Computable error bounds for asymptotic expansions formulas of distributions related to gamma functions

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1. Introduction

1. Testing equality of r covariance matrices

$$\boldsymbol{X}_{ij},\ i=1,\dots,n_j \overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}_j,\Sigma_j),\ j=1,\dots,r$$
$$H_0:\Sigma_1=\cdots=\Sigma_r$$

The modified likelihood ratio test rejects H_0 for small values of

$$V=\frac{\prod_{j=1}^{r}(\det A_j)^{n_j/2}}{(\det A)^{n/2}},$$

A_j : the matrix of sums of squares and products formed from the j–nth sample

$A=A_1+\cdots+A_r$

Moment formula (Under H_0)

$$\mathrm{E}[V^h]=\frac{\Gamma_p[n/2]}{\Gamma_p[n(1+h)/2]}\prod_{j=1}^{r}\frac{\Gamma_p[n_j(1+h)/2]}{\Gamma_p[n_j/2]},$$

$$\Gamma_p[a]=\pi^{p(p-1)/4}\prod_{k=1}^{p}\Gamma[a-(j-1)/2]$$

2. The sphericity test

$$\boldsymbol{X_i},\ i = 1, \ldots, n \overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}, \Sigma)$$
$$H_0 : \Sigma = \lambda I_p,\ \text{where } \lambda \text{ is unspecified}$$

The likelihood ratio text rejects H_0 for small values of

$$V = \frac{\det A}{\left(\frac{1}{p}\mathrm{tr}A\right)^p},$$

$A :$ the matrix of sums of squares and products

Moment formula (Under H_0)

$$\mathrm{E}[V^h] = p^{ph}\frac{\Gamma[pn/2]\Gamma_p[(n+2h)/2]}{\Gamma[p(n+2h)/2]\Gamma_p[n/2]}$$

3. MANOVA test

$$\boldsymbol{Y}_i = \mathbb{B}\boldsymbol{X}_i + \boldsymbol{\varepsilon}_i, \; i = 1, \ldots, N$$

$$\mathbb{B} : p \times r, \quad \boldsymbol{X}_i : r \times 1, \quad \boldsymbol{\varepsilon}_i \; i = 1, \ldots, N \overset{i.i.d.}{\sim} N_p(\mathbf{0}, \Sigma)$$

$$H_0 : \mathbb{B}C = O, \quad C \text{ is a known matrix of rank } q$$

The likelihood ratio text rejects H_0 for small values of

$$V = \frac{\det(B)}{\det(A+B)}$$

A : the matrix of sums of squares and products due to the hypothesis

B : the matrix of sums of squares and products due to the error

Moment formula (Under H_0)

$$\mathrm{E}[V^h] = \frac{\Gamma_p[(n+2h)/2]\Gamma_p[(n+q)/2]}{\Gamma_p[n/2]\Gamma_p[(n+q+2h)/2]}, \; n = N - r$$

2. General formulas of asymptotic expansion

Classical large sample approximation Box (1949, Biometrika)

$$\mathrm{E}[V^h] = K \times \left[\frac{\prod_{j=1}^{p} {y_j}^{y_j}}{\prod_{i=1}^{q} {x_i}^{x_i}} \right]^h \frac{\prod_{i=1}^{q} \Gamma[x_i(1+h)+\xi_i]}{\prod_{j=1}^{p} \Gamma[y_j(1+h)+\eta_j]}$$

$$\sum_{j=1}^{p} y_j = \sum_{k=1}^{q} x_k, \quad x_k = a_k M, \quad y_j = b_j M, \quad M \to \infty \ (n \to \infty)$$

$$T = -2\rho \log V, \quad \beta_k = (1-\rho) x_k = O(1), \ \epsilon_j = (1-\rho) y_j = O(1)$$

$$\psi_T(t) = \log \mathrm{E}[e^{itT}] = -\frac{f}{2} \log(1-2it) + \sum_{k=1}^{s} \frac{\gamma_k}{M^k} \{(1-2it)^{-k} - 1\} + O(M^{-(s+1)})$$

$$f = -2 \left[\sum_{k=1}^{q} \xi_k - \sum_{j=1}^{p} \eta_j - \frac{1}{2}(q-p) \right],$$

$$\gamma_k = \frac{(-1)^{k+1}}{k(k+1)} \left[\sum_{i=1}^{q} \frac{B_{k+1}(\beta_i + \xi_i)}{(\rho x_i)^k} - \sum_{j=1}^{p} \frac{B_{k+1}(\epsilon_j + \eta_j)}{(\rho y_j)^k} \right],$$

Expansion of the characteristic function

$$\phi_T(t) = \exp\{\psi_T(t)\} = \phi_s(t) + O(M^{-(s+1)}),$$

$$\phi_s(t) = (1-2it)^{-f/2}$$

$$\cdot \left\{1 + \sum_{k=1}^{s} \frac{1}{M^k} \sum_{m=1}^{k} \frac{1}{m!} \sum_{\substack{l_1+\cdots+l_m=k \\ l_i \geq 1}} \prod_{i=1}^{m} \gamma_{l_i}[(1-2it)^{-l_i} - 1]\right\}$$

Asymptotic expansion of the distribution function

$$\mathrm{P}(T \leq x) = G_s(x) + O(M^{-(s+1)})$$

$$G_s(x) = \int_{-\infty}^{x} \left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \psi_s(t) dt\right\} dx \text{ (inverting formula)}$$

High dimensional Edgeworth expansion

The moment generating function of $\log V$

$$\phi_{\log V}(t) := \mathrm{E}[\exp(it \log V)] = \mathrm{E}[V^{it}]$$
$$= K \times \left[\frac{\prod_{j=1}^{p} {y_j}^{y_j}}{\prod_{i=1}^{q} {x_i}^{x_i}}\right]^{it} \frac{\prod_{i=1}^{q} \Gamma[x_i(1+it)+\xi_i]}{\prod_{j=1}^{p} \Gamma[y_j(1+it)+\eta_j]}$$

cumulants

$$\kappa_1 = \mathrm{E}[\log V] = \log \frac{\prod_{j=1}^{p} {y_j}^{y_j}}{\prod_{i=1}^{q} {x_i}^{x_i}} + \sum_{i=1}^{q} x_i \psi^{(0)}(x_i+\xi_i) - \sum_{j=1}^{p} y_j \psi^{(0)}(y_j+\eta_j)$$

$$\kappa_k = \sum_{i=1}^{q} x_i \psi^{(k-1)}(x_i+\xi_i) - \sum_{j=1}^{p} y_j \psi^{(k-1)}(y_j+\eta_j) \ (k \geq 2)$$

where $\psi^{(k)}(a) = \dfrac{d^{k+1}}{da^{k+1}} \log \Gamma[a]$

Formal Edgeworth expansion

$$\phi_{\log V}(t) \approx \varphi_s(t) := e^{-t^2/2}\left\{1 + \sum_{k=1}^{s} \frac{1}{k!} \sum_{j=0}^{s-k} \gamma_{k,j}(it)^{3k+j}, \right\}$$

$$\gamma_{k,j} = \kappa_2^{-(j+3k)/2} \sum_{s_1+\cdots+s_k=j} \frac{\kappa_{s_1+3} \cdots \kappa_{s_k+3}}{(s_1+3)! \cdots (s_k+3)!},$$

$$\mathrm{P}\left(\frac{\log V - \kappa_1}{\sqrt{\kappa_2}} \leq x\right) \approx Q_s(x) := \Phi(x) - \phi(x) \sum_{k=1}^{s} \frac{1}{k!} \sum_{j=0}^{s-k} \gamma_{k,j} h_{3k+j-1}(x),$$

h_j : the j–th order Hermite polynomial

3. Uniform error bounds

If we find a function such that

$$|\phi_{\log V}(t) - \varphi_s(t)| \leq G(t) \text{ and}$$
$$\int_{-\infty}^{\infty} \frac{G(t)}{|t|} dt \text{ is computable}$$

Then

$$\sup_x \left| \mathrm{P}\left(\frac{\log V - \kappa_1}{\sqrt{\kappa_2}} \leq x \right) - Q_s(x) \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\phi_{\log V}(t) - \varphi_s(t)|}{|t|} dt$$
$$\leq \int_{-\infty}^{\infty} \frac{G(t)}{|t|} dt \leq \infty$$

3.1. Previous results

We have found computable error bounds of the following statistics under a framework

$$p \to \infty, \quad n_i \to \infty, \quad \text{with } \frac{p}{n_i} \to c_i \in (0, 1)$$

- MANOVA test
- The modified likelihood ratio test testing $\Sigma = I_p$
- The likelihood ratio test testing the sphericity
- The modified likelihood ratio test testing $\Sigma_1 = \Sigma_2 = \cdots = \Sigma_r$

Numerical examples

★ Error bounds for testing $H_0 : \Sigma_1 = \Sigma_2$ in the case of $s = 2$

$n_1 = n_2 = 18$		$n_1 = 18, n_2 = 36$	
p	BOUND	p	BOUND
3	0.0037 (0.55)	3	0.0035 (0.65)
6	0.0013 (0.55)	6	0.0013 (0.65)
9	0.0009 (0.60)	9	0.0010 (0.65)
12	0.0010 (0.70)	12	0.0012 (0.80)
15	0.0037 (0.95)	15	0.0097 (0.95)

★ Error bounds for testing $H_0 : \Sigma = \lambda I_p$ in the case of $s = 2$

$n = 30$		$n = 60$	
p	BOUND	p	BOUND
5	0.1417 (0.75)	10	0.0255 (0.55)
10	0.0276 (0.60)	20	0.0031 (0.40)
15	0.0093 (0.60)	30	0.0008 (0.40)
20	0.0052 (0.65)	40	0.0004 (0.45)
25	0.0067 (0.90)	50	0.0004 (0.60)

★ Error bounds for testing $H_0 : \Sigma = I_p$ in the case of $s = 2$

$n = 30$		$n = 60$	
p	BOUND	p	BOUND
5	0.0886 (0.70)	10	0.0186 (0.50)
10	0.0196 (0.60)	20	0.0026 (0.40)
15	0.0074 (0.60)	30	0.0008 (0.40)
20	0.0041 (0.65)	40	0.0004 (0.40)
25	0.0050 (0.85)	50	0.0004 (0.55)

3.2. Error bounds for large sample approximation

$$\psi_T(t) = \sum_{k=1}^{q}\{g_k(t) - g_k(0)\} - \sum_{j=1}^{p}\{h_j(t) - h_j(0)\},$$

where

$$g_k(t) = 2it\rho x_k \log x_k + \log\Gamma[\rho x_k(1-2it) + \beta_k + \xi_k],$$
$$h_j(t) = 2it\rho y_j \log y_j + \log\Gamma[\rho y_j(1-2it) + \epsilon_j + \eta_j],$$

and

$$\beta_k = (1-\rho)x_k = O(1),\ \epsilon_j = (1-\rho)y_j = O(1)$$

Generalized Stirling's theorem due to Barnes (1899):

$$\log\Gamma[x+h] = \log\sqrt{2\pi} + (x+h-\frac{1}{2})\log x - x - \sum_{k=1}^{s}\frac{(-1)^k B_{k+1}(h)}{k(k+1)x^k} + R_{s+1}(x),$$

rigorous representation of $R_s(x)$ is required.

Lemma (from the infinite product definition due to Euler)

$$\log\Gamma[z+a] = \lim_{n\to\infty}\Big\{(z+a-1)\log n + \sum_{k=1}^{n}\{\log k - \log(z+a-1+k)\}\Big\}$$

Lemma generalized Euler–Maclaurin's formula

$f(x)$: complex valued function of real argument

$$G_n(x) = \sum_{l=1}^{n-1} f(x+l)$$

$\Rightarrow$

$$\sum_{k=0}^{s}\frac{G_n^{(k)}(0)}{k!}a^k = \int_1^n f(x)dx + \sum_{k=0}^{s-1}\frac{B_{k+1}(a)}{(k+1)!}\{f^{(k)}(n) - f^{(k)}(1)\} + \int_1^n \frac{B_{s+1}(a) + (-1)^s B_{s+1}(x-[x]-a)}{(s+1)!} f^{(s+1)}(x)dx$$

Lemma

$$\sum_{l=1}^{n-1} \log(z+a+l) = \int_1^n \log(z+x)dx + B_1(a) \log \frac{z+n}{z+1}$$

$$+ \sum_{m=1}^{s-1} \frac{(-1)^{m+1} B_{m+1}(a)}{m(m+1)} \left\{ \frac{1}{(z+n)^m} - \frac{1}{(z+1)^m} \right\}$$

$$+ \int_1^n \frac{C_{s+1}(x;a)}{s+1} \frac{1}{(z+x)^{s+1}} dx + R_{s,n}(z,a),$$

where

$$C_k(x;a) = B_k(x-[x]-a) - B_k(1-a),$$

$$R_{s,n}(z,a) = \sum_{l=1}^{n-1} \sum_{k=s+1}^{\infty} \frac{-1}{k} \Big(\frac{-a}{z+l} \Big)^k$$

Lemma

$$
\begin{aligned}
g_k(t) - g_k(0) = {}& B_1(\beta_k + \xi_k) \log(1 - 2it) \\
& + \sum_{m=1}^{s-1} \frac{(-1)^{m+1} B_{m+1}(\beta_k + \xi_k)}{m(m+1)(\rho x_k)^m} \left\{ \frac{1}{(1-2it)^s} - 1 \right\} \\
& + \int_0^\infty \frac{C_{s+1}(x; \beta_k + \xi_k)}{s+1} \\
& \qquad \cdot \left\{ \frac{1}{(\rho x_k + x)^{s+1}} - \frac{1}{(\rho x_k(1-2it) + x)^{s+1}} \right\} dx \\
& + \sum_{l=0}^{\infty} \sum_{m=s+1}^{\infty} \frac{-1}{m} \left\{ \left(\frac{-\beta_k - \xi_k}{\rho x_k + l} \right)^m - \left(\frac{-\beta_k - \xi_k}{\rho x_k (1 - 2il) + l} \right)^m \right\}
\end{aligned}
$$

Theorem

$$\psi_T(t) = -\frac{f}{2}\log(1-2it) + \sum_{m=1}^{s-1}\frac{\gamma_m}{M^m}\{1-(1-2it)^{-m}\} + Rem_s,$$

$$f = -2\left[\sum_{k=1}^{q}\xi_k - \sum_{j=1}^{p}\eta_j - \frac{1}{2}(q-p)\right],$$

$$\gamma_m = \frac{(-1)^{m+1}}{m(m+1)}\left[\sum_{k=1}^{q}\frac{B_{m+1}(\beta_k+\xi_k)}{(\rho x_k)^m} - \sum_{j=1}^{p}\frac{B_{m+1}(\epsilon_j+\eta_j)}{(\rho y_j)^m},\right]$$

$$
\begin{aligned}
Rem_s = & \sum_{k=1}^{q} \int_0^\infty \frac{C_{s+1}(x;\beta_k+\xi_k)}{s+1} \left\{ \frac{1}{(\rho x_k + x)^{s+1}} - \frac{1}{(\rho x_k(1-2it)+x)^{s+1}} \right\} dx \\
& - \sum_{j=1}^{p} \int_0^\infty \frac{C_{s+1}(x;\epsilon_j+\eta_j)}{s+1} \left\{ \frac{1}{(\rho y_j + x)^{s+1}} - \frac{1}{(\rho y_j(1-2it)+x)^{s+1}} \right\} dx \\
& + \sum_{k=1}^{q} \sum_{l=0}^{\infty} \sum_{m=s+1}^{\infty} \frac{-1}{m} \left\{ \Big(\frac{-\beta_k-\xi_k}{\rho x_k + l} \Big)^m - \Big(\frac{-\beta_k-\xi_k}{\rho x_k(1-2it)+l} \Big)^m \right\} \\
& - \sum_{j=1}^{p} \sum_{l=0}^{\infty} \sum_{m=s+1}^{\infty} \frac{-1}{m} \left\{ \Big(\frac{-\epsilon_j-\eta_j}{\rho y_j + l} \Big)^m - \Big(\frac{-\epsilon_j-\eta_j}{\rho y_j(1-2it)+l} \Big)^m \right\}
\end{aligned}
$$

3.3. Wilks' lambda distribution (MANOVA test)

$$\Lambda = \det\{B(W+B)^{-1}\} \sim \Lambda_p(q,n)$$
$$B \sim W_p(q,\Sigma), \quad W \sim W_p(n,\Sigma) \quad (n > p)$$

Moment formula

$$\mathrm{E}[\Lambda^h] = \prod_{j=1}^{q} \frac{\Gamma[\frac{n-p+j}{2}+h]\Gamma[\frac{n+j}{2}]}{\Gamma[\frac{n-p+j}{2}]\Gamma[\frac{n+j}{2}+h]}.$$

High dimensional approximation (Tonda and Fujikoshi (2004), Wakaki (2007))

$$\Pr\left\{ \frac{-\log\Lambda - \kappa_1}{(\kappa_2)^{1/2}} \le x \right\}$$

$$= \Phi(x) - \phi(x)\left\{ \frac{\tilde{\kappa}_3}{6} h_2(x) + \cdots \right\} + O(m^{-(s+1)/2}),$$

κ_i : cumulants , $\tilde{\kappa}_i$: standardized cumulant ,

$$m = \frac{n - p - \frac{1}{2}}{2} (\kappa_2)^{1/2}$$

$$n \to \infty, \quad (n - p) \to \infty$$

Error bound(Wakaki(2007))

$F_s(x)$: Edgeworth expansion with using $(s+2)$-th cumulants

Table 1: The error bounds for $s = 2$ and $n = 50$.

p	q	error-bound	p	q	error-bound
10	5	0.0272	10	10	0.0099
20	5	0.0110	20	10	0.0035
30	5	0.0077	30	10	0.0023
40	5	0.0100	40	10	0.0029

Error bound for the large sample approximation

Theorem

$$\psi_T(t) \equiv \log \mathrm{E}[\exp\{-Mit \log \Lambda\}]$$
$$= -\frac{pq}{2}\log(1-2it) + \sum_{k=1}^{s} \frac{\gamma_k}{M^k}\{1-(1-2it)^{-k}\}$$
$$+ \sum_{j=1}^{q} \mathrm{Rem}_{c-p+j,s+1}(it),$$

where

$$\gamma_k = -\frac{(-2)^k \sum_{j=1}^{q}\{B_{k+1}(\frac{c+j}{2}) - B_{k+1}(\frac{c-p+j}{2})\}}{k(k+1)},$$
$$c = \frac{1}{2}(p-q+1)$$

where

$$
\begin{aligned}
\mathrm{Rem}_{a,s}(it) = \int_1^\infty & \frac{C_{s+1}(x;\frac{p+a}{2}) - C_{s+1}(x;\frac{a}{2})}{s+1} \\
& \cdot \left\{ \frac{1}{(\frac{M(1-2it)}{2} + x - 1)^{s+1}} - \frac{1}{(\frac{M}{2} + x - 1)^{s+1}} \right\} dx \\
& + \sum_{l=1}^{\infty} \sum_{k=s+1}^{\infty} \frac{(-1)^{k+1}}{k} \\
& \cdot \left\{ \left(\frac{(p+a)^k - a^k}{\{M(1-2it)+2l\}^k} \right) - \left(\frac{(p+a)^k - a^k}{(M+2l)^k} \right) \right\},
\end{aligned}
$$

Note

k : odd $\Rightarrow$ $\gamma_k = 0$ (Box (1949) : $\gamma_1 = \gamma_3 = \gamma_5 = 0$)

$$\psi_T(t) = -\frac{pq}{2}\log(1-2it) + \sum_{k=1}^{s} \frac{\gamma_{2k}}{M^{2k}}\{1-(1-2it)^{-2k}\}$$
$$+ \sum_{j=1}^{q} \mathrm{Rem}_{c-p+j,2s+2}(it),$$

The residual of the expansion of the characteristic function

$$\psi(t) = \psi_0(t) + \sum_{u=1}^{s} \frac{g_u(t)}{M^{2u}} + \frac{R_{s+1}(t)}{M^{2(s+1)}} \quad (s = 0, 1, \cdots),$$

$$g_u(t) = \gamma_{2u}\{1 - (1-2it)^{-2u}\},$$

$$R_s(t) = M^{2s} \sum_{j=1}^{q} \mathrm{Rem}_{c-p+j,2s},$$

$$\phi_T(t) = \mathrm{E}[\exp\{-Mit \log \Lambda\}] = \phi_s(t) + (1-2it)^{-pq/2} \frac{R_{p,q,s}(t)}{M^{2(m+1)}}$$

$$\phi_s(t) = (1-2it)^{-pq/2}\left\{1 + \sum_{j=1}^{s} \frac{1}{M^{2j}} \sum_{k=1}^{j} \frac{1}{k!} \sum_{\substack{l_1+\cdots+l_k=j \\ l_i \geq 1}} \prod_{i=1}^{k} g_{l_i}\right\},$$

$$R_{p,q,s}(t) = \{R_1(t)\}^{s+1} E_{1,s+1}\left[\frac{R_1(t)}{M^2}\right]$$

$$+ R_{s+1} + \sum_{k=2}^{s} \frac{1}{k!}\left\{\sum_{j=0}^{k-2} \sum_{\substack{l_i : \sum_{i=1}^{k-1-j} l_i \\ \leq s-1-j}} \left(\prod_{i=1}^{k-1-j} g_{l_i}\right) R_{s+1-j-\sum l_i} R_1^j\right.$$

$$\left. + R_{s-k+2} R_1^{k-1}\right\}, \qquad E_{1,s}(x) = \frac{1}{x^s}\left\{e^x - \sum_{k=1}^{s-1} \frac{x^k}{k!}\right\}$$

Lemma

$$\left|\sum_{j=1}^{q} \mathrm{Rem}_{c-p+j,m}(it)\right| \le \frac{1}{M^m} K_{p,q,m}(t), \quad |R_s(t)| \le K_{p,q,2s}(t)$$

$$K_{p,q,m}(t) = \min\left\{ \frac{|2t|}{\sqrt{1+4t^2}} A_{p,q,m}, B_{p,q,m} \right\} + C_{p,q,m} \min\left\{ \frac{|2t|}{\sqrt{1+4t^2}}, \frac{2}{m+1} \right\}$$

$$A_{p,q,m} = \frac{1}{2}\sum_{j=1}^{q}\Big\{(c+j)^{m+1}L_{1,m}\Big(\frac{c+j}{M}\Big) - (c-p+j)^{m+1}L_{1,m}\Big(\frac{c-p+j}{M}\Big)\Big\},$$

$$B_{p,q,m} = \sum_{j=1}^{q}\Big\{(c+j)^{m+1}L_{2,m}\Big(\frac{c+j}{M}\Big) - (c-p+j)^{m+1}L_{2,m}\Big(\frac{c-p+j}{M}\Big)\Big\},$$

$$C_{p,q,m} = \frac{2^m}{m}\max_{0\le x\le 1}\left|\sum_{j=1}^{q} C_{m+1}\Big(x;\frac{c+j}{2}\Big) - C_{m+1}\Big(x;\frac{c-p+j}{2}\Big)\right|,$$

$$L_{1,m}(x) = \frac{1}{x^m}\Big\{-\log(1-x) - \sum_{k=1}^{m-1}\frac{x^k}{k}\Big\},$$

$$L_{2,m}(x) = \frac{1}{x^{m+1}}\Big\{(1-x)\log(1-x) + x - \sum_{k=1}^{m-1}\frac{x^{k+1}}{k(k+1)}\Big\}.$$

Theorem

$$\sup_x |\Pr\{-M\log\Lambda \le x\} - F_s(x)| \le \frac{1}{2\pi M^{2(s+1)}} \int_{-\infty}^{\infty} \frac{G_{p,q,s}(t)}{|t|} dt,$$

$$F_s(x) = \int_{-\infty}^{x} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_s(t) dt \right\} dx$$

$$G_{p,q,s}(t) = \{K_{p,q,2}(t)\}^{s+1} E_{1,s+1}\left[\frac{K_{p,q,2}(t)}{M^2}\right] + K_{p,q,2s+2}(t)$$
$$+ \sum_{k=2}^{s} \frac{1}{k!} \left\{ \sum_{j=0}^{k-2} \sum_{\substack{l_i : \sum_{i=1}^{k-1-j} l_i \\ \le s-1-j}} \left| \prod_{i=1}^{k-1-j} g_{l_i} \right| K_{p,q,2(s+1-j-\sum l_i)}(t) K_{p,q,2}(t)^j \right.$$
$$\left. + K_{p,q,2(s-k+2)}(t) K_{p,q,2}(t)^{k-1} \right\},$$

case of $s = 1$

$$F_1(x) = G_f(x) + \frac{\gamma_2}{M^2}\{G_f(x) - G_{f+4}(x)\},$$

$$G_{p,q,1}(t) = (1+4t^2)^{-pq/4}\left\{K_{p,q,4}(t) + \{K_{p,q,2}(t)\}^2 E_{1,2}\left[\frac{K_{p,q,2}(t)}{M^2}\right]\right\}$$

case of $s = 2$

$$F_2(x) = G_f(x) + \frac{\gamma_2}{M^2}\{G_f(x) - G_{f+4}(x)\}$$
$$+\frac{1}{M^4}\{(-c_1 - c_2)G_f(x) + c_1 G_{f+4}(x) + c_2 G_{f+8}(x)\}.$$

$$c_1 = -\gamma_2(1+\gamma_2), \quad c_2 = \frac{\gamma_2^2}{2} - \gamma_4,$$

$$\gamma_4 = \frac{pq(159 - 50p^2 + 3p^4 - 50q^2 + 10p^2q^2 + 3q^4)}{1920},$$

$$G_{p,q,2}(t) = (1+4t^2)^{-pq/4}\left\{K_{p,q,6}(t) + \right.$$
$$\left.\frac{|\gamma_2|}{2}|1-(1-2it)^{-2}|K_{p,q,4}(t) + \{K_{p,q,2}(t)\}^3 E_{1,3}\left[\frac{K_{p,q,2}(t)}{M^2}\right]\right\}.$$

3.4. Comparison of error bounds between two types of expansions

Table 2: $n = 50$.HD(High dimension)($s = 2$), LS(Large sample)($s = 1$)

p	q	HD	LS
10	5	0.0272	0.000783
20	5	0.0110	0.0170
30	5	0.0077	0.251
40	5	0.0100	> 1

p	q	HD	LS
10	10	0.0099	0.00235
20	10	0.0035	0.0357
30	10	0.0023	0.517
40	10	0.0029	> 1

Table 3: $n = 50, p = 20, q = 5$

s	HD
0	0.0406
1	0.0181
2	0.0110
3	0.0078
4	0.0061

s	LS
0	0.0571
1	0.0170
2	0.00187

End of this talk. Thank you.