on m_q & N_f

- $N_f = 2+1$ with physics strange quark mass
- $N_t = 4$ & 6, → continuum limit
- pion mass ≃ 150 – 500 MeV with kaon mass ≃ 500 MeV
- scale setting by static quark potential: r_0 (Sommer scale)
- $N_s/N_t = 2 - 4$
- Improved Staggered quark : p4fat3 action

These calculations require high performance computers !!

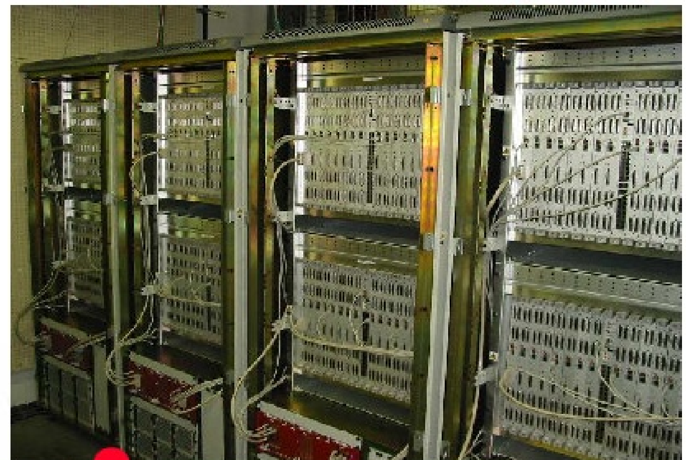
Our computers

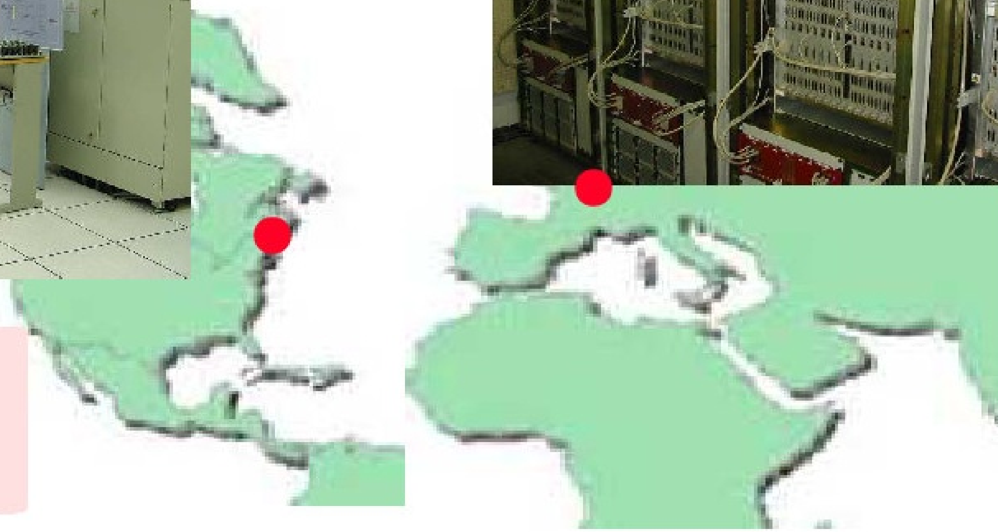
US/RBRC QCDOC

20.000.000.000.000 ops/sec

BI – apeNEXT

5.000.000.000.000 ops/sec



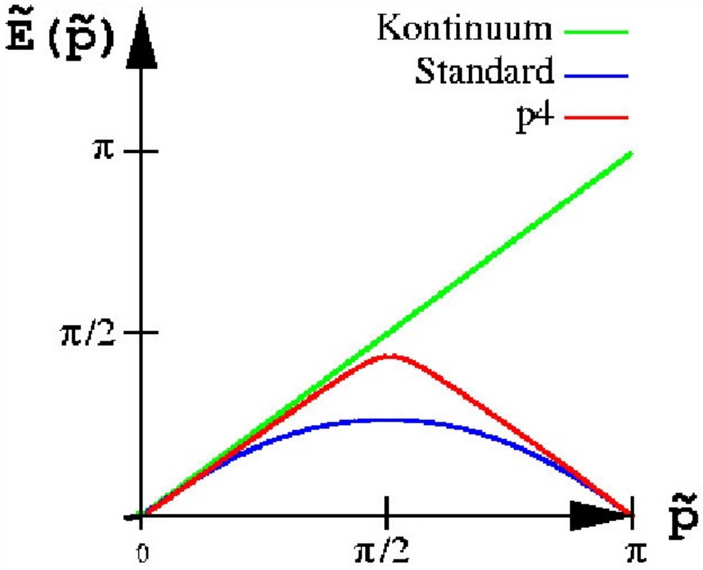


- critical temperature
- equation of state
- finite density QCD

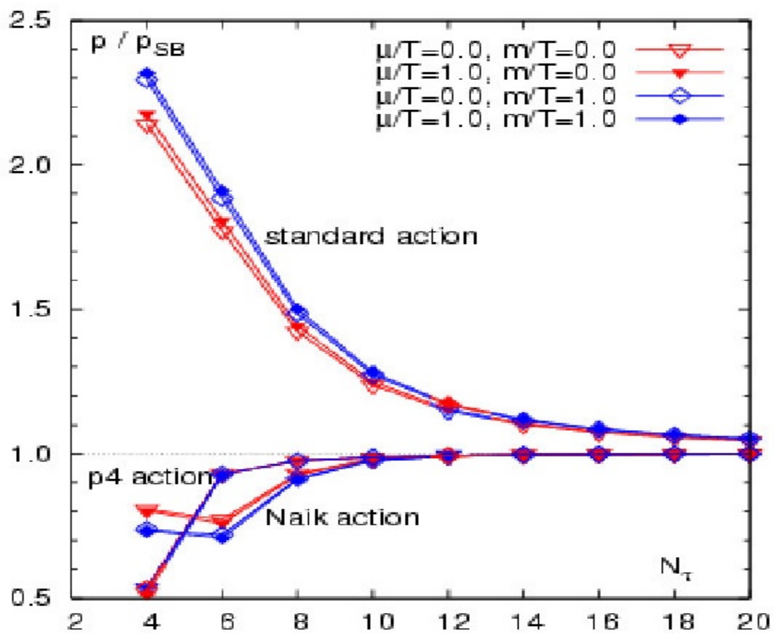
http://quark.phy.bnl.gov/~hotqcd

Properties of the p4fat3 action

Dispersion relation



pressure in high T limit



The free quark propagator is rotational invariant up to $O(p^4)$

Bulk thermodynamic quantities show drastically reduced cut-off effects

flavor sym. is also improved by fat link

Contents of this talk

Simulation parameters

Input parameters in Lattice QCD

β : (= $6/g^2$) gauge coupling (→ lattice spacing)

m_l : light quark masses (up & down)

m_s : strange quark mass

- Critical β search at T>0 calculations
  - $N_t = 4$, $m_s = 0.065$, $V = 8^3, 16^3$, $m_l/m_s = 0.05-0.4$
  - $N_t = 6$, $m_s = 0.04$, $V = 16^3$, $m_l/m_s = 0.1-0.4$

  (about 10 β values × 20K – 60K config. for each simulations)

  to check m_s dependence on T_c → N_t=4, m_s=0.1, m_l/m_s=0.2–0.5

- Scale settings at each critical β on 16^3x32 lattices (T=0)

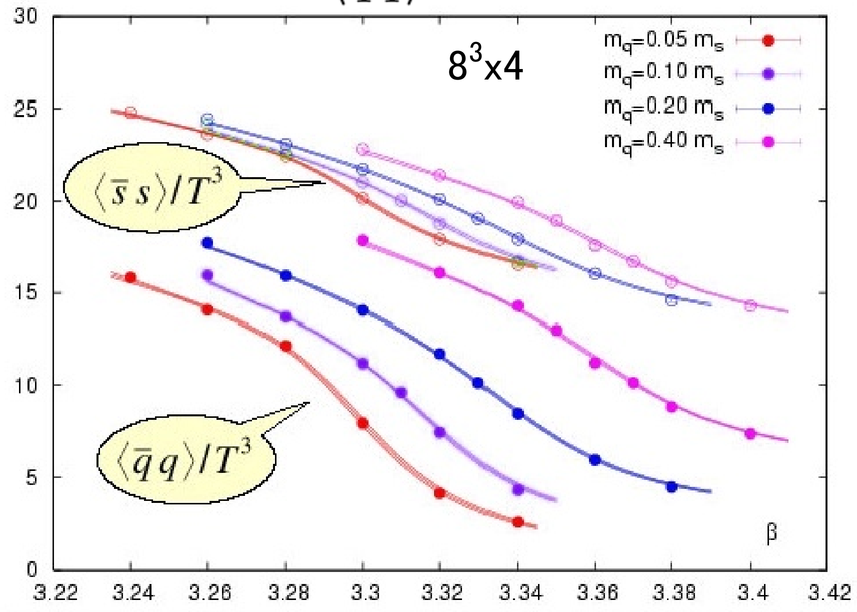
Exact algorithm (RHMC) is used
(Path integral is rigorously performed → only stat. error).

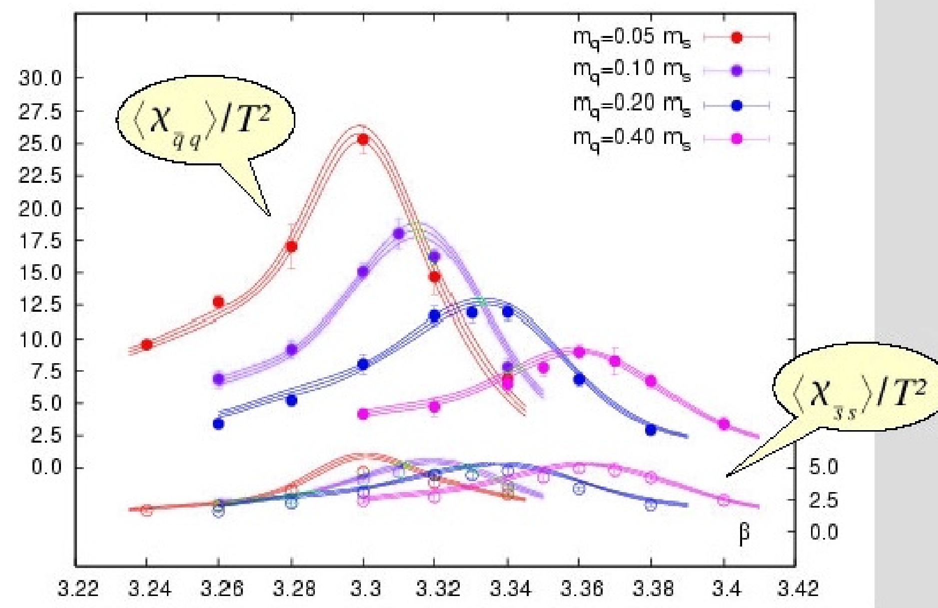
Critical β search

Order parameters : $\langle\bar{q}q\rangle, \langle\bar{s}s\rangle, \langle L\rangle$

chiral condensate: $\langle\bar{q}q\rangle$

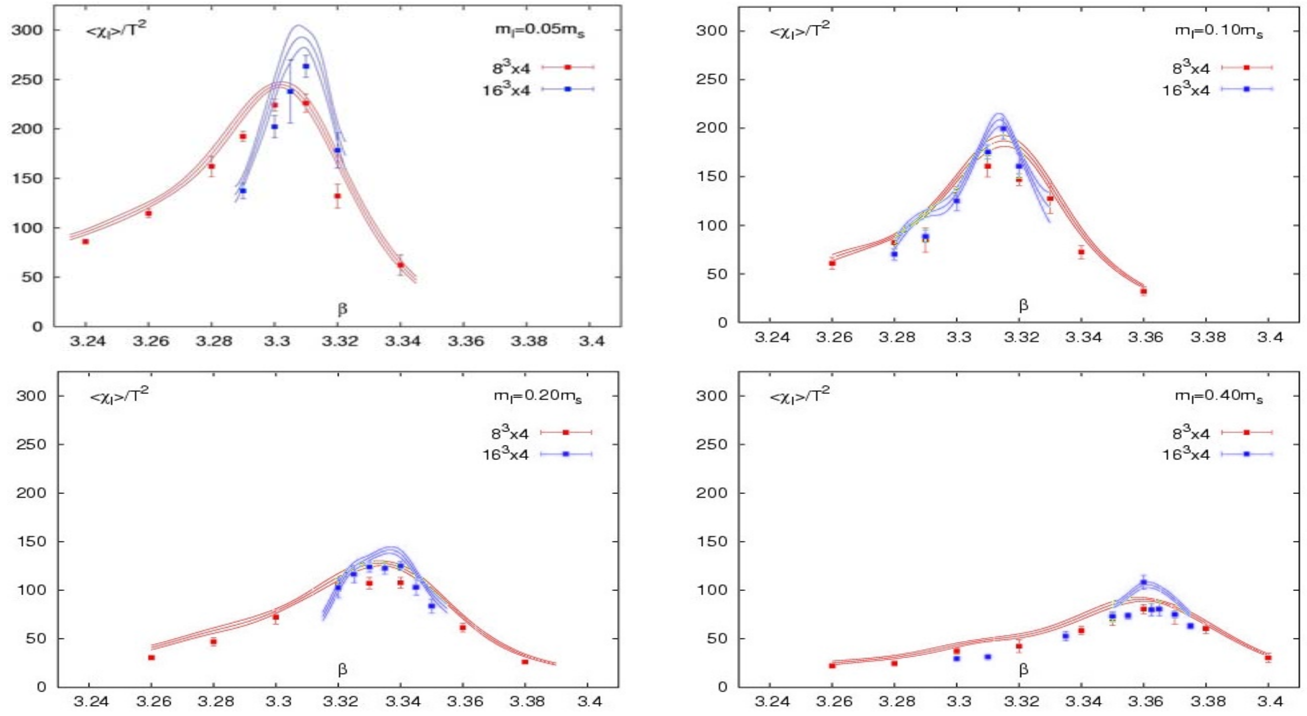
chiral susceptibility: $\langle\chi_{\bar{q}q}\rangle \equiv \langle(\bar{q}q)^2\rangle - \langle\bar{q}q\rangle^2$





- multi-histogram method (Ferrenberg-Swendson) is used
- β_c are determined by peak positions of the susceptibilities

Volume dependence of β_c



- No large change in peak height & position

  → consistent with crossover transition rather than true transition

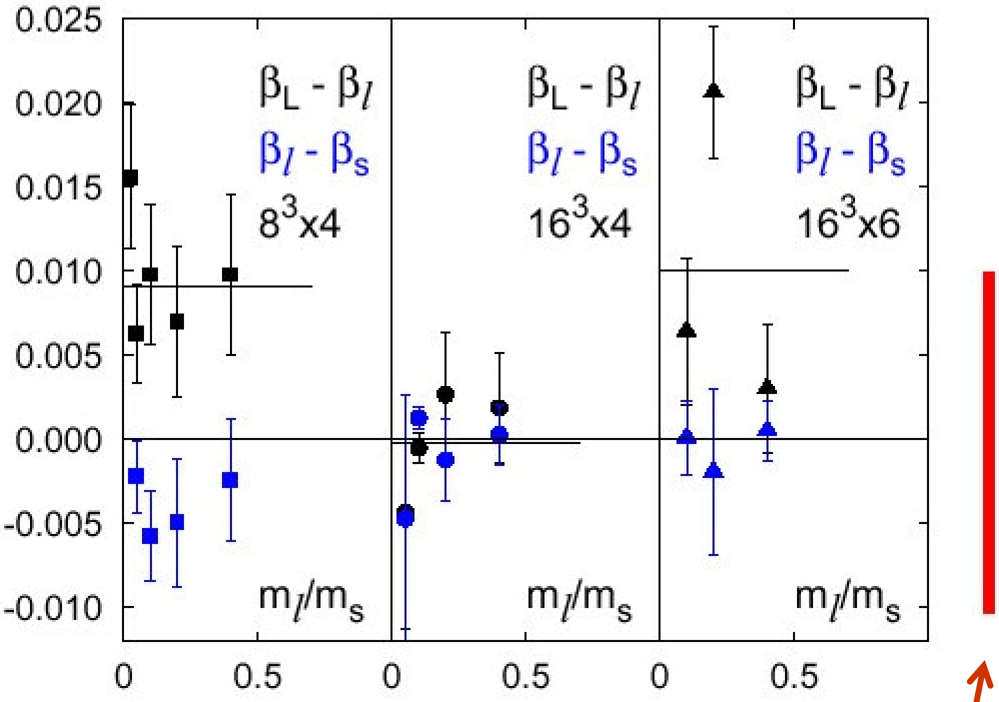
- Reliable calculation of susceptibilities requires large statistics

  at least tens thousands of trajectories are necessary at T>0

Ambiguites in β_c

β_l, β_s: peak position of chiral (strange quark) susceptibility.
β_L : peak position of Polyakov loop susceptibility

- no quark mass dependence
- the difference is negligible at 16^3x4 (N_s/N_t=4)
- the difference at 16^3x 6 are taken into account as a systematic error in β_c



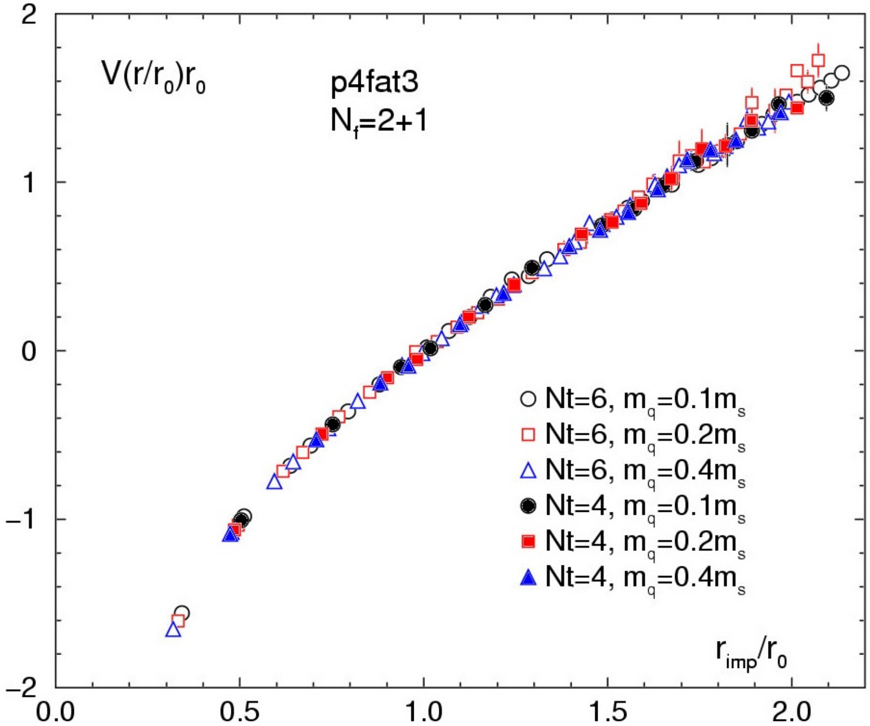
2.5% (N_t=4) or 4% (N_t=6)
error band ↔ 5 or 8 MeV

Scale setting at T=0

Lattice scale is determined by a static quark potential V(r)

$$r^2 \frac{dV_{\bar{q}q}(r)}{dr}\Big|_{r=r_0} = 1.65$$

Sommer scale : r_0



$$V_3(r) = -\frac{\alpha}{r_I} + \sigma r_I + C$$

$$V_4(r) = -\frac{\alpha}{r} + \sigma r_I - \alpha' \left(\frac{1}{r_I} - \frac{1}{r}\right) + C$$

where, r_I is the improve dist.

- systematic errors of r_0 and σ

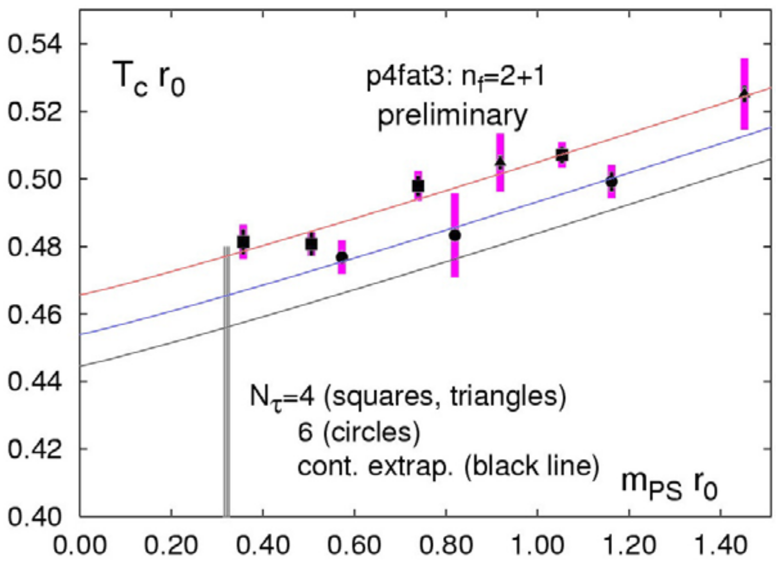
  → diff. between $V_3(r)$ & $V_4(r)$

  diff. in various fit range

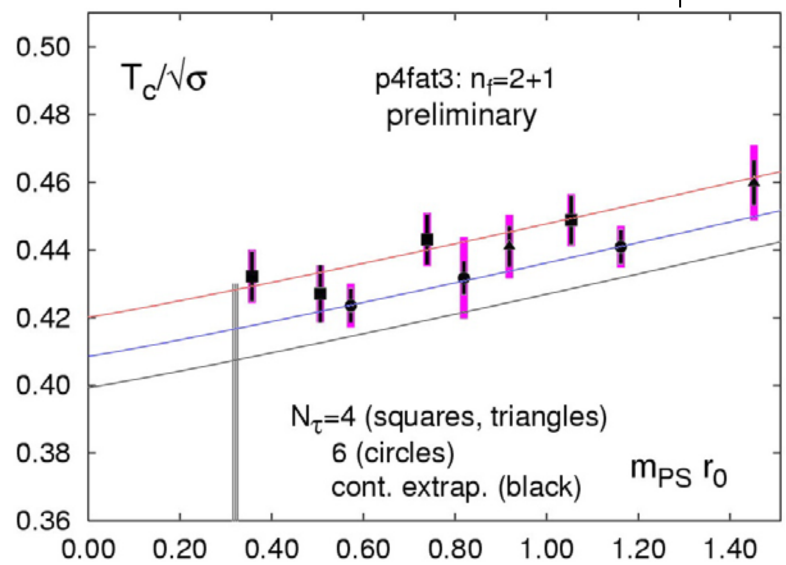
  (r_{min}=0.15−0.3fm, r_{max}=0.7−0.9fm)

Critical temperature

in units of Sommer scale r_0:



in units of string tension σ:



extrapolation to chiral & continuum limit

$$(T_c r_0)_{N_\tau} = (T_c r_0)_{cont.} + b(m_{ps} r_0)^d + c/N_\tau^2$$

(d = 1.08 for O(4), 2nd order
2 for 1st order)

$$T_c r_0 = 0.456(7)^{+3}_{-1}, \quad T_c/\sqrt{\sigma} = 0.408(7)^{+3}_{-1}$$ at phys. point,

[fit form dependence → d=1(lower), 2(upper error)]

T_c in physical units

Finally we obtain T_c=192(7)(4)MeV

using r_0=0.469(7)fm determined from quarkonium spectroscopy

A. Gray et al. Phys. Rev. D72 (2005) 094507

Why we use r_0 for scale setting ?

[1] We can, of course, use other observables, e.g. m_ρ

but it is difficult to control

stat. & syst. error of m_ρ on course lattice

[2] r_0 seems to be the best controlled lattice observable

for scale setting to determine the T_c

The physical value of r_0 have been deduced from

lattice calculations through a comparison with

bottomonium level splitting by MILC Colalb.

→ also consistent with exp. value in light sector,

e.g. f_π, f_K

T_c in physical units

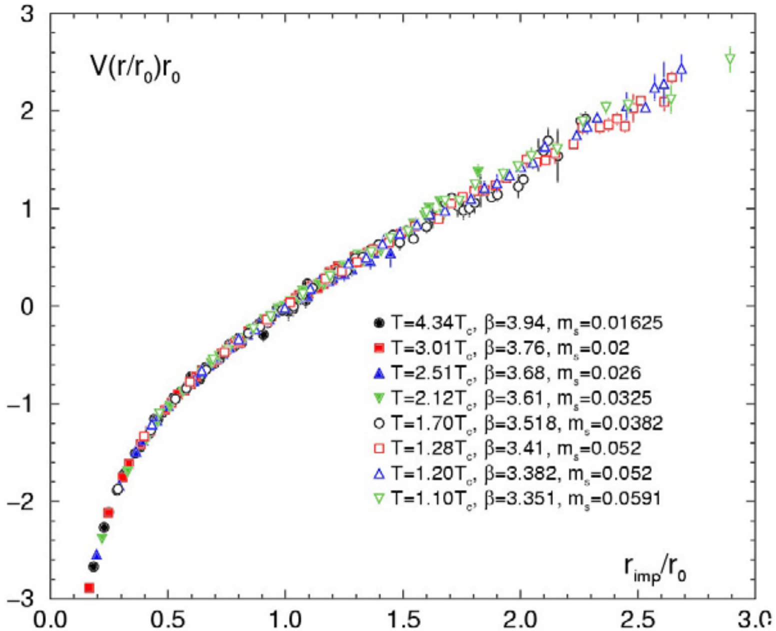
Finally we obtain T_c=192(7)(4)MeV

using r_0=0.469(7)fm determined from quarkonium spectroscopy

A. Gray et al. Phys. Rev. D72 (2005) 094507

Why we use r_0 for scale setting ?

[3] small scaling violation from "a" at $T_c(N_t=4)$



T=0 calculations at corresponding lattice spacing for various Temp.

no significant cut-off dependence when cut-off vaires by a factor 4

Summary for T_c

N_f=2+1 simulation with almost realistic quark masses at N_t=4, 6

- transition temperature in our study

  $T_c r_0$=0.456(7), (T_c=192(7)(4)MeV using r_0=0.469(7)fm)

  – $T_c r_0$ is consistent with previous p4 result
  difference in T_c mainly comes from physical value of r_0

  – most systematic uncertainties are taken into account
  remaining uncertainty is in continuum extrapolation

  – however, our value is about 10% larger than MILC result
  and about 20% larger than Wuppertal group result
  → you should check the Frithjof's talk in QM2006

- toward the finial result from the first principle calculation

  – remaining uncertainty has to be evaluated with N_t=8 and more

  – consistency with Wilson type fermion (Domain-wall, overlap) results

Contents of this talk

Equation of State

Preliminary !!

Equation of State at N_t=4 lattices（Nt=6 is on progress)

by using the Integral method

on a Line of Constant Physics (LCP)

T>0 calculations are performed on 16^3x4 lattices

Temp. range is T/Tc = 0.8 – 4.3（12 data points now)

zero temp. subtraction is calculated on 16^3x32 lattices

Integral method

For example: calculation of pressure p

$$p = -f = \frac{T}{V} \ln Z$$ for large homogeneous system

Lattice QCD can not calculate the partition function $\ln Z$

however can calculate its derivative $\frac{\partial}{\partial \beta} \ln Z = -\left\langle \frac{\partial S_{QCD}}{\partial \beta} \right\rangle$

One can get p as an integral of derivative of p

high temp.

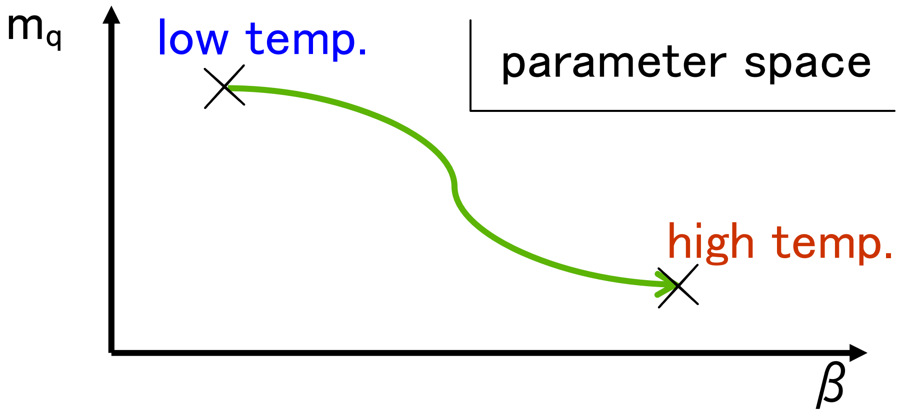
low temp.
with p≃0

$$\left.\frac{p}{T^4}\right|_{\beta_0}^{\beta} = \frac{1}{VT^3} \int_{\beta_0}^{\beta} d\beta' \frac{\partial}{\partial \beta'} \ln Z$$

$$= -N_t^4 \int_{\beta_0}^{\beta} d\beta' \frac{1}{N_s^3 N_t} \left(\langle S_{QCD} \rangle_{T>0} - \langle S_{QCD} \rangle_{T=0} \right)$$

Line of Constant Physics

Now we have three parameters: β, m_l and m_s

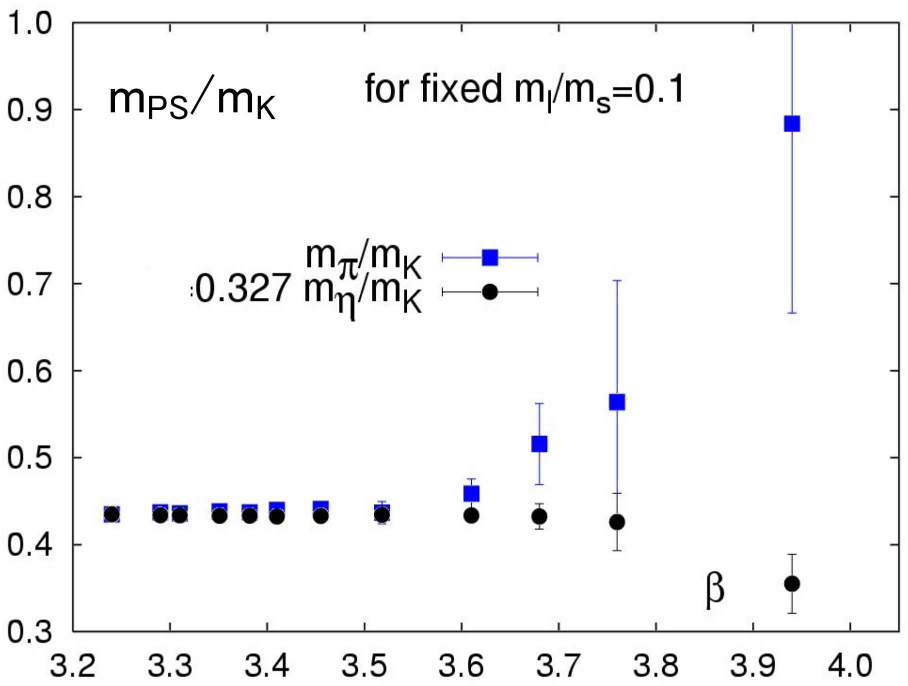


The physics (observables) should be kept along the integral path,
only lattice spacing (temperature) varies along the path
➔ Line of Constant Physics (LCP)

β is only parameter on the LCP
m_l and m_s are function of β

Our approach for LCP

We define the LCP to keep (m_{PS}/m_K & $m_{PS}r_0$)
→ two parameters (m_l & m_s) are determined



m_{PS}/m_K depends on only Δ
$\Delta = m_l/m_s$

We consider the LCP with (m_l, Δ)
instead of (m_l, m_s)

We assume that
Δ is constant on our LCP !!

it reduces comp. cost
and source of sys. error

Our approach for LCP

- The other parameter to determin the LCP : $\hat{m}_l = \hat{m}_l(\beta)$

  $\hat{m}_l$ is determined by the condition for 'm$_{PS}$r$_0$'

  calculations of m$_{PS}$ and r$_0$ at various params.
  → fits with appropriate ansatz

$$\begin{cases} am_{\mathrm{PS}}(\beta, \hat{m}_l, \Delta) \\ r_0/a(\beta, \hat{m}_l, \Delta) \end{cases}$$

- To calculate the interaction measure (ε −3p)/T some beta-functions are needed.

$$R_\beta = a \frac{\mathrm{d}\beta}{\mathrm{d}a}\bigg|_{\hat{m}_l, \hat{m}_s} = \frac{a}{r_0} \left(\frac{\partial(a/r_0)}{\partial\beta}\bigg|_{LCP} \right)^{-1}$$

$$R_{\hat{m}_l} = \left(\frac{\partial \hat{m}_l}{\partial\beta} \right)_\Delta$$

$$R_\Delta = \left(\frac{\partial\Delta}{\partial\beta} \right)_{\hat{m}_l}$$

Our approach for LCP

- The other parameter to determin the LCP :

  $\hat{m}_l$ is determined by the condition for 'm$_{PS}$r$_0$'

calculations of m_{PS} and r_0 at various params.
→ fits with appropriate ansatz

- To calculate the interaction measure (ε −3p)/T
  some beta-functions are needed.

$$R_\beta = a\frac{d\beta}{da}\Big|_{\hat{m}_l,\hat{m}_s} = \frac{a}{r_0}\left(\frac{\partial(a/r_0)}{\partial\beta}\Big|_{LCP}\right)^{-1}$$

$$R_{\hat{m}_l} = \left(\frac{\partial\hat{m}_l}{\partial\beta}\right)_\Delta$$

$$R_\Delta = \left(\frac{\partial\Delta}{\partial\beta}\right)_{\hat{m}_l}$$ = 0 on our LCP

←

$r_0/a(\beta, \hat{m}_l, \Delta)$

$\hat{m}_l = \hat{m}_l(\beta)$

details are rather technical
I skip these steps

Integral method with (m_l, Δ)

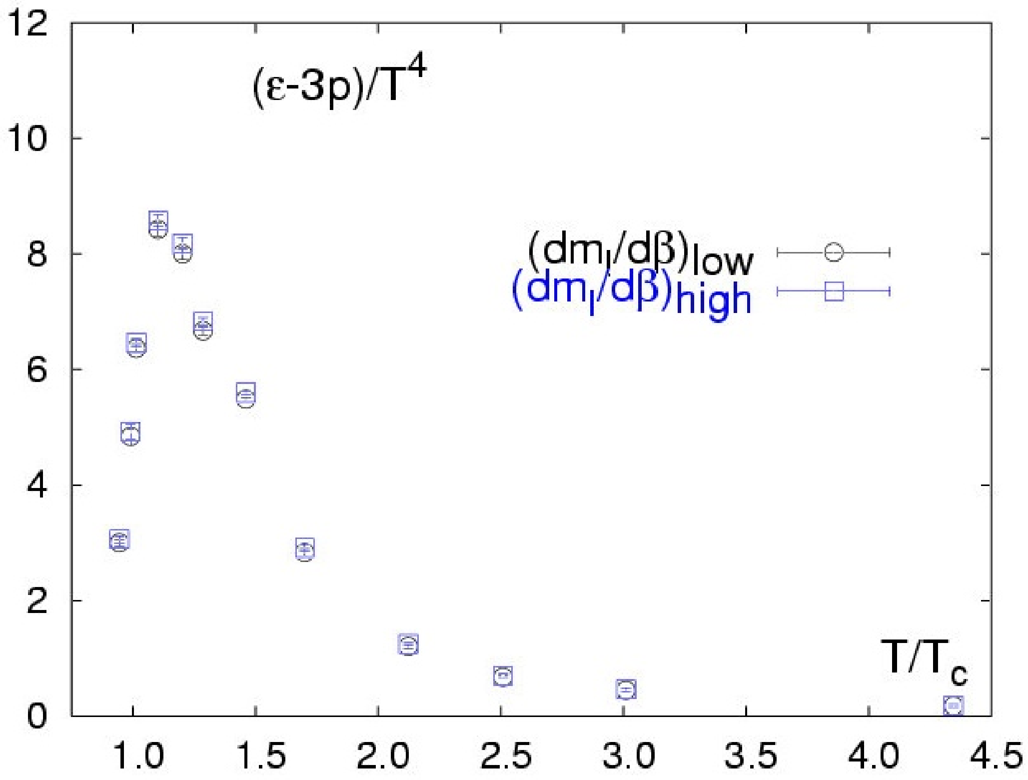
$$\left.\frac{p}{T^4}\right|_{\beta_0}^{\beta} = N_\tau^4 \int_{\beta_0}^{\beta} d\beta' \left[\frac{1}{N_\sigma^3 N_\tau}(\langle S_g\rangle_0 - \langle S_g\rangle_T)\right.$$
$$-(2(\langle\bar\psi\psi\rangle_{l0} - \langle\bar\psi\psi\rangle_{lT}) + \Delta(\langle\bar\psi\psi\rangle_{s0} - \langle\bar\psi\psi\rangle_{sT}))\left(\frac{\partial \hat{m}_l}{\partial\beta'}\right)_\Delta$$
$$\left.-\hat{m}_l\left((\langle\bar\psi\psi\rangle_{s0} - \langle\bar\psi\psi\rangle_{sT})\right)\left(\frac{\partial\Delta}{\partial\beta'}\right)_{\hat{m}_l}\right]$$

$$\frac{\epsilon - 3p}{T^4} = T\frac{d}{dT}\left(\frac{p}{T^4}\right) = a\frac{d\beta}{da}\frac{\partial p/T^4}{\partial\beta}$$
$$= \left(\frac{\epsilon-3p}{T^4}\right)_g + \left(\frac{\epsilon-3p}{T^4}\right)_{\hat{m}_l} + \left(\frac{\epsilon-3p}{T^4}\right)_\Delta$$

$$\left(\frac{\epsilon-3p}{T^4}\right)_g = \left(\frac{N_\tau}{N_\sigma}\right)^3\left(\frac{d\beta}{da}\right)(\langle S_g\rangle_0 - \langle S_g\rangle_T)$$
$$\left(\frac{\epsilon-3p}{T^4}\right)_{\hat{m}_l} = N_\tau^4\left(\frac{d\beta}{da}\right)\left(\frac{\partial\hat{m}_l}{\partial\beta}\right)\left[2\left(\langle\bar\psi\psi\rangle_{l,0} - \langle\bar\psi\psi\rangle_{l,T}\right) + \Delta\left(\langle\bar\psi\psi\rangle_{s,0} - \langle\bar\psi\psi\rangle_{s,T}\right)\right]$$
$$\left(\frac{\epsilon-3p}{T^4}\right)_\Delta = N_\tau^4\left(\frac{d\beta}{da}\right)\left(\frac{\partial\Delta}{\partial\beta}\right)\left(\langle\bar\psi\psi\rangle_{s,0} - \langle\bar\psi\psi\rangle_{s,T}\right)$$

We need beta–functions : $R_\beta = \frac{d\beta}{da},\ R_{\hat{m}_l} = \left(\frac{\partial\hat{m}_l}{\partial\beta}\right)_\Delta,\ \left[R_\Delta = \left(\frac{\partial\delta}{\partial\beta}\right)_{\hat{m}_l}\right]$

Interaction measure



- small statistical error with current statistics

  T>0 : 10,000–30,000 traj.

  T=0 : 2,000– 9,000 traj.

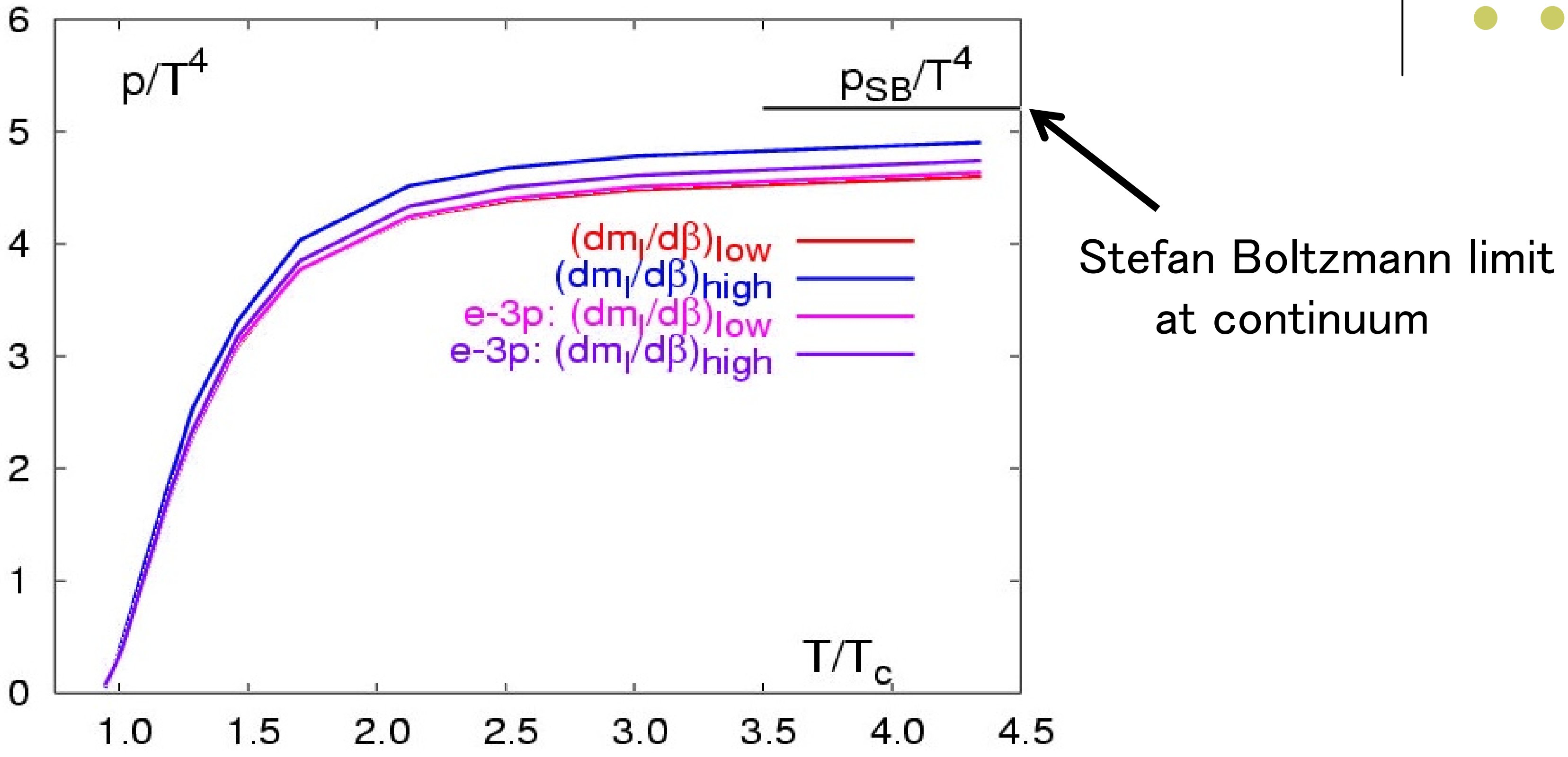
- uncertainty of beta-func. is well controlled.

β	m_l	$m_\pi r_0$	$m_\eta r_0$	d_1	d_2	d_4
3.31	0.0065	0.505	1.538	0.0216	1.2087	1.9184
3.455	0.004	0.573	1.723	0.0266	1.0886	1.9267
3.61	0.00325	0.681	1.970	0.0594	0.2294	1.0679

← "low"

← "high"

Pressure



$(dm_l/d\beta)$: $\left.\frac{p}{T^4}\right|_{\beta_0}^{\beta} = \frac{1}{VT^3}\int_{\beta_0}^{\beta} d\beta' \left[\frac{\partial \ln Z}{\partial \beta'} + \frac{\partial \ln Z}{\partial \hat{m}_l}\frac{\partial \hat{m}_l'}{\partial \beta'}\right]$

e−3p: $(dm_l/d\beta)$: $\left.\frac{p}{T^4}\right|_{T_0}^{T} = \int_{T_0}^{T} dT' \left(\frac{\epsilon - 3p}{T'^3}\right) = \int_{T_0}^{T} dT' \frac{d}{dT'}\left(\frac{p}{T'^4}\right)$

Summary for EoS

N_f=2+1 simulation with almost realistic quark masses at N_t=4, 6

- Equation of state
  - We calculate EoS on a Line of Constant Physics at N_t=4
  - using $\Delta = m_l/m_s$
  - N_t=6 is on progress

Conclusion

N_f=2+1 simulation with almost realistic quark masses at N_t=4, 6

- critical temperature

  $T_c r_0$=0.456(7), (T_c=192(7)(4)MeV using r_0=0.469(7)fm)

  – most systematic uncertainties are taken into account
    remaining uncertainty is in continuum extrapolation

- Equation of state
  - We calculate EoS on a Line of Constant Physics at N_t=4
  - using $\Delta=m_l/m_s$
  - N_t=6 is on progress

- toward the finial result from the first principle calculation
  - remaining uncertainty has to be evaluated with N_t=8 and more
  - consistency with Wilson type fermion (Domain-wall, overlap) results

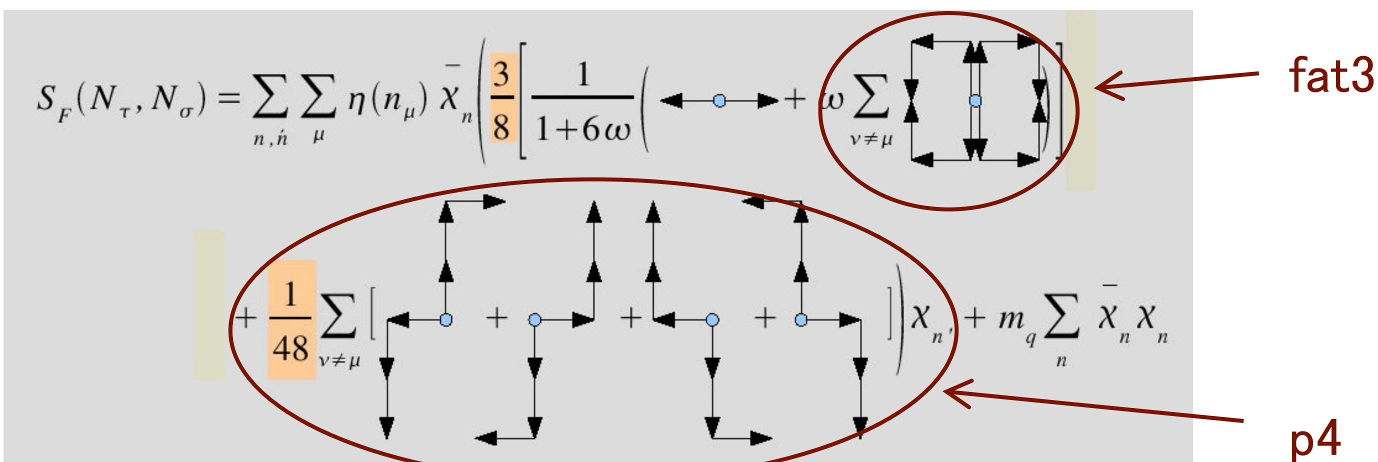
appendix

Choice of Lattice Action

- gluonic part : Symanzik improvement scheme
  - remove cut-off effects of $O(a^2)$
  - tree level improvement $O(g^0)$
- fermion part : improved Staggered fermion : p4fat3 action

Karsch, Heller, Sturm (1999)

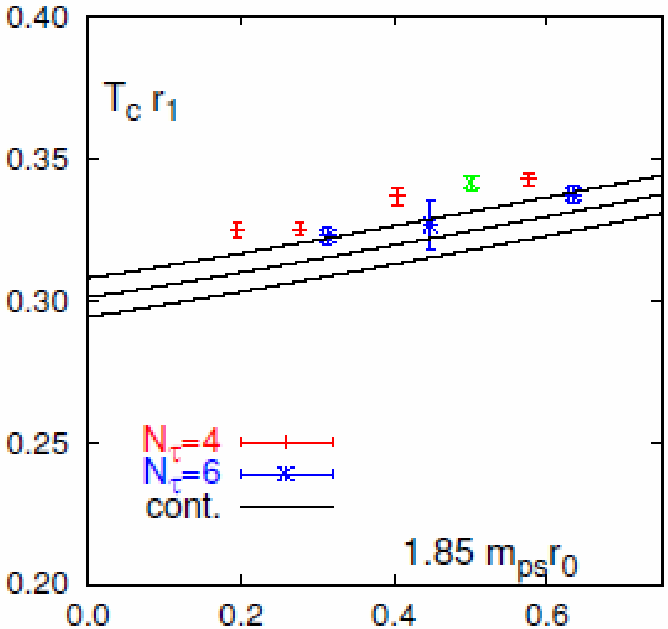
  - remove cut-off effects & improve rotational sym.
  - smeared 1-link term improves flavor symmetry



A new determination of the transition temperature in QCD

- calculation of transition temperature with almost physical quark masses and different lattice cut-off values

  $\Rightarrow$ extrapolation to physical limit ($m_\pi = 135$ MeV) and continuum limit ($a \rightarrow 0$)



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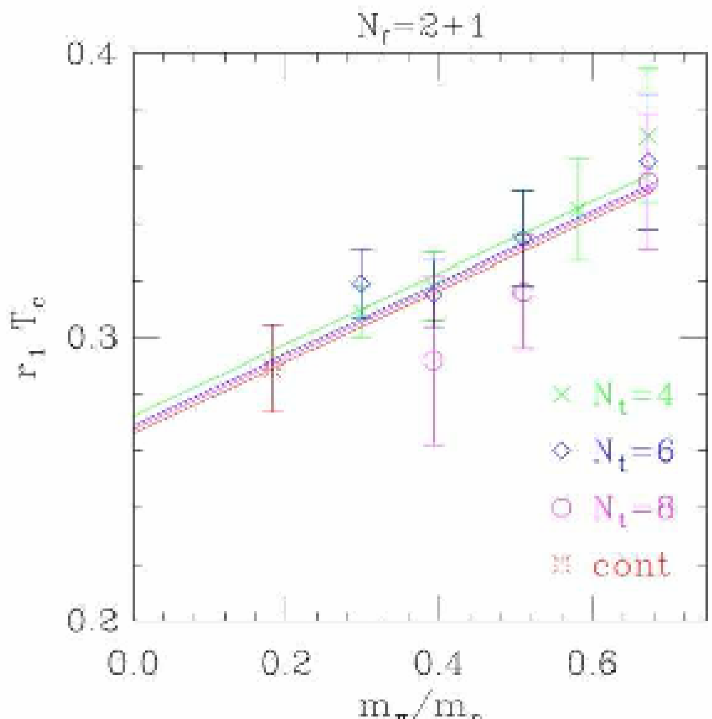


MILC, PRD71 (2005) 034504 (figure unpublished)

Simulation parameters

- Critical β search at T > 0

N_τ	$\hat{m}_s$	$\hat{m}_l$	V	#β values	max.# conf.
4	0.1	0.5 $\hat{m}_s$	8^3	10	40,000
		0.2 $\hat{m}_s$	8^3	6	12,000
4	0.065	0.4 $\hat{m}_s$	8^3, 16^3	10, 11	30,000, 60,000
		0.2 $\hat{m}_s$	8^3, 16^3	8, 7	30,000, 60,000
		0.1 $\hat{m}_s$	8^3, 16^3	9, 6	34,000, 50,000
		0.05 $\hat{m}_s$	8^3, 16^3	8, 5	30,000, 42,000
6	0.0040	0.4 $\hat{m}_s$	16^3	11	20,000
		0.2 $\hat{m}_s$	16^3	9	60,000
		0.1 $\hat{m}_s$	16^3	7	60,000

(*) conf. = 0.5 MD traj.

to check
m_s dependence for Tc

- T=0 scale setting at $\beta_c(N_t)$ on 16^3x32

N_τ	$\hat{m}_s$	$\hat{m}_l$	β	# conf.	m_{ps}/m_v	a [fm]
4	0.1	0.5 $\hat{m}_s$	3.409	600	0.520(2)	0.2273(4)
		0.2 $\hat{m}_s$	3.371	238	0.372(5)	0.2336(7)
4	0.065	0.4 $\hat{m}_s$	3.362	500	0.410(2)	0.2312(7)
		0.2 $\hat{m}_s$	3.335	400	0.303(7)	0.2365(6)
		0.1 $\hat{m}_s$	3.310	750	0.212(7)	0.2458(5)
		0.05 $\hat{m}_s$	3.300	400	0.154(5)	0.2475(8)
6	0.0040	0.4 $\hat{m}_s$	3.500	294	0.461(4)	0.1558(7)
		0.2 $\hat{m}_s$	3.470	500	0.343(6)	0.1617(5)
		0.1 $\hat{m}_s$	3.455	410	0.248(4)	0.1670(5)

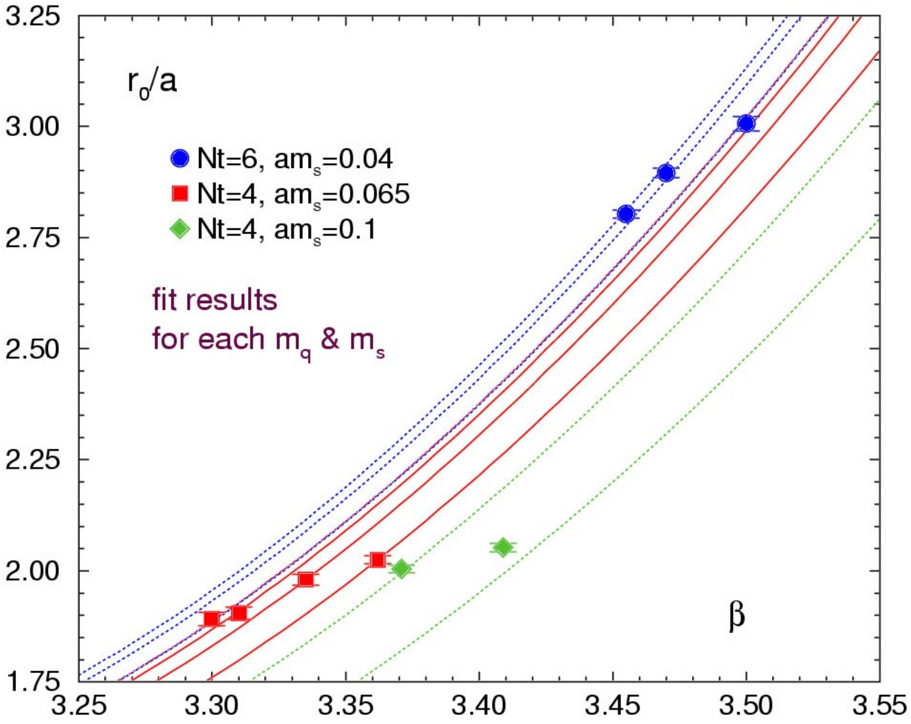
(*) conf. = 5 MD traj.
after thermalization

Exact RHMC
is used.

β & m_q dependence of r_0

RG inspired ansatz with 2−loop beta−function $R(\beta)$

$$(r_0/a)^{-1} = R(\beta)\left(1 + B\left(\frac{R(\beta)}{R(3.4)}\right)^2 + C\left(\frac{R(\beta)}{R(3.4)}\right)^4\right) e^{A(2\hat{m}_l+\hat{m}_s)+D}$$



$A = -1.45(5),\ \ B = 1.20(17)$
$C = \ 0.21(6),\ \ D = -2.45(5)$
$\chi^2/dof = 0.9$

The fit result is used

(1) correction for the diff. between β_c & simulation β at T=0

(2) conversion of sys. + stat. error of β_c into error of r_0/a

Line of Constant Physics (LCP)

The other parameter to determin the LCP : $\hat{m}_l = \hat{m}_l(\beta)$

$\hat{m}_l$ is determined by the condition for 'm$_{PS}$r$_0$'

We know a good parametrization:

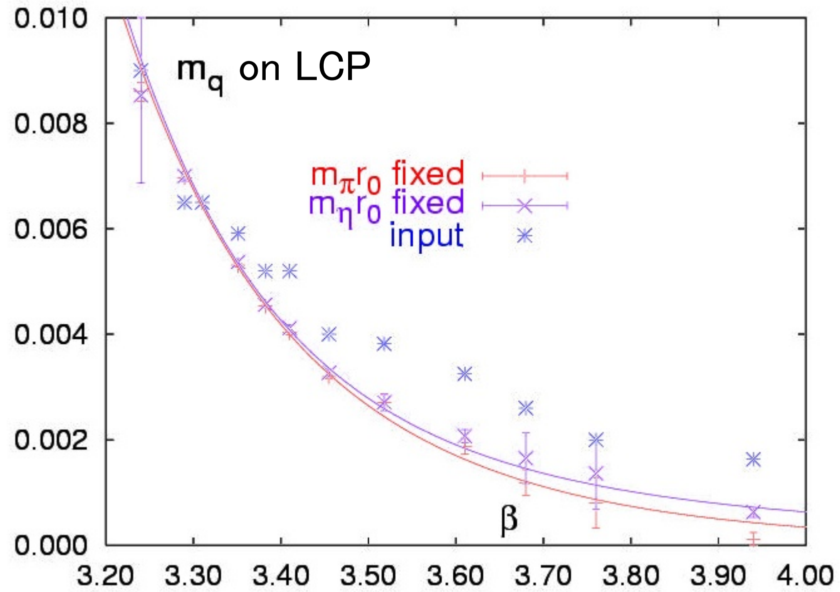
$$m_\pi a = f(\beta)\sqrt{\hat{m}_l}$$
$$r_0/a = h(\beta)\exp\left(-A\hat{m}_l(2+\Delta)\right)$$

$\hat{m}_l$ for fixed m$_{PS}$r$_0$

at each β (and Δ)

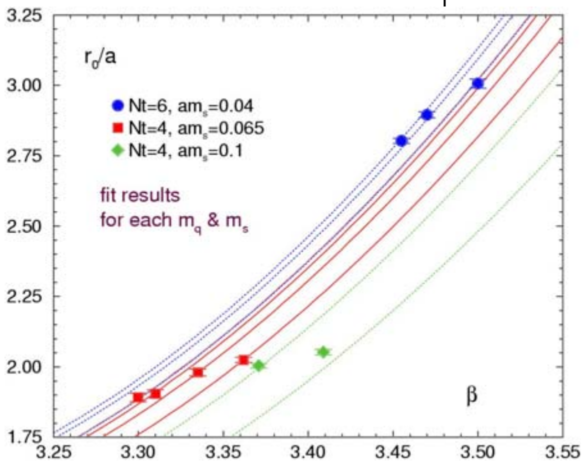
fit with Allton inspired ansatz:

$$\hat{m}_l(\beta) = d_0\left(\frac{6b_0}{\beta}\right)^{-4/9} \times R(\beta)\left(1 + d_2\hat{a}^2(\beta) + d_4\hat{a}^4(\beta)\right)$$



uncertainty remains in a choice of fixed m$_{PS}$r$_0$
→ improved by tuning of input m$_l$

Beta-function $-R_\beta-$



$$R_\beta = a\frac{d\beta}{da}\bigg|_{\hat{m}_l,\hat{m}_s} = \frac{a}{r_0}\left(\frac{\partial(a/r_0)}{\partial\beta}\bigg|_{LCP}\right)^{-1}$$

$$\frac{a}{r_0} = e^{A\hat{m}_l(2+\Delta)}R(\beta)\left(1+B\hat{a}^2(\beta)+C\hat{a}^4(\beta)\right)e^D$$

$$\frac{\partial(a/r_0)}{\partial\beta} = \hat{e}(\beta)\frac{a}{r_0} + R(\beta)\left(2B\hat{e}(\beta)\hat{a}^2(\beta)+4C\hat{e}(\beta)\hat{a}^4(\beta)\right)e^{D+A\hat{m}_l(2+\Delta)}$$

where $\hat{e}(\beta) = -\frac{1}{12b_0}+\frac{b_1}{2b_0^2\beta}$ and $\hat{a}(\beta) = R(\beta)/R(3.4)$

Finally we obtain

$$R_\beta = \frac{1+B\hat{a}^2(\beta)+C\hat{a}^4(\beta)}{\hat{e}(\beta)\left(1+3B\hat{a}^2(\beta)+5C\hat{a}^4(\beta)\right)}$$

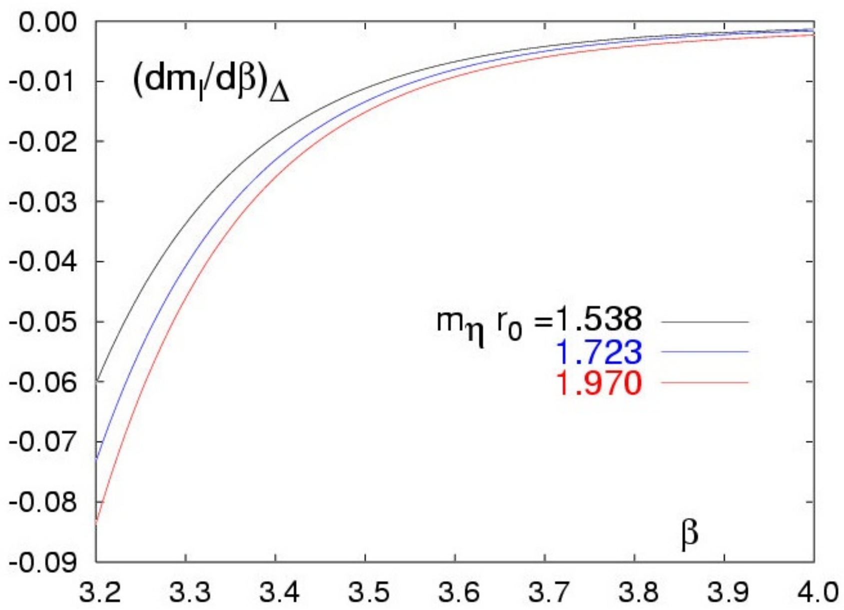
Beta-function –R_m–

fit of m_l on LCP

Allton inspired ansatz $$\hat{m}_l(\beta) = d_0 \left(\frac{6b_0}{\beta}\right)^{-4/9} R(\beta)\left(1 + d_2\hat{a}^2(\beta) + d_4\hat{a}^4(\beta)\right)$$

$$\left(\frac{\partial \hat{m}_l}{\partial \beta}\right)_\Delta = \hat{m}_l\left(\hat{e}_m(\beta) + \hat{e}(\beta)\frac{2d_2\hat{a}^2(\beta) + 4d_4\hat{a}^4(\beta)}{1 + d_2\hat{a}^2(\beta) + d_4\hat{a}^4(\beta)}\right)$$

where $$\hat{e}_m(\beta) = -\frac{1}{12b_0} + \left(\frac{b_1}{2b_0^2} + \frac{4}{9}\right)\frac{1}{\beta}$$



β	m_l	$m_\pi r_0$	$m_\eta r_0$	d_1	d_2	d_4
3.31	0.0065	0.505	1.538	0.0216	1.2087	1.9184
3.455	0.004	0.573	1.723	0.0266	1.0886	1.9267
3.61	0.00325	0.681	1.970	0.0594	0.2294	1.0679