ANALYTIC CONTINUATION OF BOREL SUM OF FORMAL SOLUTION OF SEMILINEAR PARTIAL DIFFERENTIAL EQUATION

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Abstract. We study the Borel summability and the analytic behavior of the Borel sum of a formal solution of first order semilinear system with a singular perturbative parameter. By virtue of the representation formula of the Borel sum of a formal series solution expanded in terms of a parameter, we show that the analytic continuation of the Borel sum with respect to the parameter to a regular point in a singular direction coincides with the solution of the initial value problem expanded in the space variable. We also show that a similar phenomenon occurs outside the origin of the independent variables.

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1. Introduction

In this paper we study analytic behaviors of the Borel sum of a formal solution of a semilinear system with a singular perturbative parameter. As for the Borel summability of formal solutions of partial differential equations we cite the results by Lutz-Miyake-Schäfke and Balser et al for a heat operator (cf. [6], [2]) as well as the recent papers by Michalik, Ichinobe. ([4] , [8]). Another class of Borel summable operators which are perturbations of ordinary differential equations we refer [9] and [10], while for a first order equation we refer [3]. On the other hand, the summability of partial differential equations with a singular perturbative parameter was studied by Malek et al., where asymptotic solutions were constructed. (cf. [5], [7] and [11]). In general, the summability breaks down at a singular direction. The analytic continuation of a Borel sum to a point in a singular direction is not known well. In the case of partial differential equations, the set of singular directions may contain an open cone as we will see later. We shall study the behaviors of the Borel sum as the paramteter with which we take a Borel sum approaches to a singular direction.

In order to study the analytic behaviors of the Borel sum in a singular direction we introduce the countable set E_0 of possible singular points of the Borel sum. Then we give an alternative expression (2.12) of the Borel sum. Then we show that the analytic continuation of the Borel sum with respect to the parameter from the point in a summable direction to the one in the complement of E_0 coincides with a convergent solution expanded by independent varibles. Because the convergent solution is naturally related with an initial value problem, one sees that the solution constructed by (2.4) and its Borel sum give a sufficiently general class of solutions having asymptotic expansion with respect to a parameter. We also show a similar phenomenon at other point $x = a$, $a \neq 0$.

This paper is organized as follows. In section 2 we state our main results. In section 3 we construct the formal solution and prove some lemmas which are necessary in the proof of the main theorem. In section 4 we prove our theorem.

2. Statement of results

Let $x = (x^{(1)}, \dots, x^{(n)}) \in \mathbb{C}^n$, $n \geq 1$ be the variable in $\mathbb{C}^n$. For $\lambda_j \in \mathbb{C}$, $\lambda_j \neq 0$ $(j = 1, 2, \dots, n)$, define

$$\mathcal{L} := \sum_{j=1}^{n} \lambda_j x^{(j)} \frac{\partial}{\partial x^{(j)}}. \tag{2.1}$$

Let $N \geq 1$ be an integer and let $u = (u^{(1)}, \dots, u^{(N)}) \in \mathbb{C}^N$ and $f(x,u) = (f^{(1)}(x,u), \dots, f^{(N)}(x,u))$ be the holomorphic vector function in some neighborhood of the origin of $\mathbb{C}^n \times \mathbb{C}^N$ which is entire with respect to u in $\mathbb{C}^N$. We consider the semilinear system of equations

$$\eta^{-1} \mathcal{L} u^{(\nu)} = f^{(\nu)}(x,u), \quad \nu = 1, 2, \dots, N, \tag{2.2}$$

with $u(0) = 0$, where $\eta \in \mathbb{C} \setminus \{0\}$ is a complex parameter. We assume

$$f(0,0) = 0, \qquad \det(\nabla_u f(0,0)) \neq 0 \tag{2.3}$$

where $\nabla_u f(0,0)$ denotes the Jacobi matrix of $f(x,u)$ with respect to u at the point $x = 0, u = 0$.

We construct the formal power series solution $u = v(x,\eta)$ of (2.2) in the form

$$v(x,\eta) = \sum_{\nu=0}^{\infty} \eta^{-\nu} v_\nu(x) = v_0(x) + \eta^{-1} v_1(x) + \cdots, \tag{2.4}$$

where the series is a formal power series of η^{-1} with coefficients $v_\nu(x)$ being holomorphic vector functions of x in some open set independent of ν. We denote by Ω_0 the open connected set containing the origin on which every coefficient $v_\nu(x)$ is defined. (cf. Proposition 1).

The formal Borel transform of $v(x,\eta)$ is defined by

$$B(v)(x,\zeta) := \sum_{\nu=0}^{\infty} v_\nu(x) \frac{\zeta^\nu}{\Gamma(\nu+1)}, \tag{2.5}$$

where $\Gamma(z)$ is the Gamma function. For an opening $\theta > 0$ and the bisecting direction ξ, define the sector $S_{\theta,\xi}$ by

$$S_{\theta,\xi} = \left\{ z \in \mathbb{C};\ |\arg z - \xi| < \frac{\theta}{2} \right\}. \tag{2.6}$$

We say that $v(x,\eta)$ is 1-Borel summable in the direction ξ with respect to η if $B(v)(x,\zeta)$ converges in some neighborhood of the origin of (x,ζ), and there exist a neighborhood U of the origin $x = 0$ and a $\theta > 0$ such that $B(v)(x,\zeta)$ can be analytically continued to $(x,\zeta) \in U \times S_{\theta,\xi}$ and of exponential growth of order 1 with respect to ζ in $S_{\theta,\xi}$. For the sake of simplicity we denote the analytic continuation with the same notation $B(v)(x,\zeta)$. The Borel sum $V(x,\eta)$ of $v(x,\eta)$ is, then, given by the Laplace transform

$$V(x,\eta) := \int_{L_\xi} \eta e^{-\zeta\eta} B(v)(x,\zeta) d\zeta \tag{2.7}$$

where the integral is taken on the ray starting from the origin to the infinity in the direction ξ.

We assume that $\nabla_u f(0,0)$ is a diagonal matrix,

$$\nabla_u f(0,0) = \operatorname{diag}(\mu_1, \ldots, \mu_N), \tag{2.8}$$

and that, for every ν, $1 \leq \nu \leq N$, the Hessian matrix

$$\nabla_u^2 f^{(\nu)}(0,0) := \left(\frac{\partial^2 f^{(\nu)}}{\partial u^{(j)} \partial u^{(k)}}(0,0) \right)_{j\downarrow 1,\ldots,N;\, k\rightarrow 1,\ldots,N} \tag{2.9}$$

vanishes. Moreover, we assume

$$\operatorname{Re}\mu_k > 0,\ \operatorname{Re}\lambda_j > 0, \quad \operatorname{Re}\frac{\lambda_j}{\mu_k} > 0, \quad (j = 1,\ldots,n;\ k = 1,\ldots,N). \tag{2.10}$$

Let C_0 be the smallest convex closed cone with vertex at the origin containing λ_j $(j = 1, 2, \dots, n)$ and λ_j/μ_k $(j = 1, \dots, n; k = 1, 2, \dots, N)$.

Write

$$C_0 = \{z \in \mathbb{C}; -\theta_1 \le \arg z \le \theta_2\} \tag{2.11}$$

for some nonnegative θ_1 and θ_2 such that $-\pi/2 < -\theta_1 \le \theta_2 < \pi/2$. We say that $u(x, \eta)$ is the power series solution of (2.2) if $u(x, \eta)$ is a solution of (2.2) which can be expanded in the convergent power series of x in some neighborhood of the origin

$$u(x, \eta) = \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha|>0} u_\alpha(\eta) x^\alpha, \tag{2.12}$$

where every component of $u_\alpha(\eta)$ is a function of η and $\mathbb{Z}_+^n$ is the n-product of the set of nonnegative integers $\mathbb{Z}_+$. Then we have

Theorem 1. *Suppose (2.3), (2.8), (2.9) and (2.10). Then there exists a neighborhood U of $x = 0$ such that $v(x, \eta)$ with $v_0(0) = 0$ is 1-Borel summable in the direction ξ' with $|\xi' - \xi| < \theta/2$, $\theta = \pi - \theta_1 - \theta_2$ and $\xi = \pi + \frac{\theta_2 - \theta_1}{2}$ when $x \in U$. The Borel sum $V(x, \eta)$ of $v(x, \eta)$ is holomorphic and satisfies* (2.2) *when $(x, \eta) \in U \times S_{\pi+\theta,\xi}$. Moreover, $V(x, \eta)$ coincides with $u(x, \eta)$ in $(x, \eta) \in U \times S_{\pi+\theta,\xi}$.*

Note that $S_{\pi+\theta,\xi}$ is equal to $\mathbb{C} \setminus C_0$. By Theorem 1 we see that the Borel sum $V(x, \eta)$ is analytic in η when $\eta \in S_{\pi+\theta,\xi} = \mathbb{C} \setminus C_0$ and $x \in U$. We will study the behaviors of $V(x, \eta)$ when $x \in U$ and $\eta \in C_0$. Let E_0 be given by

$$E_0 := \left\{z \in \mathbb{C}; \det\left(z^{-1}\langle\lambda, \alpha\rangle Id - \nabla_u f(0, 0)\right) = 0, \alpha \in \mathbb{Z}_+^n, |\alpha| \ge 1\right\}. \tag{2.13}$$

Then we have

Corollary 2. *Suppose (2.3), (2.8), (2.9) and (2.10). Let $K \subset \mathbb{C} \setminus E_0$ be a compact set. Then there exist a neighborhood W of the origin of $\mathbb{C}^n$ and a neighborhood $\tilde{K}$ of K such that $\tilde{K} \cap S_{\pi+\theta,\xi} \ne \emptyset$ for which $V(x, \eta)$ is analytic with respect to η and x in $(x, \eta) \in W \times \tilde{K}$. For every $x \in W$, $V(x, \eta)$ is analytically continued as a single-valued holomorphic function of η from $S_{\pi+\theta,\xi}$ to $\tilde{K} \cup S_{\pi+\theta,\xi}$. The analytic continuation of $V(x, \eta)$ coincides with $u(x, \eta)$ in (2.12).*

Remark 1. *We can show that if the condition $Re\,\lambda_j > 0$ $(j = 1, \dots, n)$ is satisfied, then E_0 is a discrete infinite set in the right half-plane $Re\, z > 0$, while E_0 accumulates to $z = \infty$. Hence, $V(x, \eta)$ has essential singularity at $\eta = \infty$ in general.*

Remark 2. *We will give a brief explanation of the case when x is not necessarily in a neighborhood of the origin $a = 0$. The situation is quite different between the cases $a = 0$ and $a \ne 0$. For simplicity, assume that $f(x, u)$ is an entire function of x and u. Instead of (2.3), we suppose that there exists $b \in \mathbb{C}^N$ such that $f(a, b) = 0$ and* $\det(\nabla_u f(a, b)) \ne 0$. *Then there exists a formal series solution $v(x, \eta)$ of (2.2) with the form (2.4) defined at $x = a$ such that $v_0(a) = b$. In order to show the summability of the series, assume that $\nabla_u f(x, v_0(x))$ is a diagonal matrix in some neighborhood of $x = a$.*

Under these assumptions we can prove that $v(x,\eta)$ is 1-Borel summable with respect to η in the direction $\arg\xi$, $\pi/2 < \arg\xi < 3\pi/2$. *Clearly, the set* $\pi/2 < \arg\xi < 3\pi/2$ *is equal to $S_{\pi,\pi}$, which is the set corresponding to the case where C_0 is a positive real line. As for the proof of this statement we refer to [11, Theorem 3]. If $V(x,\eta)$ is the 1-Borel sum of $v(x,\eta)$ defined in some neighborhood of a and $\eta \in S_{2\pi,\pi}$, then $V(x,\eta)$ is characterized as the solution of (2.2) having an asymptotic expansion with respect to η when $\eta \in S_{\theta,\pi}$, $\eta \to \infty$ and x is in some neighborhood of a for some $\theta > \pi$.*

3. Construction of formal solution

Construction of formal solution. By substituting the expansion (2.4) into (2.2) with $u = v(x,\eta)$, we obtain

$$\mathcal{L}v = \sum_{\nu=0}^{\infty} \mathcal{L}v_\nu(x)\eta^{-\nu}, \tag{3.1}$$

$$\begin{aligned} f(x,v) &= f(x, v_0 + v_1\eta^{-1} + v_2\eta^{-2} + \cdots) \\ &= f(x,v_0) + \eta^{-1} v_1 (\nabla_u f)(x,v_0) + O(\eta^{-2}). \end{aligned} \tag{3.2}$$

By comparing the coefficients of η, we obtain, for $\eta^0 = 1$,

$$f(x, v_0(x)) = 0 \tag{3.3}$$

and for η^{-1}

$$\mathcal{L}v_0 = v_1(\nabla_u f)(x, v_0). \tag{3.4}$$

In order to determine $v_\nu(x)$ $(\nu \geq 2)$ we compare the coefficients of $\eta^{-\nu}$ of both sides of (2.2). Differentiate (3.2) $(\nu-1)$-times with respect to $\varepsilon = \eta^{-1}$ and put $\varepsilon = 0$. Then we obtain

$$\mathcal{L}v_{\nu-1} = v_\nu(\nabla_u f)(x,v_0) + (\text{terms consisting of } v_\ell^{(j)},\ 1 \leq j \leq N,\ \ell < \nu), \tag{3.5}$$

where $v_\ell(x) = (v_\ell^{(1)}, \dots, v_\ell^{(N)})$.

First, note that there exists an analytic solution $v_0(x)$, $v_0(0) = 0$ of (3.3) in some domain containing the origin $x = 0$ by (2.3). The next theorem gives the existence of a formal solution.

Proposition 1. *Assume (2.3) and $v_0(0) = 0$. Then every coefficient of (2.4) is uniquely determined as a holomorphic function in some neighborhood of $x = 0$ independent of ν.*

Proof. By (2.3) and the implicit function theorem, $v_0(x)$ is uniquely determined as the holomorhic function at the origin such that $v_0(x) = O(|x|)$. Suppose that $v_k(x)$ is determined up to some $\ell - 1$ in some neighborhood of the origin. Because $v_k(x)$ are determined recursively by differentiations and algebraic manupulations, the recurrence formula for $v_\ell(x)$ implies that $v_\ell(x)$ is holomorphic in some neighborhood of the origin independent of ν. □

Remark 3. *Let $\Omega_0 \subset \mathbb{C}^n$ be the domain containing the origin on which every coefficient of $v(x,\eta)$ is defined. We define*

$$\Sigma_0 := \{x;\ \det\left((\nabla_u f)(x, v_0(x))\right) = 0,\ f(x, v_0(x)) = 0, v_0(0) = 0\}. \tag{3.6}$$

Let $\widetilde{\Omega_0 \setminus \Sigma_0}$ be the universal covering space of $\Omega_0 \setminus \Sigma_0$. Then every coefficient of $v(x,\eta)$ is analytically continued from the origin to $\widetilde{\Omega_0 \setminus \Sigma_0}$, provided that $f(x,u)$ is an entire function of $x \in \mathbb{C}^n$ and $u \in \mathbb{C}^N$.

Function space For $T > 0$ we denote the polydisk D_T by $D_T = \{|x_1| < T\} \times \cdots \times \{|x_n| < T\}$. We define the set of functions $H(T)$ holomorphic in D_T and continuous up to the boundary by

$$H(T) = \left\{ u(x) = \sum_{\alpha \in \mathbb{Z}_+^n} u_\alpha x^\alpha; \|u\|_T := \sum_\alpha |u_\alpha| T^{|\alpha|} < \infty \right\}. \tag{3.7}$$

Note that $\|\cdot\|_T$ is a norm on $H(T)$. The n- product of $H(T)$ is denoted by $(H(T))^n$ with a standard norm of the product space. For the sake of simplicity we denote the norm in $(H(T))^n$ by the same letter $\|\cdot\|_T$. We can easily show that, for every $u, v \in (H(T))^n$ we have $\|uv\|_T \le \|u\|_T \|v\|_T$.

For the proof of the main theorem we prepare a lemma. Let $g(x,u) = (g^{(1)}(x,u), g^{(2)}(x,u), \dots, g^{(N)}(x,u))$ be such that $g^{(j)}(x,u)$ is holomorphic in some neighborhood of the origin $x = 0$ and an entire function of u. Let $\nabla_u g(x,0) = (\nabla_u g^{(1)}(x,0), \nabla_u g^{(2)}(x,0), \dots, \nabla_u g^{(N)}(x,0))$ and, for $1 \le \nu, j \le N$ let $g_j^{(\nu)}(x,0) = (\partial g^{(\nu)}/\partial u^{(j)})(x,0)$ be the j-th component of $\nabla_u g^{(\nu)}(x,0)$. Next, let $\nabla_u^2 g(x,0)$ be the tensor whose ν-th component is given by the Hessian matrix $\nabla_u^2 g^{(\nu)}(x,0) = (g_{i,j}^{(\nu)}(x,0))_{i,j}$, where $g_{i,j}^{(\nu)}(x,0) = (\partial^2 g^{(\nu)}/\partial u^{(i)} \partial u^{(j)})(x,0)$. We denote by $(w\nabla_u^2 g(x,0), \tilde{w})$ the vector whose ν-th component is the bilinear form given by $(w\nabla_u^2 g^{(\nu)}(x,0), \tilde{w})$. Then we have

Lemma 1. *For every $w, \tilde{w} \in H(T)^N$ we have*

$$\|w\nabla_u g(\cdot,0)\|_T \le \|w\|_T \sum_{j,\nu=1}^N \|g_j^{(\nu)}(\cdot,0)\|_T, \tag{3.8}$$

$$\|(w\nabla_u^2 g(\cdot,0), \tilde{w})\|_T \le \|w\|_T \|\tilde{w}\|_T \sum_{i,j,\nu=1}^N \|g_{i,j}^{(\nu)}(\cdot,0)\|_T. \tag{3.9}$$

Proof. For $w = (w^{(1)}, \dots, w^{(N)}) \in H(T)^N$ we have

$$\begin{aligned} \|w\nabla_u g(\cdot,0)\|_T &\le \sum_{\nu=1}^N \|\sum_{j=1}^N w^{(j)} g_j^{(\nu)}\|_T \le \sum_{\nu=1}^N \sum_{j=1}^N \|w^{(j)}\|_T \|g_j^{(\nu)}\|_T \\ &\le \|w\|_T \sum_{\nu=1}^N \sum_{j=1}^N \|g_j^{(\nu)}\|_T. \end{aligned} \tag{3.10}$$

Similarly, for $w, \tilde{w} \in H(T)^N$ we have

$$\|(w\nabla_u^2 g(\cdot,0),\tilde{w})\|_T \le \sum_{\nu=1}^N \|\sum_{i,j=1}^N w^{(i)} g_{i,j}^{(\nu)} \tilde{w}^{(j)}\|_T \tag{3.11}$$

$$\le \sum_{\nu=1}^N \sum_{i,j=1}^N \|w^{(i)}\|_T \|g_{i,j}^{(\nu)}\|_T \|\tilde{w}^{(j)}\|_T \le \|w\|_T \|\tilde{w}\|_T \sum_{\nu=1}^N \sum_{i,j=1}^N \|g_{i,j}^{(\nu)}\|_T.$$

This ends the proof.

4. Proof of Theorem 1 and Corollary 2

For the proof of Theorem 1 we prepare propositions and lemmas. If we set $\eta e^{-i\theta} = \tilde{\eta}$ with $\theta = (\theta_2 - \theta_1)/2$, then λ_j and η are replaced by $\lambda_j e^{-i\theta}$ and $\tilde{\eta}$, respectively, and C_0 is transformed to $S_{\theta_1+\theta_2,0}$. In the following, we rewrite $\tilde{\eta}$ with η, and we assume that $C_0 = S_{\theta_1+\theta_2,0}$ for the sake of simplicity. For $\varepsilon_1 > 0$ sufficiently small, let E_{0,ε_1} be the ε_1- neighborhood of E_0, and, for $r > 0$, define the set S by

$$S = S(\varepsilon_1) := (\{\eta \in \mathbb{C}; |\eta| \le r\} \setminus E_{0,\varepsilon_1}) \cup S_{2\pi-\theta_1-\theta_2,\pi}. \tag{4.1}$$

Then we have

Proposition 2. *There exist an $\varepsilon_1 > 0$ and a neighborhood U of the origin $x = 0$ such that if $\eta \in S = S(\varepsilon_1)$ and $x \in U$, then the series (2.12) converges.*

Proof of Proposition 2. We will construct the solution $v \equiv u = (u^{(1)}, \dots, u^{(N)})$ of (2.2) in (2.12) assuming that $\eta \notin E_0$. We rewrite (2.2) in the following form

$$\eta^{-1}\mathcal{L}u - u\nabla_u f(0,0) = f(x,u) - u\nabla_u f(0,0) =: R(x,u). \tag{4.2}$$

By substituting (2.12) into (4.2), we obtain the recurrence relation for $u_\alpha(\eta)$

$$u_\alpha(\eta)(\eta^{-1}\langle\lambda,\alpha\rangle Id - \nabla_u f(0,0)) = \frac{\partial^\alpha}{\partial x^\alpha} f(x,0)|_{x=0} \quad (|\alpha| = 1), \tag{4.3}$$

$$u_\alpha(\eta)(\eta^{-1}\langle\lambda,\alpha\rangle Id - \nabla_u f(0,0)) = \cdots, \tag{4.4}$$

where the dots denotes the terms $u_\beta(\eta)$, $(|\beta| < |\alpha|)$ which are calculated inductively. Indeed, the operator $\eta^{-1}\mathcal{L} - \nabla_u f(0,0)$ preserves every monomial x^α. On the other hand, by the assumption $f(0,0) = 0$ and $u(0) = 0$ there appear terms $u_\beta(\eta)$ $(|\beta| < |\alpha|)$ from $f(x,u)$ in the right-hand side of (4.4). By the assumptions $\eta \notin E_0$ and (2.8), every $u_\alpha(\eta)$ is determined successively by (4.4) multiplied by $(\eta^{-1}\langle\lambda,\alpha\rangle Id - \nabla_u f(0,0))^{-1}$.

Let $\alpha \in \mathbb{Z}_+^n, |\alpha| \ge 1$, $1 \le j \le N$ be given. Because $\lambda_k/\mu_j \in C_0$, $\mu_j \ne 0$ and C_0 is a convex cone, we have $\langle\lambda/\mu_j,\alpha\rangle \in C_0$. Therefore, if $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$, $\eta \ne 0$, then $\eta^{-1}\langle\lambda/\mu_j,\alpha\rangle$ is not on the positive real axis. It follows that there exists $c_1 > 0$ such that

$$\begin{aligned} |\eta^{-1}\langle\lambda,\alpha\rangle - \mu_j| &= |\mu_j||\eta^{-1}\langle\lambda/\mu_j,\alpha\rangle - 1| \ge c_1, \\ &\qquad \forall\eta \in S_{2\pi-\theta_1-\theta_2,\pi}, j = 1,\dots,N,\ \forall\alpha \in \mathbb{Z}_+^n, |\alpha| \ge 1. \end{aligned} \tag{4.5}$$

This also proves that $S_{2\pi-\theta_1-\theta_2,\pi} \cap E_0 = \emptyset$.

We remark that (4.5) holds not only for $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ but also for $\eta \in S$. Indeed, if $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$, then the assertion follows from (4.5). Noting that the function of η, $|\eta^{-1}\langle\lambda,\alpha\rangle - \mu_j|$ is continuous and does not vanish on the compact set $\{\eta \in \mathbb{C}; |\eta| \le r\} \setminus E_{0,\varepsilon_1}$ we have the assertion. In the following we assume that (4.5) holds for $\eta \in S$.

Let $\eta \neq 0$. Define $P := \eta^{-1}\mathcal{L} - \nabla_u f(0,0)$. We take $T > 0$ so small that $R(x,0) \in (H(2T))^n$. We define $u_k(x) \equiv u_k(x,\eta) \in (H(T))^N$ $(k = 0,1,2,\dots)$, $u_k(x,\eta) = (u_k^{(1)}(x,\eta),\dots,u_k^{(N)}(x,\eta))$ by

$$u_0 = 0,\ Pu_1 = R(x,0), \tag{4.6}$$

$$Pu_k = R(x,u_0+\cdots+u_{k-1}) - R(x,u_0+\cdots+u_{k-2}),\ k = 2,3,\dots \tag{4.7}$$

We admit that $u_k(x) \in (H(T))^N$ $(k = 0,1,2,\dots)$, and that the series $\sum_{k=1}^{\infty} u_k =: u$ converges in $(H(T))^N$ uniformly. By (4.7) we have

$$P\sum_{k=1}^{\ell} u_k = R(x, u_1+\cdots+u_{\ell-1}).$$

Hence, by letting $\ell \to \infty$, we see that u satisfies $Pu = R(x,u)$. Because the Taylor expansion at $x = 0$ of an analytic solution is uniquely determined by virtue of (4.5), it follows that u coincides with (2.12) and the series (2.12) converges.

In order to show that $u_k(x) \in (H(T))^N$ we will show that there exists $\varepsilon < 1/2$ such that

$$\|u_k\|_T \le \varepsilon^k, \qquad k = 1,2,\dots \tag{4.8}$$

By (2.3) we have $R(x,0) = O(|x|)$. Hence there exists $K > 0$ independent of $T > 0$ such that $\|f(\cdot,0)\|_T \le KT$. By (4.5), (4.6) and simple computations we see that

$$\|u_1\|_T = c_1^{-1}\|R\|_T \le c_1^{-1}KT = \frac{\varepsilon}{2},$$

where $\varepsilon := 2c_1^{-1}TK$.

Next we have

$$u_2 = (R(x,u_1) - R(x,0))P^{-1} = \left(\int_0^1 u_1 \nabla_u R(x,u_1\theta)d\theta\right) P^{-1}. \tag{4.9}$$

By definition we have

$$R(x,u) = f(x,0) + u(\nabla_u f(x,0) - \nabla_u f(0,0)) + \frac{1}{2}(u\nabla_u^2 f(x,0),u) + \tilde{R}(x,u) \tag{4.10}$$

where $\nabla_u^2 f(\cdot,0)$ is the vector whose ν-th component is given by the Hessian matrix $\nabla_u^2 f^{(\nu)}(\cdot,0)$ for $1 \le \nu \le N$ and $\tilde{R}(x,u) = O(|u|^3)$.

We will estimate $u_1\nabla_u R(x,u_1\theta)$. In view of (4.10) we first consider

$$u_1\nabla_u(u(\nabla_u f(x,0) - \nabla_u f(0,0))) = u_1(\nabla_u f(x,0) - \nabla_u f(0,0)).$$

By Lemma 1 we have

$$\|u_1(\nabla_u f(\cdot,0) - \nabla_u f(0,0))\|_T \le KT\|u_1\|_T \tag{4.11}$$

for some $K > 0$. Next we consider

$$u_1 \nabla_u ((u \nabla_u^2 f(x,0), u))|_{u=u_1\theta} = 2\theta (u_1 \nabla_u^2 f(x,0), u_1).$$

By assumption (2.9) and Lemma 1 we have

$$\|(u_1 \nabla_u^2 f(\cdot, 0), u_1)\|_T \le KT \|u_1\|_T^2. \tag{4.12}$$

As for the estimate of $u_1 \nabla_u \tilde{R}(x,u)|_{u=u_1\theta}$, one may consider every component of $u_1 \nabla_u \tilde{R}(x,u)|_{u=u_1\theta}$. Because $\tilde{R}(x,u) = O(|u|^3)$, we have $u_1 \nabla_u \tilde{R}(x,u)|_{u=u_1\theta} = O(|u_1|^3)$. Hence, we have

$$\|u_1 \nabla_u \tilde{R}(x, u_1\theta)\|_T \le K\varepsilon^3 \theta \tag{4.13}$$

for possibly different constant $K > 0$.

Hence we get, from (4.11), (4.12), (4.13) and (4.9),

$$\begin{aligned} \|u_2\|_T &\le c_1^{-1} \int_0^1 \|u_1 \nabla_u R(x, u_1\theta)\|_T d\theta \\ &\le c_1^{-1} \varepsilon (KT + 2KT\varepsilon) + c_1^{-1} \int_0^1 \|u_1 \nabla_u \tilde{R}(x, u_1\theta)\|_T d\theta \\ &\le c_1^{-1} \varepsilon (KT + 2KT\varepsilon) + c_1^{-1} \int_0^1 K\varepsilon^3 \theta d\theta. \end{aligned} \tag{4.14}$$

By setting $\varepsilon = 2c_1^{-1}KT$, the right-hand side of (4.14) is bounded by $\varepsilon^2(\frac{1}{2} + \varepsilon + Kc_1^{-1}\varepsilon/2)$. We take ε sufficiently small such that $\frac{1}{2} + \varepsilon + Kc_1^{-1}\varepsilon/2 \le 1$. Then we have (4.8) for $k \le 2$. By the same argument we can show that u_k satisfies the estimate (4.8). Hence the sum $\sum_{k=1}^{\ell} u_k$ is uniformly bounded in ℓ and converges in $(H(T))^N$. We note that the convergence is uniform with respect to $\eta \in S$ if $|\eta| \ge \eta_0 > 0$ for every $\eta_0 > 0$ because the constants K, T and ε can be chosen uniformly $\eta \in S$ with $|\eta| \ge \eta_0 > 0$. Especially the limit function is holomorphic in $\eta \in S$. This completes the proof.

Let $v_\nu(x)$ $(\nu = 0, 1, 2, \ldots, N)$ be given by Proposition 1. Then we have

Proposition 3. *For every N, $N = 0, 1, 2, \ldots$, there exists $R_N(x,\eta)$ being holomorphic when x is in some neighborhood of the origin and $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ such that*

$$u(x,\eta) = \sum_{\nu=0}^{N} v_\nu(x)\eta^{-\nu} + R_N(x,\eta)\eta^{-N-1}, \quad \eta \in S_{2\pi-\theta_1-\theta_2,\pi}. \tag{4.15}$$

First we will show (4.15) for $N = 0$.

Lemma 2. *(4.15) holds for $N = 0$.*

Proof. In view of the proof of Proposition 2 we have $u = \sum_{k=1}^{\infty} u_k(x,\eta)$ uniformly in $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$, $|\eta| \ge \eta_0$ for every $\eta_0 > 0$. In view of (4.6) and (4.7), we see that the degree of u_1 with respect to x is greater than or equal to 1 because P preserves every monomials x^α. It follows from (4.9) that the degree of u_2 is greater than or equal to 2. Similarly, we can show that the degree of u_j with respect to x is greater than or equal to j. We note that when solving (4.7), every coefficient of x^α in u_j is obtained by multiplying the corresponding coefficient in the right-hand side of (4.7) by $(\eta^{-1}\langle\lambda,\alpha\rangle Id - \nabla_u f(0,0))^{-1}$. In view of the definition of $u_j(x,\eta)$, every component of $u_\alpha(\eta)$ in the expansion (2.12) is a polynomial of $(\eta^{-1}\langle\lambda,\beta\rangle - \mu_k)^{-1}$ for some k, $1 \le k \le n$ and $\beta \in \mathbb{Z}_+^n$, $|\beta| \le |\alpha|$. Because $u = \sum_{j=1}^\infty u_j(x,\eta)$ converges uniformly in $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$, $|\eta| \ge \eta_0$ we have

$$\lim_{\eta\in S_{2\pi-\theta_1-\theta_2,\pi},\eta\to\infty} u(x,\eta) = \lim_\eta \sum_{j=1}^\infty u_j(x,\eta) = \sum \lim_\eta u_j(x,\eta). \tag{4.16}$$

By virtue of (4.6) and (4.7), the limit $\lim_\eta u_j(x,\eta)$ is computed if one replaces $(\eta^{-1}\langle\lambda,\beta\rangle - \mu_k)^{-1}$ in the coefficients of the expansion of $u_j(x,\eta)$ with $-\mu_k^{-1}$. Therefore we have $\lim_{\eta\to\infty} u_j(x,\eta) = u_j(x,\infty)$. We denote the right-hand side of (4.16) by w for the sake of simplicity. By the definition of R we see that (3.3) is written as $-v_0\nabla_u f(0,0) = R(x,v_0)$. On the other hand, by letting $\eta\to\infty$ in (4.6) and (4.7) and summing up the relations we see that w satisfies $-w\nabla_u f(0,0) = R(x,w)$. By the uniqueness of the solution of (3.3) such that $w(0)=0$ we see that the right-hand side of (4.16) is equal to $v_0(x)$.

Next we set

$$Q(x,\eta) = u(x,\eta) - v_0(x) = \sum_\alpha (u_\alpha(\eta) - u_\alpha(\infty))x^\alpha, \tag{4.17}$$

where $v_0(x) = \sum u_\alpha(\infty)x^\alpha$ is the expansion of $v_0(x)$. We recall the following formula

$$(\mu_k - \eta^{-1}\langle\lambda,\beta\rangle)^{-1} = \mu_k^{-1}\sum_{j=0}^N \left(\frac{\langle\lambda,\beta\rangle}{\mu_k\eta}\right)^j + \left(\frac{\langle\lambda,\beta\rangle}{\mu_k\eta}\right)^{N+1}(\mu_k - \eta^{-1}\langle\lambda,\beta\rangle)^{-1}, \tag{4.18}$$

where $N = 0,1,2,\ldots,$ and $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$. Because every component of $u_\alpha(\eta)$ is a polynomial of $(\eta^{-1}\langle\lambda,\beta\rangle - \mu_k)^{-1}$, we get, from (4.18) that

$$u_\alpha(\eta) - u_\alpha(\infty) = \eta^{-1}\tilde{u}_\alpha(\eta)$$

for some $\tilde{u}_\alpha(\eta) = (\tilde{u}_\alpha^{(1)}(\eta),\ldots,\tilde{u}_\alpha^{(N)}(\eta))$. It follows that

$$Q(x,\eta) = \eta^{-1}\tilde{Q}(x,\eta),\ \tilde{Q}(x,\eta) = \sum_\alpha \tilde{u}_\alpha(\eta)x^\alpha, \quad \eta \in S_{2\pi-\theta_1-\theta_2,\pi}, |\eta| > \eta_0, \tag{4.19}$$

where the equality is understood as a formal power series of x.

We will look for the equation of $\tilde{Q}(x,\eta)$. In view of (4.17) and (4.19), put $u = v_0 + \eta^{-1}\tilde{Q}$. Substitute u into (4.2). Then we have the equation for $\tilde{Q}$ with $\Lambda := \nabla_u f(0,0)$

$$\begin{aligned} \eta^{-1}\mathcal{L}v_0 + \eta^{-2}\mathcal{L}\tilde{Q} &= (v_0 + \eta^{-1}\tilde{Q})\Lambda + R(x, v_0 + \eta^{-1}\tilde{Q}) \\ &= \eta^{-1}\tilde{Q}\Lambda + R(x, v_0+\eta^{-1}\tilde{Q}) - R(x,v_0) \\ &= \eta^{-1}\tilde{Q}\Lambda + \eta^{-1}\int_0^1 \tilde{Q}\cdot\nabla_u R(x, v_0 + \theta\eta^{-1}\tilde{Q})d\theta. \end{aligned} \tag{4.20}$$

It follows that

$$\eta^{-1}\mathcal{L}\tilde{Q} - \tilde{Q}\nabla_u f(0,0) = -\mathcal{L}v_0 + \int_0^1 \tilde{Q}\cdot\nabla_u R(x, v_0+\theta\eta^{-1}\tilde{Q})d\theta. \tag{4.21}$$

This equation has a similar form as the equation for $u(x,\eta)$. Indeed, the linear part has the same form as (4.2). The nonlinear term in the right-hand side is bounded in η when $\eta \to \infty$, $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ because the nonlinear term in the right-hand side of (4.21) contains η^{-1} as a function of η and $R(x,u)$ is a vector whose components are entire functions of u. By the same argument as in the proof of the convergence of $u(x,\eta)$ we see that $\tilde{Q}(x,\eta)$ is holomorphic when x is in some neighborhood of the origin $x=0$ and $\eta \in S$. We also note that the convergence is uniform with respect to $\eta \in S$. Moreover, $\tilde{Q}(x,\eta)$ is bounded when $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$, $\eta \to \infty$. This proves (4.15) for $N=0$.

Lemma 3. *(4.15) holds for $N=1$.*

Proof. We will show (4.15) for $N=1$. Define $R_0(x,\eta) := \tilde{Q}(x,\eta)$. We want to show

$$\lim_{\eta\to\infty,\eta\in S_{2\pi-\theta_1-\theta_2,\pi}} \tilde{Q}(x,\eta) = U_1(x). \tag{4.22}$$

In order to show that the limit in (4.22) converges uniformly in some neighborhood of $x=0$ we will construct the approximate sequence $\tilde{Q}_n(x,\eta)$ of the equation (4.21) as in Proposition 2. Suppose that there exists $c_0 > 0$ such that the convergence of the approximate sequence $\{\tilde{Q}_n\}$, $\tilde{Q} = \lim \tilde{Q}_n$ is uniform with respect to $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$, $|\eta| \geq c_0$. Then one can show (4.22) as follows.

By assumption, for a given $\varepsilon > 0$ there exist $N_0 > 0$ and a neighborhood W_0 of $x=0$ such that, for any $m \geq N_0$, $n \geq N_0$, $x \in W_0$, and $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ with $|\eta| \geq c_0$ we have

$$\left|\tilde{Q}_n(x,\eta) - \tilde{Q}_m(x,\eta)\right| < \varepsilon. \tag{4.23}$$

Hence we see that $\lim_{m\to\infty} \tilde{Q}_m(x,\eta) =: \tilde{Q}(x,\eta)$ exists. On the other hand, by the definition of the sequence, $\{\tilde{Q}_n(x,\eta)\}$ we have, for each n the limit

$$\lim_{\eta\to\infty,\eta\in S_{2\pi-\theta_1-\theta_2,\pi}} \tilde{Q}_n(x,\eta) =: \tilde{Q}_n(x,\infty) \tag{4.24}$$

exists. By letting $\eta \to \infty$ in $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ in (4.23), we have, for every $m, n \geq N_0$ and $x \in W_0$

$$\left|\tilde{Q}_n(x,\infty) - \tilde{Q}_m(x,\infty)\right| \leq \varepsilon. \tag{4.25}$$

It follows that the limit $\lim_{n\to\infty} \tilde{Q}_n(x,\infty) =: U_1(x)$ exists uniformly for $x \in W_0$. By putting $n = N_0$ in (4.25) and letting $m \to \infty$ we have, for every $x \in W_0$

$$\left|\tilde{Q}_{N_0}(x,\infty) - U_1(x)\right| \leq \varepsilon. \tag{4.26}$$

On the other hand, by setting $n = N_0$ in (4.23) and letting $m \to \infty$ we have, for any $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ with $|\eta| \geq c_0$ and every $x \in W_0$

$$\left|\tilde{Q}_{N_0}(x,\eta) - \tilde{Q}(x,\eta)\right| \leq \varepsilon. \tag{4.27}$$

By (4.24) there exists $c_1 > 0$ such that, for every $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ with $|\eta| \geq c_1$ and every $x \in W_0$

$$\left|\tilde{Q}_{N_0}(x,\eta) - \tilde{Q}_{N_0}(x,\infty)\right| < \varepsilon. \tag{4.28}$$

Therefore for $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ with $|\eta| \geq \max\{c_0, c_1\}$ and every $x \in W_0$ we have

$$\begin{aligned} \left|\tilde{Q}(x,\eta) - U_1(x)\right| &\leq \left|\tilde{Q}(x,\eta) - \tilde{Q}_{N_0}(x,\eta)\right| \\ + \ &\left|\tilde{Q}_{N_0}(x,\eta) - \tilde{Q}_{N_0}(x,\infty)\right| + \left|\tilde{Q}_{N_0}(x,\infty) - U_1(x)\right| < 3\varepsilon. \end{aligned} \tag{4.29}$$

Hence we have (4.22).

Finally, in order to show the uniform convergence of the approximate sequence it is sufficient to verify that (4.8) holds uniformly in $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ with $|\eta| \geq c_0$ for some $c_0 > 0$. Indeed, we apply the arugument of the proof of Proposition 2 to (4.21). Set $w = \tilde{Q}$ and denote the right-hand side of (4.21) by $F(x, w, 1/\eta)$.

In order to have the estimate (4.8) for $k = 1$, we need the estimate $\|F(\cdot, 0, \eta^{-1})\|_T \leq KT$. This follows from the relations $F(x, 0, \eta^{-1}) = -\mathcal{L}v_0$ and $v_0(0) = 0$. Next we consider $\nabla_w F(x, 0, \eta^{-1}) - \nabla_w F(0, 0, \eta^{-1})$. In view of (4.10) we have

$$\begin{aligned} &\nabla_w F(x, 0, \eta^{-1}) - \nabla_w F(0, 0, \eta^{-1}) \\ = \ &\int_0^1 \nabla_u R(x, v_0(x)) d\theta - \int_0^1 \nabla_u R(0, v_0(0)) d\theta \\ = \ &\int_0^1 \nabla_u R(x, v_0(x)) d\theta = \nabla_u R(x, v_0(x)). \end{aligned}$$

Because $\nabla_u R(0, v_0(0)) = 0$, we obtain a similar estimate like (4.11).

Finally we estimate $(w_1 \nabla_w^2 F(x, 0, \eta^{-1}), w_1)$. For this purpose we may consider the next terms, for $1 \leq \nu \leq N$

$$\begin{aligned} &\left(w_1 \nabla_w^2 \int_0^1 w \cdot \left(\nabla_u R^{(\nu)}(x, v_0 + \theta\eta^{-1}w)\right) d\theta\Big|_{w=0}, w_1\right) \\ = \ &2 \int_0^1 w_1 \cdot w_1 \nabla_w \left(\nabla_u R^{(\nu)}(x, v_0 + \theta\eta^{-1}w)\right) d\theta\Big|_{w=0} \\ = \ &2 \int_0^1 (w_1 \theta \eta^{-1} \nabla_u^2 R^{(\nu)}(x, v_0), w_1) d\theta = \frac{1}{\eta}(w_1 \nabla_u^2 R^{(\nu)}(x, v_0), w_1). \end{aligned}$$

By assumption we have $\nabla_u^2 R^{(\nu)}(0, v_0(0)) = 0$. Hence we have the estimate like (4.12). Similarly one can estimate the third order term. By these estimates one can see that the constant $K > 0$ in the proof of (4.8) is uniform in η, $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ with $|\eta| \geq c_0$, which implies the desired estimate.

By virtue of (4.22) we have the following expression

$$R_0(x,\eta) = U_1(x) + \eta^{-1} R_1(x,\eta). \tag{4.30}$$

We substitute

$$u(x,\eta) = v_0(x) + \eta^{-1} R_0(x,\eta) = v_0(x) + \eta^{-1} U_1(x) + \eta^{-2} R_1(x,\eta) \tag{4.31}$$

into the equation (4.2) and we compare the coefficients of η^{-1}. Then we obtain

$$\mathcal{L}v_0 = U_1\Lambda + U_1\nabla_u R(x, v_0). \tag{4.32}$$

This is the same equation as (3.4) with v_1 replaced by U_1 because $(\nabla_u f)(x, v_0) = \Lambda + \nabla_u R(x, v_0)$. By the uniqueness of $v_1(x)$ in the formal solution, we see that $U_1(x) = v_1(x)$.

We look for the equation of $R_1(x, \eta)$. Set $u = v_0 + v_1\eta^{-1} + R_1\eta^{-2}$, or equivalently, $\tilde{Q} = v_1(x) + \eta^{-1}R_1(x, \eta)$. By substituting $\tilde{Q}$ into (4.21) we obtain

$$\begin{aligned} &\eta^{-1}\mathcal{L}(v_1 + \eta^{-1}R_1) \\ = \ & -\mathcal{L}v_0 + (v_1 + \eta^{-1}R_1)\Lambda + \int_0^1 (v_1 + \eta^{-1}R_1) \cdot \nabla_u R(x, v_0 + \theta\eta^{-1}(v_1 + \eta^{-1}R_1))d\theta. \end{aligned} \tag{4.33}$$

Using the relation

$$-\mathcal{L}v_0 + v_1\Lambda + v_1\nabla_u R(x, v_0) = 0, \tag{4.34}$$

the right-hand side of (4.33) is equal to

$$\begin{aligned} \eta^{-1}R_1\Lambda \ &+ \int_0^1 v_1 \cdot \left(\nabla_u R(x, v_0 + \theta\eta^{-1}(v_1 + \eta^{-1}R_1)) - \nabla_u R(x, v_0)\right) d\theta \\ &+ \ \eta^{-1}\int_0^1 R_1 \cdot \nabla_u R(x, v_0 + \theta\eta^{-1}(v_1 + \eta^{-1}R_1))d\theta. \end{aligned} \tag{4.35}$$

If we substitute the relation

$$\begin{aligned} &\nabla_u R(x, v_0 + \theta\eta^{-1}(v_1 + \eta^{-1}R_1)) - \nabla_u R(x, v_0) \\ = \ & \theta\eta^{-1}(v_1 + \eta^{-1}R_1) \cdot \int_0^1 \nabla_u^2 R(x, v_0 + t\theta\eta^{-1}(v_1 + \eta^{-1}R_1))dt \end{aligned}$$

in (4.35), then we see that $R_1(x, \eta)$ satisfies the following equation

$$\begin{aligned} &\eta^{-1}\mathcal{L}R_1 - R_1\Lambda = -\mathcal{L}v_1 \\ + \ &\int_0^1\int_0^1 v_1 \cdot \nabla_u^2 R(x, v_0 + t\theta\eta^{-1}(v_1 + \eta^{-1}R_1)) \cdot \theta(v_1 + \eta^{-1}R_1)dtd\theta \\ + \ &\int_0^1 R_1 \cdot \nabla_u R(x, v_0 + \theta\eta^{-1}(v_1 + \eta^{-1}R_1))d\theta. \end{aligned} \tag{4.36}$$

This equation has a similar form as the equation (4.2) for $u(x, \eta)$. The nonlinear term in the right-hand side together with the derivative with respect to η^{-1} are bounded in η when $\eta \to \infty$, $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ because the nonlinear term contains η^{-1} as a function of η and $R(x, u)$ is a vector whose components are entire functions of u. Therefore we see that there exists the solution $R_1(x, \eta)$ being holomorphic in x in some neighborhood of the origin and $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$. Therefore we have (4.15) for $N = 1$.

Proof of Proposition 3. We prove by induction. By Lemmas 2 and 3 we have (4.15) for $N = 0, 1$. Assume that (4.15) holds for $N = 0, 1, 2, \ldots, \nu$. We look for the equation of R_ν by substituting (4.15) with $N = \nu$ into (4.2). First we note

$$\mathcal{L}u(x,\eta) = \sum_{k=0}^{\nu} \mathcal{L}v_k \eta^{-k} + \eta^{-\nu-1}\mathcal{L}R_\nu. \tag{4.37}$$

We use the following formula for $R(x,u)$

$$\begin{aligned} R(x,u) &= R(x,v_0) + R(x,u) - R(x,v_0) \\ &= R(x,v_0) + (u-v_0)\cdot\nabla_u R(x,v_0) + g(x,u), \end{aligned} \tag{4.38}$$

where

$$g(x,u) = \int_0^1 (1-\theta)(u-v_0)\cdot\nabla_u^2 R(x, v_0 + \theta(u-v_0))\cdot(u-v_0)d\theta. \tag{4.39}$$

Set $t = \eta^{-1}$ and write

$$u = \sum_{k=0}^{\nu} v_k(x)\eta^{-k} + R_\nu \eta^{-\nu-1} = \sum_{k=0}^{\nu} v_k(x)t^k + R_\nu t^{\nu+1}. \tag{4.40}$$

We substitute (4.40) into (4.2) and compare terms with the power $t^{\nu+1}$. We recall that the terms with the power t^k for $k \le \nu$ vanish by the definition of v_k's in (3.4) and (3.5). In view of (4.37) and (4.38), the terms with the power $t^{\nu+1}$ appearing from the left-hand side of (4.2) are given by $\eta^{-1}\mathcal{L}R_\nu - R_\nu\Lambda + \mathcal{L}v_\nu$. On the other hand, the term which appears from $(u-v_0)\cdot\nabla_u R(x,v_0)$ is $R_\nu \nabla_u R(x,v_0)$. Hence we consider terms which appear from $g(x,u)$. We use Taylor's formula for a smooth function $h(t)$

$$h(t) = \sum_{\ell=0}^{\nu} \frac{t^\ell}{\ell!}\left(\frac{d}{dt}\right)^{\ell} h(0) + \frac{t^{\nu+1}}{\nu!}\int_0^1 (1-\theta)^\nu \left(\frac{d}{dt}\right)^{\nu+1} h(\theta t)d\theta.$$

Applying Taylor's formula for $h(t) := g(x,u)$, with u given by (4.40), we see that R_ν satisfies

$$\eta^{-1}\mathcal{L}R_\nu - R_\nu\Lambda = -\mathcal{L}v_\nu + R_\nu\nabla_u R(x,v_0) + Q_\nu, \tag{4.41}$$

where the nonlinear term Q_ν is given by

$$Q_\nu = \frac{1}{\nu!}\int_0^1 (1-s)^\nu \left(\frac{\partial}{\partial\tau}\right)^{\nu+1} g\left(x, \sum_k v_k\tau^k + R_\nu\tau^{\nu+1}\right)\Bigg|_{\tau=st} ds. \tag{4.42}$$

We note that the linear part of (4.41) has the same form as in (4.36) by the similar computations as for R_1. The nonlinear term Q_ν is bounded by some constant depending on ν when $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ and $\eta \to \infty$ because it is analytic in $t = \eta^{-1}$ at $t = 0$ and $R(x,u)$ is entire in u. Therefore, by the same argument as in the above, we see that $R_\nu(x,\eta)$ is the solution of (4.41) when $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ and x is in some neighborhood of the origin $x = 0$ possibly depending on ν. Moreover, by the uniform convergence in η of the approximate sequence the limit

$$\lim_{\eta\to\infty,\eta\in S_{2\pi-\theta_1-\theta_2,\pi}} R_\nu(x,\eta) = U_{\nu+1}(x) \tag{4.43}$$

exists in some neighborhood of the origin $x = 0$ possibly depending on ν. Because the coefficients of the Taylor expansion in x of $R_\nu(x, \eta)$ are polynomials of η^{-1} and $(\mu_k - \eta^{-1}\langle\lambda, \beta\rangle)^{-1}$, by expanding $(\mu_k - \eta^{-1}\langle\lambda, \beta\rangle)^{-1}$ in the power series of η^{-1} we have the expression for $R_\nu(x, \eta)$

$$R_\nu(x, \eta) = U_{\nu+1}(x) + \eta^{-1} R_{\nu+1}(x, \eta). \tag{4.44}$$

By substituting (4.15) with $N = \nu$ and (4.44) into (4.2), we see, from the uniqueness of $v_{\nu+1}$, that $U_{\nu+1}(x) = v_{\nu+1}(x)$ in some neighborhood of the origin depending on ν. Because $v_{\nu+1}(x)$ is defined in some neighborhood of the origin independent of ν by the definition of $v_{\nu+1}$, $U_{\nu+1}$ is analytically continued to some neighborhood of the origin, $x = 0$ independent of ν. Similarly, $R_{\nu+1}(x, \eta)$ is defined in some neighborhood of the origin, $x = 0$ independent of ν and $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$ in view of (4.44) because $R_\nu(x, \eta)$ has the property by the induction on ν. Therefore we have (4.15) with $N = \nu + 1$. This ends the proof.

Let $v_\nu(x)$ be given by (2.4). By Proposition 1 $v_\nu(x)$ is analytic at the origin $x = 0$. Let $v_\nu(x) = \sum_\alpha v_{\nu,\alpha} x^\alpha$ be its Taylor expansion at the origin. Let $u_\alpha(\eta) = (u_\alpha^{(1)}, \dots, u_\alpha^{(N)})$ be given by (2.12). Then we have

Proposition 4. *For every α in (2.12), $u_\alpha(\eta)$ is analytic at $\eta = \infty$. Let $u_\alpha(\eta) = \sum_{\nu=0}^\infty \tilde{v}_{\nu,\alpha} \eta^{-\nu}$ be its expansion at $\eta = \infty$. Then we have*

$$\tilde{v}_{\nu,\alpha} = v_{\nu,\alpha}, \quad \textit{for every} \ \ \nu. \tag{4.45}$$

Proof. First we look for the alternative expression of $u_\alpha(\eta)$. By substituting the expansions (2.12) and $R(x, z) = \sum_{\gamma,\delta} R_{\gamma,\delta} z^\gamma x^\delta$ into (4.2), we obtain the following relation

$$\sum_\alpha u_\alpha(\eta^{-1}\langle\lambda, \alpha\rangle Id - \Lambda) x^\alpha = \sum_{\gamma,\delta} R_{\gamma,\delta} (\sum_\alpha u_\alpha x^\alpha)^\gamma x^\delta. \tag{4.46}$$

It follows that

$$u_\alpha(\eta^{-1}\langle\lambda, \alpha\rangle Id - \Lambda) = \sum{}' R_{\gamma,\delta} \prod_{j=1}^{N} \prod_{i=1}^{\gamma(j)} u_{\alpha(j,i)}^{(j)}, \tag{4.47}$$

where $\gamma = (\gamma(1), \gamma(2), \dots, \gamma(N))$, and the summation $\sum'$ is taken over all combinations

$$\sum_{j=1}^{N} \sum_{i=1}^{\gamma(j)} \alpha(j, i) = \alpha - \delta, \ \alpha(j, i) \in \mathbb{Z}_+^n, \delta \in \mathbb{Z}_+^n, \ |\alpha(j, i)| \geq 1. \tag{4.48}$$

These relations imply that there appear no unknown quantities $u_\alpha^{(k)}$ in the right-hand side of (4.47). Indeed, we obtain

$$u_\alpha^{(k)} = (\eta^{-1}\langle\lambda, \alpha\rangle - \mu_k)^{-1} R_{0,\alpha}^{(k)} \quad (|\alpha| = 1, 0 = (0, 0, \dots, 0) \in \mathbb{C}^n). \tag{4.49}$$

When $|\alpha| \geq 1$, we use (4.47) recurrently, and we obtain, for $k = 1, 2, \ldots, N$

$$u_\alpha^{(k)} = \sum{}'' (\eta^{-1}\langle\lambda, \alpha\rangle - \mu_k)^{-1} R^{(k)}_{\gamma^{(1)},\delta^{(1)}} \left(\prod_{\nu=1}^{\ell} R^{(j(\nu))}_{\gamma^{(\nu+1)},\delta^{(\nu+1)}} \right) \times \left(\prod_{\nu=1}^{\ell} \prod_{j_\nu=1}^{N} \prod_{i_\nu=1}^{\gamma^{(\nu)}(j_\nu)} (\eta^{-1}\langle\lambda, \alpha(j_\nu, i_\nu, \nu)\rangle - \mu_{j_\nu})^{-1} \right) \tag{4.50}$$

where $\gamma^{(\nu)} = (\gamma^{(\nu)}(1), \gamma^{(\nu)}(2), \ldots, \gamma^{(\nu)}(N))$, and $R^{(j(\nu))}_{\gamma^{(\nu+1)},\delta^{(\nu+1)}}$ is the $j(\nu)$-th component of the vector $R_{\gamma^{(\nu+1)},\delta^{(\nu+1)}}$. The summation $\sum''$ is taken over combinations

(4.51)

$$\sum_{j_1=1}^{N} \sum_{i_1=1}^{\gamma^{(1)}(j_1)} \alpha(j_1, i_1, 1) = \alpha - \delta^{(1)}, \quad |\gamma^{(1)}| + |\delta^{(1)}| \leq |\alpha|,$$

$$\sum_{j_2=1}^{N} \sum_{i_2=1}^{\gamma^{(2)}(j_2)} \alpha(j_2, i_2, 2) = \alpha(j_1, i_1, 1) - \delta^{(2)}, \quad |\gamma^{(2)}| + |\delta^{(2)}| \leq |\alpha(j_1, i_1, 1)|,$$

$\cdots$

$$\sum_{j_\ell=1}^{N} \sum_{i_\ell=1}^{\gamma^{(\ell)}(j_\ell)} \alpha(j_\ell, i_\ell, \ell) = \alpha(j_{\ell-1}, i_{\ell-1}, \ell-1) - \delta^{(\ell)}, \; |\gamma^{(\ell)}| + |\delta^{(\ell)}| \leq |\alpha(j_{\ell-1}, i_{\ell-1}, \ell-1)|,$$

$$0 = \alpha(j_\ell, i_\ell, \ell) - \delta^{(\ell+1)}, \quad \gamma^{(\ell+1)} = 0,$$

especially, $\sum_{\nu=1}^{\ell+1} \delta^{(\nu)} = \alpha$ and $\ell \leq |\alpha|$. Note that the integer ℓ in (4.50) is the number of times the substitutions made by (4.47).

By (4.49) and (4.50) we see that $u_\alpha(\eta)$ is analytic at $\eta = \infty$ with the expansion $u_\alpha(\eta) = \sum_{\nu=0}^{\infty} \tilde{v}_{\nu,\alpha}\eta^{-\nu}$. Then we have

$$u(x, \eta) = \sum_{\alpha} \sum_{\nu=0}^{\infty} \tilde{v}_{\nu,\alpha}\eta^{-\nu}x^\alpha. \tag{4.52}$$

Note that the series (4.52) is a formal series, because the radius of convergence of $u_\alpha(\eta)$ at $\eta = \infty$ is not uniform in α. On the other hand, by inserting the expansion of $v_\nu(x)$ into the right-hand side of (4.15) we see that the coefficient of $x^\alpha\eta^{-\nu}$ is equal to $v_{\nu,\alpha}$. Hence we have (4.45). This ends the proof.

We define the path Γ as follows. Let $0 < \tau < \pi/2 - \theta_1$ and $R > 0$ be the number chosen later. The path Γ starts from $\eta = \infty$ and goes on the ray from ∞ to the origin in the domain $\operatorname{Im}\eta > 0$, $\operatorname{Re}\eta > 0$ up to some point $Re^{i\pi/2 - i\tau}$, then goes along the circle with center at the origin and radius R counterclockwise to the point $Re^{i3\pi/2+i\tau}$, and goes to $\eta = \infty$ on the ray from the origin to ∞ in the domain $\operatorname{Im}\eta < 0$, $\operatorname{Re}\eta > 0$. We take $\tau > 0$ so small and $R > 0$ so large such that Γ encircles the pole of u_α. Then we have

Lemma 4. *There exists $\rho_1 > 0$ such that*

$$|u_\alpha(\eta)| \le \rho_1^{|\alpha|+1}, \ \forall \alpha \in \mathbb{Z}_+^n, \ \eta \in \Gamma. \tag{4.53}$$

Proof. We may estimate the right-hand side of (4.50). First we consider $R^{(k)}_{\gamma^{(1)},\delta^{(1)}} \prod_{\nu=1}^{\ell} R^{(j(\nu))}_{\gamma^{(\nu+1)},\delta^{(\nu+1)}}$. By (4.10) and the assumption $f(0,0) = 0$ and the condition $u(0) = 0$ we see that $R(x, u(x))$ vanishes at $x = 0$. Therefore, by the scale change of variables $x = \varepsilon y$, for every given $0 < \varepsilon_1 < 1$ there exists $K_0 > 0$ such that, for every $1 \le \nu \le \ell$, $1 \le j(\nu) \le N$, $\gamma^{(\nu+1)} \in \mathbb{Z}_+^N$ and $\delta^{(\nu+1)} \in \mathbb{Z}_+^n$ we have

$$|R^{(j(\nu))}_{\gamma^{(\nu+1)},\delta^{(\nu+1)}}| \le K_0 \varepsilon_1^{|\gamma^{(\nu+1)}|+|\delta^{(\nu+1)}|}, \tag{4.54}$$

and for every $\gamma^{(1)} \in \mathbb{Z}_+^N$ and $\delta^{(1)} \in \mathbb{Z}_+^n$ we have

$$|R^{(k)}_{\gamma^{(1)},\delta^{(1)}}| \le K_0\, \varepsilon_1^{|\gamma^{(1)}|+|\delta^{(1)}|}.$$

Therefore we have

$$|R^{(k)}_{\gamma^{(1)},\delta^{(1)}} \prod_{\nu=1}^{\ell} R^{(j(\nu))}_{\gamma^{(\nu+1)},\delta^{(\nu+1)}}| \le K_0^{\ell+1} \prod_{\nu=1}^{\ell+1} \varepsilon_1^{|\gamma^{(\nu)}|+|\delta^{(\nu)}|} = K_0^{\ell+1} \varepsilon_1^{1+\sum_{\nu=1}^{\ell} |\gamma^{(\nu)}|+|\delta^{(\nu)}|}. \tag{4.55}$$

It follows from (4.50) and (4.5) that

$$\left| (\eta^{-1}\langle \lambda, \alpha\rangle - \mu_k)^{-1} \left(\prod_{\nu=1}^{\ell} \prod_{j_\nu=1}^{N} \prod_{i_\nu=1}^{\gamma^{(\nu)}(j_\nu)} (\eta^{-1}\langle \lambda, \alpha(j_\nu, i_\nu, \nu)\rangle - \mu_{j_\nu})^{-1} \right) \right| \tag{4.56}$$

$$\begin{aligned} &\le \ c_1^{-1} \prod_{\nu=1}^{\ell} \prod_{j_\nu=1}^{N} \prod_{i_\nu=1}^{\gamma^{(\nu)}(j_\nu)} c_1^{-1} \\ &\le \ c_1^{-1} \prod_{\nu=1}^{\ell} \prod_{j_\nu=1}^{N} c_1^{-\gamma^{(\nu)}(j_\nu)} = c_1^{-1} \prod_{\nu=1}^{\ell} c_1^{-|\gamma^{(\nu)}|} = c_1^{-1-\sum_\nu |\gamma^{(\nu)}|}. \end{aligned}$$

In order to estimate the modulus of the right-hand side of (4.50) we consider the number of combinations $\{\alpha(j_\nu, i_\nu, \nu)\}_{\nu=1}^{\ell}$ in (4.51) for every ℓ. Because the number of combinations is multiplied at most by N when j_ν runs through 1 to N, we may fix some j_ν, $1 \le j_\nu \le N$ and count the number of combinations. Abbreviating the index j_ν and rearranging $\{\alpha(\cdot, i_\nu, \nu)\}$, we define

$$\begin{aligned} &\alpha^{(1)} := \alpha(\cdot, i_{\ell-1}, \ell-1), \ \alpha^{(2)} := \alpha(\cdot, i_{\ell-2}, \ell-2) - \alpha(\cdot, i_{\ell-1}, \ell-1), \\ &\alpha^{(3)} := \alpha(\cdot, i_{\ell-3}, \ell-3) - \alpha(\cdot, i_{\ell-2}, \ell-2), \quad \cdots, \\ &\alpha^{(\ell-1)} := \alpha(\cdot, i_1, 1) - \alpha(\cdot, i_2, 2), \ \alpha^{(\ell)} := \alpha - \alpha(\cdot, i_1, 1). \end{aligned}$$

If we set

$$\beta^{(1)} = \alpha, \beta^{(2)} = \sum_{m=1}^{\ell-1} \alpha^{(m)}, \cdots, \beta^{(\ell-1)} = \alpha^{(2)} + \alpha^{(1)}, \beta^{(\ell)} = \alpha^{(1)}, \tag{4.57}$$

then the number of sequences $\{\alpha(\cdot, i_\nu, \nu)\}_\nu$ are bounded by the number of paths on $\mathbb{Z}_+^n$ which starts from α and arrives at some point in $\mathbb{Z}_+^n$ with length greater than or equal to 1 under the condition that on every point γ on the path, the length $|\gamma|$ is strictly decreasing. Clearly, such number can be bounded by

$$n^{|\alpha|-|\beta^{(2)}|}n^{|\beta^{(2)}|-|\beta^{(3)}|} \times \cdots \times n^{|\beta^{(\ell-1)}|-|\beta^{(\ell)}|}n^{|\beta^{(\ell)}|-1} = n^{|\alpha|-1} \le n^{|\alpha|}. \tag{4.58}$$

If we recall that j_ν runs through 1 to N, then the number of combinations of $\{\alpha(j_\nu, i_\nu, \nu)\}_{\nu=1}^{\ell}$ is bounded by $n^{N|\alpha|}$.

By (4.55), (4.56) and the condition $\varepsilon_1 < 1$ the modulus of the left-hand side of (4.50) is bounded by

$$\begin{aligned}
&\sum{}'' K_0^{\ell+1}\varepsilon_1^{1+\sum_{\nu=1}^{\ell}|\gamma^{(\nu)}|+|\delta^{(\nu)}|}c_1^{-1-\sum_\nu|\gamma^{(\nu)}|} \\
\le\ & n^{N|\alpha|}\frac{K_0\varepsilon_1}{c_1}\sum_{\ell=1}^{\infty}\sum_{n_1,\dots,n_\ell=1}^{\infty}\sum_{|\gamma^{(\nu)}|=n_\nu}K_0^{\ell}(\varepsilon_1/c_1)^{\sum_{\nu=1}^{\ell}n_\nu} \\
\le\ & n^{N|\alpha|}\frac{K_0\varepsilon_1}{c_1}\sum_{\ell=1}^{\infty}\sum_{n_1,\dots,n_\ell=1}^{\infty}K_0^{\ell}(\varepsilon_1/c_1)^{\sum_{\nu=1}^{\ell}n_\nu}N^{\sum_{\nu=1}^{\ell}n_\nu} \\
\le\ & n^{N|\alpha|}\frac{K_0\varepsilon_1}{c_1}\sum_{\ell=1}^{\infty}\sum_{n_1,\dots,n_\ell=1}^{\infty}K_0^{\ell}(N\varepsilon_1/c_1)^{\sum_{\nu=1}^{\ell}n_\nu} \\
\le\ & n^{N|\alpha|}\frac{K_0\varepsilon_1}{c_1}\sum_{\ell=1}^{\infty}\left(\frac{K_0N\varepsilon_1}{c_1-N\varepsilon_1}\right)^{\ell},
\end{aligned} \tag{4.59}$$

provided $(N\varepsilon_1)/c_1 < 1$. Because the right-hand side converges if $\left(\frac{K_0N\varepsilon_1}{c_1-N\varepsilon_1}\right) < 1$, we obtain (4.53) by taking ε_1 sufficiently small. This ends the proof.

Proof of Theorem 1. We shall prove the absolute convergence of the formal Borel transform (2.5). Consider the sum

$$\sum_\alpha\sum_{\nu=0}^{\infty}v_{\nu,\alpha}\frac{\zeta^\nu}{\nu!}x^\alpha = \sum_\alpha\sum_{\nu=0}^{\infty}\tilde{v}_{\nu,\alpha}\frac{\zeta^\nu}{\nu!}x^\alpha. \tag{4.60}$$

Because $u_\alpha(\eta)$ is holomorhic at $\eta = \infty$, the formal Borel transform of $u_\alpha(\eta)$ appearing in the right-hand side of (4.60) is equal to the Borel transform of the Hankel type for every α, namely

$$\hat{u}_\alpha(\zeta) = \frac{1}{2\pi i}\int_\Gamma \eta^{-1}u_\alpha(\eta)e^{\zeta\eta}d\eta, \tag{4.61}$$

where $\mathrm{Re}\,\zeta < 0$, and the path Γ is given in Lemma 4.

Because $u_\alpha(\eta)$ is the polynomial of η^{-1} and $(\mu_k - \eta^{-1}\langle\lambda, \beta\rangle)^{-1}$ for some β, it follows that $\hat{u}_\alpha(\zeta)$ is an entire function of ζ. We will estimate the growth of (4.61) when ζ tends to infinity in some small sector in the direction π. We restrict ζ on a small sector containing negative real axis. Clearly, if η lies on the ray of Γ we see that $e^{\zeta\eta}$ is bounded because $\mathrm{Re}\,(\zeta\eta) < 0$. Therefore, in terms of (4.53) there exist constants

$K_2 > 0$, $C_3 > 0$ and $C_4 > 0$ and a sector Σ_0 with vertex at the origin containing negative real axis independent of α so that

$$|\hat{u}_\alpha(\zeta)| \le K_2 \exp\left(C_3|\alpha| + C_4|\zeta|\right) \quad \text{for all } \zeta \in \Sigma_0. \tag{4.62}$$

We shall study Σ_0. We recall that $C_0 = S_{\theta_1+\theta_2,0}$ and $E_0 \subset C_0$. Because the path Γ is taken so that it encircles E_0 inside, one can take the rays of Γ arbtrarily close to the boundary of C_0. Because every $\zeta \in \Sigma_0$ should satisfy $\mathrm{Re}\,(\zeta\eta) < 0$ for any η on the ray of Γ, we see that Σ_0 contains the sector $S_{\pi-\theta_1-\theta_2,\pi}$.

By using the Cauchy estimate we can easily show that

$$|\tilde{v}_{\nu,\alpha}| \le K_3 \nu! \exp\left(C_3|\alpha| + C_4\nu\right), \quad \nu = 1, 2, \dots, \tag{4.63}$$

for, possibly another constants $C_3 > 0$, $C_4 > 0$ and $K_3 > 0$. By (4.63), we see that the right-hand side of (4.60) absolutely converges in some neighborhood of the origin $x = 0$, $\zeta = 0$. By (4.60) the formal Borel transform of (2.5) absolutely converges in some neighborhood of the origin $x = 0$, $\zeta = 0$.

In view of the definition of the Borel transform of $u_\alpha(\eta)$, the right-hand side of (4.60) is equal to $\sum_\alpha \hat{u}_\alpha(\zeta)x^\alpha$. Because $\hat{u}_\alpha(\zeta)$ is holomorphic in Σ_0 and satisfies (4.62) we see that $\sum_\alpha \hat{u}_\alpha(\zeta)x^\alpha$ is holomorphic in (x, ζ) when x is in some neighborhood of the origin and $\zeta \in \Sigma_0$. Moreover, by (4.62) it is of exponential growth when $\zeta \in \Sigma_0$, $\zeta \to \infty$. Because $\hat{u}_\alpha(\zeta)$ is of exponential growth of order 1 in Σ_0 by (4.62), the Laplace transform of the right-hand side of (4.60) is equal to $u_\alpha(\eta)$. Hence the Borel sum $V(x, \eta)$ of $v(x, \eta)$ is equal to $u(x, \eta)$ when $\eta \in S_{2\pi-\theta_1-\theta_2,\pi}$. Because Σ_0 contains $S_{\pi-\theta_1-\theta_2,\pi}$, we have the 1-Borel summability of $v(x, \eta)$ in $S_{\pi-\theta_1-\theta_2,\pi}$.

Finally, we show our theorem in the general case $\theta_1 \ne \theta_2$. In view of the proof of Proposition 2, we have $\eta e^{-i\theta} = \tilde{\eta}$ with $\theta = (\theta_2 - \theta_1)/2$. Hence we have the 1-Borel summability when $\tilde{\eta} \in S_{\pi-\theta_1-\theta_2,\pi}$. By returning to η we have the 1-Borel summability when $\eta \in e^{i\theta}S_{\pi-\theta_1-\theta_2,\pi} = S_{\pi-\theta_1-\theta_2,\pi+\theta}$. This ends the proof of Theorem 1.

Proof of Corollary 2. Let $V(x, \eta)$ be the Borel sum of the formal solution given in Theorem 1. $V(x, \eta)$ coincides with $u(x, \eta) = \sum_{\alpha\in\mathbb{Z}_+^n,|\alpha|\ge 1} u_\alpha(\eta)x^\alpha$ when $x \in S_{2\pi-\theta_1-\theta_2,\pi}$. In the proof of Proposition 2 we proved that $u(x, \eta)$ is analytic in $(x, \eta) \in W \times S$, where S is given by (4.1). This implies the assertion. This ends the proof.

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