確率統計基礎講義A　補足

歪正規分布の特性関数　$X \sim \mathrm{SN}(\lambda)$ の特性関数は

$$\mathcal{C}(t) = \exp\bigl(-\frac{1}{2}t^2\bigr)\Bigl\{1 + i\sqrt{\frac{2}{\pi}}\int_0^{t\lambda/\sqrt{1+\lambda^2}} e^{u^2/2}du\Bigr\}\ (t \in \mathbb{R})$$

証明

$$C(t) = \int_{-\infty}^{\infty}\cos(tx)\phi(x)\Phi(\lambda x)dx,\ S(t) = \int_{-\infty}^{\infty}\sin(tx)\phi(x)\Phi(\lambda x)dx$$

とおくと，

$$\mathcal{C}(t) = 2\{C(t) + iS(t)\}$$

$\phi'(x) = -x\phi(x)$ を用いると

$$\begin{aligned}
C'(t) &= \int_{-\infty}^{\infty}\sin(tx)(-x)\phi(x)\Phi(\lambda x)dx \\
&= \bigl[\sin(tx)\phi(x)\Phi(\lambda x)\bigr]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}t\cos(tx)\phi(x)\Phi(\lambda x)dx - \int_{-\infty}^{\infty}\sin(tx)\phi(x)\lambda\phi(\lambda x)dx \\
&= -tC(t) \quad \text{（第３項は奇関数の積分）}
\end{aligned}$$

より

$$\log\frac{C'(t)}{C(t)} = -t,\ C(t) = K_1e^{-t^2/2}$$

$$C(0) = \frac{1}{2}\ \text{より}\ K_1 = \frac{1}{2}$$

同様に

$$\begin{aligned}
S'(t) &= \int_{-\infty}^{\infty}\cos(tx)x\phi(x)\Phi(\lambda x)dx \\
&= \bigl[-\cos(tx)\phi(x)\Phi(\lambda x)\bigr]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}t\sin(tx)\phi(x)\Phi(\lambda x)dx + \int_{-\infty}^{\infty}\cos(tx)\phi(x)\lambda\phi(\lambda x)dx \\
&= -tS(t) + \int_{-\infty}^{\infty}\cos(tx)\phi(x)\lambda\phi(\lambda x)dx
\end{aligned}$$

$S(t) = K(t)e^{-t^2/2}$ とおくと

$$\begin{aligned}
S'(t) + tS(t) = K'(t)e^{-t^2/2} &= \int_{-\infty}^{\infty}\cos(tx)\phi(x)\lambda\phi(\lambda x)dx \\
&= \frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\cos(tx)\exp\Bigl\{-\frac{1+\lambda^2}{2}x^2\Bigr\}dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda}{2\pi\sqrt{1+\lambda^2}}\int_{-\infty}^{\infty}\cos\Big(\frac{ty}{\sqrt{1+\lambda^2}}\Big)e^{-y^2/2}dy \\
&= \frac{\lambda}{\sqrt{(2\pi)(1+\lambda^2)}}\exp\Big\{-\frac{t^2}{2(1+\lambda^2)}\Big\}
\end{aligned}
$$

よって

$$
\begin{aligned}
K(t) &= \frac{\lambda}{\sqrt{(2\pi)(1+\lambda^2)}}\int_0^t \exp\Big\{\frac{\lambda^2 t^2}{2(1+\lambda^2)}\Big\}dt + K_2 \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{t\lambda/\sqrt{1+\lambda^2}} e^{u^2/2}du \quad (S(0)=0 \text{ より } K_2{=}0)
\end{aligned}
$$

以上より

$$
\mathcal{C}(t) = e^{-t^2/2} + ie^{-t^2/2}\sqrt{\frac{2}{\pi}}\int_0^{t\lambda/\sqrt{1+\lambda^2}} e^{u^2/2}du
$$

半正規分布の特性関数　$Z \sim N(0,1)$ とするとき $X = |Z|$ の特性関数は

$$
\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int_0^\infty e^{itx}e^{-x^2/2}dx = \sqrt{\frac{2}{\pi}}\{C(t)+iS(t)\}, \\
&C(t) = \int_0^\infty \cos(tx)e^{-x^2/2}dx,\ S(t) = \int_0^\infty \sin(tx)e^{-x^2/2}dx
\end{aligned}
$$

$$
\begin{aligned}
C'(t) &= \int_0^\infty \sin(tx)(-x)e^{-x^2/2}dx \\
&= \big[\sin(tx)e^{-x^2/2}\big]_0^\infty - t\int_0^\infty \cos(tx)e^{-x^2/2}dx = -tC(t) \\
C(t) &= K_1 e^{-t^2/2}, \quad K_1 = C(0) = \sqrt{\frac{\pi}{2}} \\
S'(t) &= \int_0^\infty \cos(tx)xe^{-x^2/2}dx \\
&= \big[\cos(tx)(-e^{-x^2/2})\big]_0^\infty - t\int_0^\infty \sin(tx)e^{-x^2/2}dx = -tS(t)+1 \\
S(t) &= K(t)e^{-t^2/2} \text{ とおくと} \\
S'(t) &+ tS(t) = K'(t)e^{-t^2/2} = 1 \\
K(t) &= K_2 + \int_0^t e^{u^2/2}du, \quad S(0)=0 \text{ より } K_2 = 0
\end{aligned}
$$

よって

$$
\sqrt{\frac{2}{\pi}}\{C(t)+iS(t)\} = e^{-t^2/2}\bigg\{1 + i\sqrt{\frac{2}{\pi}}\int_0^t e^{u^2/2}du\bigg\}
$$

乱数発生法　$Z_1, Z_2 \overset{i.i.d.}{\sim} N(0,1)$ とする。

$$X = \frac{\mathrm{sign}(\lambda)}{\sqrt{1+\lambda^2}}(Z_1 + |\lambda Z_2|)$$

の特性関数は

$$\begin{aligned}\mathcal{C}(t) &= \exp\Big\{-\frac{t^2}{2(1+\lambda^2)}\Big\} \times \exp\Big\{-\frac{\lambda^2 t^2}{2(1+\lambda^2)}\Big\}\Big\{1 + i\sqrt{\frac{2}{\pi}}\int_0^{t\lambda/\sqrt{1+\lambda^2}} e^{u^2/2} du\Big\} \\ &= \exp\big(-\frac{1}{2}t^2\big)\Big\{1 + i\sqrt{\frac{2}{\pi}}\int_0^{t\lambda/\sqrt{1+\lambda^2}} e^{u^2/2} du\Big\}\end{aligned}$$

したがって，$X \sim$ SN(λ) である。

モーメント

$$\begin{aligned}\frac{d^r}{dt^r} &= \Big\{1 + i\sqrt{\frac{2}{\pi}}\int_0^{t\lambda/\sqrt{1+\lambda^2}} e^{u^2/2} du\Big\}\frac{d^r}{dt^r}e^{-t^2/2} \\ &\quad + \sum_{k=1}^{r} {}_rC_k \Big\{\frac{d^{r-k}}{dt^{r-k}}e^{-t^2/2}\Big\}\Big\{i\sqrt{\frac{2}{\pi}}\frac{d^k}{dt^k}\int_0^{t\lambda/\sqrt{1+\lambda^2}} e^{u^2/2} du\Big\} \\ &= \Big\{1 + i\sqrt{\frac{2}{\pi}}\int_0^{t\lambda/\sqrt{1+\lambda^2}} e^{u^2/2} du\Big\}\frac{d^r}{dt^r}e^{-t^2/2} \\ &\quad + \sum_{k=1}^{r} {}_rC_k \Big\{\frac{d^{r-k}}{dt^{r-k}}e^{-t^2/2}\Big\}\Big\{i\sqrt{\frac{2}{\pi}}\frac{r}{\sqrt{1+r^2}}\frac{d^{k-1}}{dt^{k-1}}\exp\Big\{\frac{t^2 r^2}{2(1+r^2)}\Big\}\}du\Big\} \\ \mathrm{E}[X^r] &= i^{-r}\Big\{(-1)^r H_r(0) + \sum_{k=1}^{r}\sqrt{\frac{2}{\pi}}\Big(\frac{r}{\sqrt{1+r^2}}\Big)^k i^k (-1)^{k-1} H_{k-1}(0)\Big\}\end{aligned}$$