GLOBAL SOLVABILITY OF MONGE-AMPÈRE TYPE EQUATIONS

Masafumi Yoshino[1]

Department of Mathematics, Chuo University, Higashinakano Hachioji, Tokyo 192-0393, Japan (E-mail: yoshinom@tamacc.chuo-u.ac.jp)

Dedicated to Professor Norio Shimakura on his sixtieth birthday

Abstract

We study the global solvability of Monge-Ampère equations of mixed type by "blowing up" the problem onto the torus embedded at the singular point of the equations.

1 Introduction

In this paper we are interested in the global solvability of fully nonlinear equations including Monge-Ampère equations of mixed type degenerating on normally crossing lines. As far as the author knows these types of equations are not well solved even in the anaytic class due to the lack of estimates of the linearized equations. We shall present a new method for such equations. Our

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main idea is to reduce the equations, by use of the Cauchy-Riemann equation, onto the (product of) torus embedded at points on which the equations change type. The resultant equations on the torus contain no singularities. After solving them we extend the solutions inside the torus by harmonic extensions, and the maximal principle yields the unique solution for the given problems. This method enables us to work outside "singular" set instead of working directly in a neighborhood of such a point. Because we can obtain a good estimate of the linearized operators on the torus we can use an elementary iteration scheme instead of a Nash-Moser one although the original equations degenerate.

Our main results in this paper are the global solvability of a Monge-Ampère type equations $M(u) = f$ in a bounded complete Reinhardt domain in $\mathbf{C}^n$ not necessarily convex. (cf. §2 and §5). More precisely, let the order of a formal power series f, $\operatorname{ord} f$ be defined as the smallest degree of its constituent monomials, that is, the smallest integer k such that $\partial_x^\alpha f_0(0) \neq 0$ for some $|\alpha| = k$ and $\partial_x^\beta f_0(0) = 0$ for all $|\beta| \leq k-1$. For u_0 and f_0 satisfying $M(u_0) = f_0$, we want to solve $M(u_0 + v) = f_0 + g$ for analytic g satisfying $\operatorname{ord} g \geq \operatorname{ord} f_0$. Here the localizing function u_0 corresponds to initial values in the case of initial value problems, while in some cases the equations are Tricomi mixed type at $u = u_0$ in the real domain. In §2 we show the global solvability under the Riemann-Hilbert factorization condition of the corresponding symbol on tori (Toeplitz symbol) in case $n = 2$. In §5 we study the global solvability in the case $n > 2$ under certain ellipticity conditions like (5.4) and (5.20) of the reduced symbol on tori.

Another interesting application of the method in this paper is the convergence of all formal power series solutions of fully nonlinear equations not satisfying a Poincaré condition. (cf. [3], [5], [6]). This is an extension of a classical theorem of Kashiwara-Kawai-Sjöstrand's (cf. [4]) for linear equations to fully nonlinear equations. In fact, we have different phenomena in the nonlinear case. (See Corollary 2.5 and §6).

In §6 we study the global solvability of Monge-Ampère equations in an unbounded outer domain. We will show new phenomena as to the structure of regular and singular solutions in the case $n = 2$. Namely, we will show the (essentially) linear structure of solutions of $M(u) = 0$ at infinity (cf. Theorem 6.2.), the parametrization theorem of all solutions of $M(u) = f_0 + g$ by a linear subspace (Theorems 6.5). As to the solvability, we will show unique global solvability of mixed type equations under a Riemann-Hilbert factorization condition (Theorem 6.8) and the one under a so-called spectral condition (Corollary 6.9. See also Remark 6.10).

2 Global Solvability and Riemann-Hilbert factorization

1.1. Statement of results. For $x = (x_1, x_2) \in \mathbf{C}^2$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbf{Z}_+^2$ we define $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}$ and $|\alpha| = \alpha_1 + \alpha_2$, where $\mathbf{Z}_+$ denotes the set of nonnegative integers. We set $\mathbf{R}_+^2 = \{(\eta_1, \eta_2) \in \mathbf{R}^2; \eta_1 \geq 0, \eta_2 \geq 0\}$. For $a = (a_1, a_2) \in \mathbf{C}^2$, let $D_a := \{|z_1| \leq |a_1|\} \times \{|z_2| \leq |a_2|\}$ and $\mathbf{T}_a^2 = \{|z_1| = |a_1|\} \times \{|z_2| = |a_2|\}$ be the closed disk and the torus, respectively. We say that $\Omega \subset \mathbf{C}^2$ is a Reinhardt domain if $\mathbf{T}_a^2 \subset \Omega$ for each $a = (a_1, a_2) \in \Omega$. If, in addition, $D_a \subset \Omega$, we say that Ω is complete. Let Ω be a bounded complete Reinhardt domain containing the origin not necessarily convex. We denote by $\mathcal{O}(\Omega)$, the set of holomorphic functions in Ω.

For $m \in \mathbf{Z}, m \geq 1$, let $M(u)$ be a fully nonlinear operator

$$M(u) := \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(\partial^\alpha u)(\partial^\beta u) + \sum_{|\alpha|=|\beta|=m} b_{\alpha\beta} x^\alpha \partial^\beta u, \quad a_{\alpha\beta}, b_{\alpha\beta} \in \mathbf{C}. \tag{2.1}$$

Let $u_0(x)$ be a homogeneous polynomial of degree $2m$, and set $f_0(x) = M(u_0)$. For $g \in \mathcal{O}(\Omega)$ such that $\operatorname{ord} g > 2m$ we consider

$$M(u_0 + w) = f_0(x) + g(x), \qquad \text{in } \ \Omega. \tag{2.2}$$

Let $P := M_{u_0} = \sum_\alpha (\partial M/\partial z_\alpha)(u_0)\partial^\alpha$ be the linearized operator of M at $u = u_0$. We denote by $p_m(x, \xi)$ the principal symbol of P, where $\xi = (\xi_1, \xi_2)$ is the covariable of x. The Toeplitz symbol $\sigma(z, \eta)$, ($z \in \Omega$, $\eta \in \mathbf{R}_+^2$) at u_0 is defined by

$$\sigma(z, \eta) = p_m(z_1, z_2; z_1^{-1}\eta_1, z_2^{-1}\eta_2), \tag{2.3}$$

namely, we set $x = (z_1, z_2)$ and $\xi = (z_1^{-1}\eta_1, z_2^{-1}\eta_2)$ in $p_m(x, \xi)$.

We denote by $\tilde{\Omega}$ the real representation of Ω, and by $\partial\tilde{\Omega}$ its boundary.

We assume the following conditions

$$(A.1) \qquad \sigma(z, \eta) \neq 0 \quad \forall z = (z_1, z_2) \in \Omega,\ (|z_1|, |z_2|) \in \partial\tilde{\Omega}, |z_j| > 0, j = 1, 2,$$
$$\forall \eta \in \mathbf{R}_+^2, |\eta| = 1.$$

$$(A.2) \qquad \operatorname{ind}_1\sigma = \operatorname{ind}_2\sigma = 0.$$

Here $\operatorname{ind}_1 \sigma$ (resp. $\operatorname{ind}_2 \sigma$) is defined by

$$\operatorname{ind}_1\sigma = \frac{1}{2\pi i}\oint_{|\zeta_1|=1} d_{\zeta_1} \log \sigma(R_1\zeta_1, R_2 z_2, \xi), \quad (R_1, R_2) \in \partial\tilde{\Omega}. \tag{2.4}$$

Remark. The right-hand side of (2.4) is an integer-valued continuous function of z_2 ($|z_2| = R_2$), ξ ($\xi \in \mathbf{R}_+^2$) and $R = (R_1, R_2) \in \partial\tilde{\Omega}$ ($R_j > 0$). By the connectedness, the integral is independent of z_2, ξ and $R = (R_1, R_2) \in \partial\tilde{\Omega}$. The condition (A.1)-(A.2) is called a Riemann-Hilbert factorization condition.

Then we have

Theorem 2.1 *Suppose (A.1) and (A.2). Let $\Omega' \subset\subset \Omega$ be arbitrarily given. Then there exist $\varepsilon > 0$ and an integer $N \geq 2m$ such that, for any $g \in \mathcal{O}(\Omega)$ satisfying $\sup_\Omega |g(x)| < \varepsilon$ and $\operatorname{ord} g \geq N$, (2.2) has a unique analytic solution $w \in \mathcal{O}(\Omega')$ satisfying $\operatorname{ord} w \geq N$.*

We define the set of holomorphic functions W_R by

$$W_R := \{u = \sum_{\eta \geq 0} u_\eta x^\eta : \|u\|_R := \sum_\eta |u_\eta| R^\eta < \infty\}. \tag{2.5}$$

In case $\partial\tilde{\Omega}$ consists of one point, $R = (R_1, R_2), R_j > 0$ we have

Theorem 2.2 *Suppose that $\partial\tilde{\Omega} = \{(R_1, R_2)\}$ $(R_j > 0)$, and assume (A.1) and (A.2). Then there exist $\varepsilon > 0$ and an integer $N \geq 2m$ depending only on u_0 such that, for any $g \in W_R$ satisfying $\|g\|_R < \varepsilon$ and $\operatorname{ord} g \geq N$, (2.2) has a unique analytic solution $w \in W_R$ satisfying $\operatorname{ord} w \geq N$.*

Remark 2.3 *In case the local solvability of (2.2) is concerned we need not the smallness of g, and u_0 may be any polynomial of degree $2m$. Indeed, if we replace u_0 in (2.3) with $2m$ homogeneous part of u_0, a similar argument as in the proof of Theorem 2.2 yields the following*

Proposition 2.4 *Suppose (A.1) and (A.2). Then there exist an integer $N \geq 2m$ depending only on u_0 such that, for any g analytic in some neighborhood of the origin satisfying $\operatorname{ord} g \geq N$, (2.2) has a unique analytic solution w in some neighborhood of the origin such that $\operatorname{ord} w \geq N$.*

We shall prove a nonlinear version of a Kashiwara-Kawai-Sjöstrand theorem.

Corollary 2.5 *Suppose (A.1) and (A.2). Then, for every g analytic at the origin such that $\operatorname{ord} g \geq 2m$ all formal power series solutions of (2.2) converge in some neighborhood of the origin.*

Remark 2.6 *a) The condition (A.1) in Corollary 2.5 cannot be dropped in general. (See Corollary 6.6 and Remark 6.7.)*

b) In 1974, Kashiwara-Kawai-Sjöstrand showed the convergence of all formal power series solutions of the following linear partial differential equations of regular singular type, $\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) x^\alpha (\partial/\partial x)^\beta u = f(x)$, where m is an integer and $a_{\alpha\beta}(x)$ and $f(x)$ are analytic in some neighborhood of the origin *of $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$. They gave a sufficient condition for the convergence of all formal power series solutions, a certain ellipticity of the equation. (cf. (0.2) in [4]). It contains, as a special case, a so-called Poincaré condition. Corollary 2.5 gives a generalization of [4] to fully nonlinear equations.*

Let $M \equiv M(u)$ be a Monge-Ampère operator in $(x, y) \in \mathbf{R}^2$

$$M(u) := u_{xx}u_{yy} - u_{xy}^2 + c(x, y)u_{xy}, \tag{2.6}$$

where $c(x, y)$ is a homogeneous polynomial of degree 2, and we abbreviate $u_{xx} = \partial_x^2 u$, $u_{yy} = \partial_y^2 u$, and so on. Let $u_0(x, y)$ be a homogeneous polynomial of degree 4 and set $f_0(x, y) = M(u_0)$. For an analytic function $g(x, y)$ such that $\operatorname{ord} g > 4$ we consider

$$M(u_0 + w) = f_0(x, y) + g(x, y). \tag{2.7}$$

In the following we denote by P the linearized operator of $M(u)$ at $u = u_0$. We also set $R_1 = R_2 = 1$ for the sake of simplicity.

Example 2.1. Let $u_0(x, y) = x^2y^2$ and $c(x, y) = kxy$ $(k \in \mathbf{R})$. It follows that $f_0(x, y) = 4(k - 3)x^2y^2$. The operator P and the Toeplitz symbol are given, respectively, by

$$\begin{aligned} P &= 2x^2\partial_x^2 + 2y^2\partial_y^2 + (k-8)xy\partial_x\partial_y, \qquad (2.8)\\ \sigma(z, \eta) &= 2(\eta_1^2 + \eta_2^2) + (k-8)\eta_1\eta_2. \qquad (2.9) \end{aligned}$$

The condition (A.1) reads $2 + (k - 8)\eta_1\eta_2 \neq 0$ for all $\eta \in \mathbf{R}_+^2$, $|\eta| = 1$. Because $0 \le \eta_1\eta_2 \le 1/2$, it follows that (A.1) is equivalent to $k > 4$. We can easily verify (A.2) if $k > 4$. Note that if $k > 4$ (A.1) is a so-called Poincaré condition for P. We note that because the Toeplitz symbol is independent of z, we can apply Theorem 2.1 as well as Theorem 2.2 if $k > 4$.

By simple computations of the characteristic polynomials, we see that $M(u)$ at $u = u_0$ is elliptic outside the set $xy = 0$ if and only if $4 < k < 12$. Note that if $k < 4$ or $k > 12$ the equation is weakly hyperbolic and degenerates on the lines $x = 0$ and $y = 0$.

Next we will estimate the integer N in Theorem 2.2 when $k > 4$. Because P preserves homogeneous polynomials we study the injectivity of P on the set of homogeneous polynomials of degree greater than 5. By definition, a monomial $x^\nu y^\mu$ $(\nu + \mu \ge 5)$ is in the kernel of P if and only if

$$2\nu(\nu - 1) + 2\mu(\mu - 1) + (k - 8)\nu\mu \neq 0. \tag{2.10}$$

Since $k > 4$, we easily see that (2.10) implies that $k \neq 16/3, 9/2$ if $\nu = 2$ or $\mu = 2$. If $\nu = \mu = 3$ it follows from (2.10) that $k \neq 16/3$. Similarly, if $\nu = 3, \mu = 4$ or $\nu = 4, \mu = 3$ we get $k \neq 5$. More generally, if $\nu + \mu = n$

$(n \geq 5)$ the condition (2.10) is equivalent to $k \neq 8-2(n^2-n)/(\nu\mu) \leq 4+8/n$. The equality holds when $\nu = \mu = n/2$. Consequently, the injectivity of P on the set of homogeneous polynomials of degree n holds in each of the following cases: a) $k > 16/3$, $n \geq 5$, b) $k > 5$, $n \geq 7$, c) $k > 4+8/n$, $n \geq 8$.

Example 2.2. Let $u_0(x,y) = x^4 + kx^2y^2 + y^4$, $c(x,y) \equiv 0$ and $f_0(x,y) = M(u_0) = 12(2kx^4 + 2ky^4 + (12-k^2)x^2y^2)$. The operator P and its Toeplitz symbol $\sigma(z,\eta)$ are given, respectively by

$$P = 12y^2\partial_x^2 + 12x^2\partial_y^2 + 2k(y^2\partial_y^2 + x^2\partial_x^2) - 8xy\partial_x\partial_y \tag{2.11}$$

$$\sigma(z,\eta) = 2k(\eta_1^2+\eta_2^2) - 8\eta_1\eta_2 + 12(z_1^{-2}z_2^2\eta_1^2 + z_1^2z_2^{-2}\eta_2^2). \tag{2.12}$$

Clearly, Poincaré condition does not hold for P. In order to apply Theorem 2.2, the condition (A.1) with $\tilde{\Omega} = \{(1,1)\}$ reads

$$k - 4\eta_1\eta_2 + 6(\eta_1^2t^2 + \eta_2^2t^{-2}) \neq 0 \qquad \forall t \in \mathbf{C},\ |t| = 1\ \forall \eta \in \mathbf{R}_+^2,\ |\eta| = 1. \tag{2.13}$$

In case $\eta_1 = \eta_2$, it follows from $|\eta| = 1$ that $\eta_1 = \eta_2 = 1/\sqrt{2}$. Hence (2.13) implies that $k \notin [-4,8]$. If $\eta_1 \neq \eta_2$ we have that $2i\,Im\,(\eta_1^2t^2 + \eta_2^2t^{-2}) = (\eta_1^2-\eta_2^2)(t^2-t^{-2})$, which vanishes only if $t^2 = \pm 1$. Because k is real (2.13) is verified if $t^2 \neq \pm 1$. In the case $t^2 = \pm 1$, (2.13) implies that $k \neq 4\eta_1\eta_2 \pm 6$. It follows that $k \notin [-6,-4]$ and $k \notin [6,8]$ since $0 \leq \eta_1\eta_2 \leq 1/2$. Therefore (A.1) is equivalent to $k < -6$ or $k > 8$. We can easily verify (A.2) for $\eta_1 = 0$, $\eta_2 = 1$ under these conditions.

On the other hand, if we want to apply Theorem 2.1, it is necessary to verify (A.1). By simple computations this is equivalent to verify (2.13) for $|t| = \rho$, where ρ varies in $0 < \exists\rho_1 \leq \rho \leq \exists\rho_2 < \infty$. Hence the condition is valid if k is sufficiently large.

We will show that the type of M changes near the origin. Indeed, if $k > 8$ f_0 changes sign on the four lines $f_0(x,y) = 0$ in $\mathbf{R}^2$ intersecting at the origin. Hence the equation is hyperbolic - elliptic near the origin. If $k < -6$ the origin $x = y = 0$ is the only zero of f_0 in $\mathbf{R}^2$, i.e., P is degenerate hyperbolic.

Example 2.3. Let $u_0(x,y) = x^4 + bx^2y^2$, $c(x,y) = cxy$, where $b > 6$ and c is a constant chosen later. The operator P is given by $P = 2bx^2\partial_x^2 + 2(6x^2 + by^2)\partial_y^2 + (c-8b)xy\partial_x\partial_y$, which is degenerate elliptic with degeneracy on the line $x = 0$. Because $\sigma(z,\eta) = 2b(\eta_1^2+\eta_2^2) + 12\eta_2^2z_1^2z_2^{-2} + (c-8b)\eta_1\eta_2$, (A.1) reads

$$2b + 12\eta_2^2t^2 + (c-8b)\eta_1\eta_2 \neq 0, \qquad \forall t, |t| = 1,\ \forall \eta \in \mathbf{R}_+^2,\ |\eta| = 1. \tag{2.14}$$

This is easily verified if $\eta_2 = 0$ or t^2 is not real. Now, suppose that $t^2 = \pm 1$. Because $b > 6$ and $0 \leq \eta_1\eta_2 \leq 1/2$, (2.14) holds if $c - 8b > 0$. The condition (A.2) for $\eta_1 = 1, \eta_2 = 0$ is easily checked by definition. It follows from Theorem

2.2 that (2.7) has a solution if the order of g is sufficiently large. If we want to apply Theorem 2.1 we have to verify (2.14) for $|t| = \rho$ with $0 < \exists\rho_1 \leq \rho \leq \exists\rho_2 < \infty$. Clearly, this condition is valid for sufficiently large b depending on ρ_1 and ρ_2.

3 Reduction to the boundary

In this section we obtain a crucial estimate for the linearized operator P in (2.1) under (A.1) and (A.2).

2.1. Restriction to the boundary and Toeplitz operators. Let $W_R(\mathbf{T}_R^2)$ be the restriction of W_R in (2.5) to the torus $\{|x_1| = R_1\} \times \{|x_2| = R_2\}$

$$W_R(\mathbf{T}_R^2) := \left\{ u = \sum_{\eta \geq 0} u_\eta R^\eta e^{i\eta\theta}\,;\ \|u\|_R := \sum_\eta |u_\eta| R^\eta < \infty \right\}. \tag{3.1}$$

The space $W_R(\mathbf{T}_R^2)$ is a Banach space with the norm $\|\cdot\|_R$ and each $u \in W_R(\mathbf{T}_R^2)$ can be extended to an analytic function on the polydisk D_R by a harmonic extension. We denote by $L_R^1(\mathbf{T}_R^2)$ the space of integrable functions on $\{|x_1| = R_1\} \times \{|x_2| = R_2\}$. We denote by π the projection from $L_R^1(\mathbf{T}_R^2)$ to $W_R(\mathbf{T}_R^2)$.

For $\alpha \in \mathbf{N}^2, \beta \in \mathbf{N}^2$ and a smooth function u we have $x^\alpha \partial_x^\beta u = x^{\alpha-\beta} x^\beta \partial_x^\beta u$. Moreover, by the commutator relation $[t, \partial/\partial t] = -1$ we have $t^k \partial_t^k = t\partial_t(t\partial_t - 1)\cdots(t\partial_t - k + 1)$. It follows that, by setting $\delta_j = x_j \partial/\partial x_j$ $(j = 1, 2)$,

$$x^\beta \partial_x^\beta = \delta_1(\delta_1 - 1)\cdots(\delta_1 - \beta_1 + 1)\delta_2(\delta_2 - 1)\cdots(\delta_2 - \beta_2 + 1) \equiv p_\beta(\delta), \tag{3.2}$$

where $\delta = (\delta_1, \delta_2)$. Hence we can write P in the form $P = \sum_{\alpha,\beta} c_{\alpha\beta} x^{\alpha-\beta} p_\beta(\delta)$, where $c_{\alpha\beta}$ are appropriately chosen constants.

Suppose now that P acts on holomorphic functions u, $\bar{\partial}_j u = 0$, where $\bar{\partial}_j$ is the Cauchy-Riemann operator with respect to x_j. If we introduce the polar coordinates $x_j = r_j \exp(i\theta_j)$ $(j = 1, 2)$ we have that $x_j \partial_{x_j} = (r_j \partial_{r_j} - i\partial_{\theta_j})/2$, $(j = 1, 2)$. It follows from Cauchy-Riemann equation that $(r_j \partial_{r_j} + i\partial_{\theta_j})u = 0$. Because the normal derivative $r_j \partial_{r_j}$ can be expressed by a tangential derivative ∂_{θ_j}, we can restrict P to the torus $\{|x_1| = R_1\} \times \{|x_2| = R_2\}$. Namely, we can regard P as the operator on $W_R(\mathbf{T}_R^2)$. By definition we have $\pi P = P$ on $W_R(\mathbf{T}_R^2)$.

Let $< D_\theta >$ be a pseudodifferential operator on $\mathbf{T}_R^2$ with symbol $< \eta > := (1 + |\eta|^2)^{1/2}$, where η is the covariable of θ. Then the operator $\pi P < D_\theta >^{-m} = P < D_\theta >^{-m}$ on $W_R(\mathbf{T}_R^2)$ is called a Toeplitz operator on $W_R(\mathbf{T}_R^2)$. Note that the Fredholmness of $P < x\partial_x >^{-m}$ on W_R is equivalent to that of the Toeplitz operator $\pi P < D_\theta >^{-m}$ on $W_R(\mathbf{T}_R^2)$.

2.1. Fredholm property of Toeplitz operators. Let H be a Banach space with norm $\|\cdot\|$. We denote by $\mathcal{L}(H)$ the space of linear continuous operators on H. An operator $L \in \mathcal{L}(H)$ is said to be a Fredholm operator if the range LH of L is closed in H, the kernel and cokernel of L is of finite dimension, i.e., dim $KerL < \infty$ and dim $Coker\, L < \infty$, where $Coker\, L = H/LH$. We denote the space of Fredholm operators by $\Psi(H)$. For $L \in \Psi(H)$ we define the index of L by $\mathrm{ind}\, L := \dim Ker\, L - \dim CokerL$. A linear operator C is said to be a compact operator if it maps every bounded set into a precompact set. For $L \in \Psi(H)$ and a compact operator C, the operator $L + C$ is a Fredholm operator such that $\mathrm{ind}\,(L + C) = \mathrm{ind}\, L$.

The following elementary lemma is useful in the following arguments. (See also Lemma 3.4 in [6])

Lemma 3.1 *Let $q(\eta)$ be a function on $\mathbf{N}^2$ such that $\sup_{|\eta|\geq N} |q(\eta)| \to 0$ when $N \to \infty$. Then the pseudodifferential operator $q(D_\theta) : W_R(\mathbf{T}^2_R) \longrightarrow W_R(\mathbf{T}^2_R)$ is a compact operator.*

Let $\lambda_\beta(D_\theta)$ be a pseudodifferential operator with symbol $\lambda_\beta(\eta) := \eta^\beta |\eta|^{-|\beta|}$ ($\eta \neq 0$) and $\lambda_\beta(0) = 0$. By Lemma 3.1, the Fredholmness of πP is equivalent to that of the operator T,

$$T = \pi \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} e^{i(\alpha-\beta)\theta} \lambda_\beta(D_\theta) : W_R(\mathbf{T}^2_R) \longrightarrow W_R(\mathbf{T}^2_R). \tag{3.3}$$

We note that, by (2.3), $T = \pi\sigma(e^{i\theta}, D_\theta/|D_\theta|)$. For every $u \in W_R(\mathbf{T}^2_R)$ the order of u, $\mathrm{ord}\, u$ is defined as the order of $\hat{u}$, where $\hat{u}$ is an analytic extension of u to W_R. The following theorem is crucial in our argument.

Theorem 3.2 *Suppose (A.1) and (A.2). Then there exists an integer k_0 such that $\pi P < D_\theta >^{-m}$ is invertible as a map on $W_R(\mathbf{T}^2_R) \cap \{u; ord\, u \geq k_0\}$ into itself.*

For the proof we prepare

Proposition 3.3 *Suppose (A.1) and (A.2). Then $T : W_R(\mathbf{T}^2_R) \longrightarrow W_R(\mathbf{T}^2_R)$ in (3.3) is a Fredholm operator of index zero.*

The proof is done by exactly the same method as that of Theorem 3.1 in [6]. $\square$

Proof of Theorem 3.2. By (A.1), (A.2), Proposition 3.3 and Lemma 3.1 we see that T is a Fredholm operator of index zero on $W_R(\mathbf{T}^2_R)$. Because T preserves the homogeneity there exists an integer k_1 such that T is a bijection on $W_R(\mathbf{T}^2_R) \cap \{u; \mathrm{ord}\, u = n\}$ into itself if $n \geq k_1$. Because $\pi P < D_\theta >^{-m}$ $-T$ is a pseudodifferential operator with negative order times a projection we can choose k_0 such that $\pi P < D_\theta >^{-m}$ is invertible as a map on $W_R(\mathbf{T}^2_R) \cap \{u; \mathrm{ord}\, u \geq k_0\}$ into itself. $\square$

4 Proof of Theorems

Proof of Theorem 2.2. We note that $W_R(\mathbf{T}_R^2)$ is a Banach algebra by the usual multiplication of functions. We restrict the equation (2.2) on W_R to $W_R(\mathbf{T}_R^2)$. Let the polynomial $p_\beta(\delta)$ be given by (3.2). Then, by restriction we replace $\partial^\alpha u$ in (2.1) by

$$e^{-i\alpha\theta}p_\alpha(D_\theta)u = \pi e^{-i\alpha\theta}p_\alpha(D_\theta)u. \tag{4.1}$$

Because $M(u_0 + w) = M(u_0) + Qw + M(w)$ for the linearization Q of M at $u = u_0$, the restriction of (2.2) to the torus can be written in the following form

$$\pi Qw + \sum_{\alpha,\beta} a_{\alpha\beta}(\pi e^{-i\alpha\theta}p_\alpha(D_\theta)w)(\pi e^{-i\beta\theta}p_\beta(D_\theta)w) = g. \tag{4.2}$$

Let k_0 be given in Theorem 3.2 and consider (4.2) on $X := W_R(\mathbf{T}_R^2)\cap\{w; \mathrm{ord}\, w \geq k_0\}$. It follows from Theorem 3.2 that there exists $(\pi Q)^{-1} =: S$ on X. We denote the nonlinear part in (4.2) by $K(w)$ and we introduce a new unknown function v by $w = Sv$. Then by recalling $g = \pi g$ we have

$$v + K(Sv) = \pi g. \tag{4.3}$$

We define the sequence $\{v_k\}$ by $v_0 = 0$, $v_1 = \pi g - K(Sv_0) = \pi g$, $v_{k+1} = \pi g - K(Sv_k)$ for $k = 0, 1, \ldots$. If $\|g\|_R$ is sufficiently small, it follows from the definition of K that the apriori estimate $\|v_k\|_R < \rho$ $(k = 1, 2, \ldots)$ holds for small $\rho > 0$. Hence, the limit $v := \sum_{k=0}^\infty (v_{k+1} - v_k) = \lim_k v_k$ exists in X. We have

$$v = \lim v_k = \pi g - \lim K(Sv_{k-1}) = \pi g - K(Sv).$$

Hence there exists a solution $w \in X$ to (4.2).

As to the uniqueness, let $\|w_1\|_R < \rho$, $\|w_2\|_R < \rho$ be the solutions of (4.3). Then we see that $v = w_1 - w_2$ satisfies $v + \tilde{K}(Sw_1, Sw_2)v = 0$, where $\tilde{K}(Sw_1, Sw_2) := K(Sw_1) - K(Sw_2)$ is a polynomial of Sw_1 and Sw_2. Because $\|\tilde{K}(Sw_1, Sw_2)\|_R < 1$ for sufficiently small ρ, we have $\|v\|_R \leq \|\tilde{K}v\|_R \leq \|\tilde{K}\|_R\|v\|_R < \|v\|_R$. Hence $v = 0$.

Let $\hat{w}$ $(w = Sv)$ be the anaytic extension of w into the polydisk D_R. The analytic function $M(u_0+\hat{w})-f_0(x)-g(x)$ in D_R vanishes on the Silov boundary of D_R by the construction of w. Hence the maximal principle implies that it vanishes in D_R. Hence $\hat{w}$ is the solution of (2.2).

Suppose that there exist two solutions $\hat{w}_1$ and $\hat{w}_2$ of (2.2). By the unique solvability of the restricted equation we see that $\hat{w}_1 = \hat{w}_2$ on the boundary. The maximal principle implies that $\hat{w}_1 = \hat{w}_2$ in D_R. □

Proof of Theorem 2.1. By definition of u_0 the Toeplitz symbol is in variant if we replace R by $R\rho$ for $\rho \leq 1$. It follows that the conditions (A.1) and (A.2) hold true for $R = R\rho$. We cover Ω' by a finite number of polydisks D_R's such that for each D_R (A.1) and (A.2) holds. In each D_R there exists a unique solution of our equation if g is sufficiently small. In order to prolong analytic solution, suppose that there exist solutions in D_R and $D_{R'}$. By assumption the solution is unique on $D_R \cap D_{R'}$. Hence we can prolong the solution to $D_R \cup D_{R'}$. It follows that we have a unique solution in Ω'. $\square$

Proof of Corollary 2.5. Let $w = \sum_{j=2m+1}^{\infty} w_j$ be a formal solution of (2.2), where $\operatorname{ord} g \geq 2m+1$, and w_j is homogeneous degree j. For k chosen later, we set $w = w_0 + U$, where $U = \sum_{j=k}^{\infty} w_j$. In order to show the convergence of U, define a polynomial h by $M(u_0 + w_0) = f_0 + h$ and rewrite (2.2) in the form $M(u_0 + w_0 + U) = f_0 + h + g - h$. By taking k sufficiently large, one can make the order of $g - h$ arbitrarily large. Because (A.1) and (A.2) are invariant if we replace u_0 by $u_0 + w_0$, it follows from by Proposition 2.4 that the equation has a unique analytic solution U near the origin. This proves the convergence of U. $\square$

5 Extension to higher dimensions

In this section we will generalize the results in §2 to a general dimension. Let $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$ and define $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$. For $R = (R_1, \ldots, R_n)$ $(R_j > 0)$, let D_R be the disk $D_R = \{|x_1| \leq R_1\} \times \cdots \times \{|x_n| \leq R_n\}$. We define the torus $\mathbf{T}_R^n$ by $\mathbf{T}_R^n = \{|x_1| = R_1\} \times \cdots \times \{|x_n| = R_n\}$.

Let $\Omega \subset \mathbf{C}^n$ be a bounded complete Reinhardt domain containing the origin not necessarily convex. We say that Ω is a Reinhardt domain if for each $a = (a_1, \ldots, a_n) \in \Omega$ the torus $\{(a_1 e^{i\theta_1}, \ldots, a_n e^{i\theta_n}); 0 \leq \theta_j \leq 2\pi, j = 1, \ldots, n\}$ is contained in Ω. A Reinhardt domain Ω is said to be complete if for each $a = (a_1, \ldots, a_n) \in \Omega$ the disk D_R, $R = (|a_1|, \ldots, |a_n|)$ is contained in Ω. We denote by $\tilde{\Omega}$ the real representation of a Reinhardt domain, namely, $\tilde{\Omega} := \{r \in \mathbf{R}_+^n; r = (|x_1|, \ldots, |x_n|), x \in \Omega\}$. We denote by $\mathcal{O}(\Omega)$ the set of holomorphic functions in Ω.

Let $M(x, z_\alpha, |\alpha| \leq m)$ be an analytic function of (x, z_α) in $\Omega \times W$ holomorphic in $x \in \Omega$ and polynomial in z_α. We consider the operator $M(u) := M(x, \partial^\alpha u)$. Let $u_0(x) \in \mathcal{O}(\Omega)$ and set $f_0(x) = M(u_0)$. For $g \in \mathcal{O}(\Omega)$ we consider

$$M(u_0 + v) = f_0(x) + g(x). \tag{5.1}$$

Let $P := M_{u_0} = \sum_{|\alpha| \leq m} (\partial M/\partial z_\alpha)(x, u_0) \partial_x^\alpha$ be the linearized operator of $M(u)$ at $u = u_0$. We write it in the form $P = \sum_{\alpha, |\alpha| \leq m} a_\alpha(x) \partial_x^\alpha$, where $m \in \mathbf{N}$, and $a_\alpha(x)$ is holomorphic in the closure of Ω. We define the Toeplitz symbol $\sigma(z, \xi)$ by

$$\sigma(z, \xi) := \sum_{|\alpha| \leq m} a_\alpha(z) z^{-\alpha} p_\alpha(\xi) < \xi >^{-m}, \tag{5.2}$$

where $< \xi >= (1+|\xi|^2)^{1/2}$ and $p_\alpha(\xi) = \prod_{j=1}^n \xi_j(\xi_j - 1)\cdots(\xi_j - \alpha_j + 1)$. We decompose

$$\sigma(z,\xi) = \sigma'(\xi) + \sigma''(z,\xi), \quad \sigma'(\xi) = \int_{\mathbf{T}^n} \sigma(Re^{i\theta},\xi)d\theta. \tag{5.3}$$

We assume that there exist constants $c \in \mathbf{C}$, $|c| = 1$ and $d > 0$ such that

$$Re\, c\sigma'(\xi) \geq d > 0 \quad \text{for all} \quad \forall \xi \in \mathbf{Z}_+^n. \tag{5.4}$$

Then we have

Theorem 5.1 *Let $\Omega' \subset\subset \Omega$ be arbitrarily given. Suppose that (5.4) is satisfied. Then there exist d_0 and $\varepsilon > 0$ such that if $d \geq d_0$ in (5.4) the equation (5.1) in Ω' has a unique solution v in Ω' for any g holomorphic in Ω such that* $\sup_\Omega |g(x)| \leq \varepsilon$.

We define the set of holomorphic functions $W_R \equiv W_R(D_R)$ by

$$W_R(D_R) := \{u = \sum_{\eta \geq 0} u_\eta x^\eta; \|u\|_R := \sum_\eta |u_\eta| R^\eta < \infty\}. \tag{5.5}$$

We denote by $W_R(\mathbf{T}_R^n)$ the restriction of W_R to the torus $\mathbf{T}_R^n$, $|x_j| = R_j$ by Fourier series expansion. Every element in $W_R(\mathbf{T}_R^n)$ can be extended as a holomorphic function in D_R, uniquely. Let ℓ_R^1 be the space of all absolutely convergent sequences, namely $(u_\eta)_{\eta \in \mathbf{Z}^n}$ such that $\sum_\eta |u_\eta| R^\eta < \infty$. Let $\ell_{R,+}^1$ be the subspace of ℓ_R^1 such that $u_\eta = 0$ for $\eta \notin \mathbf{Z}_+^n$. We note that $W_R(\mathbf{T}_R^n)$ is isomorphic to the space $\ell_{+,R}^1$ in terms of Fourier series expansion. We denote by π the projection $\pi : \ell_R^1 \mapsto \ell_{R,+}^1$. Clearly, the map π also defines the natural projection on the space of all weighted absolutely convergent Fourier series on $\mathbf{T}_R^n$ onto $W_R(\mathbf{T}_R^n)$. Because every element of W_R has a Taylor expansion we can define a pseudodifferential operator $< D_x >$ by the same way as before.

Lemma 5.2 *Suppose that (5.4) is satisfied for some $\tilde{\Omega} = (R_1, \ldots, R_n)$ with $R_j > 0$ $(j = 1, \ldots, n)$. Then there exist d_0 and $\varepsilon > 0$ such that if $d \geq d_0$ in (5.4) the equation $Pu = f$ in W_R has a unique solution for any $f \in W_R$ such that $\|f\|_R \leq \varepsilon$.*

Proof. We restrict the operator $P < D_x >^{-m}$ to the torus by the help of $\overline{\partial}$ operator. Then the Toeplitz symbol of the restricted operator is given by (5.2). It follows from (5.4) that, if d is sufficiently large,

$$Re\, c\sigma(z,\xi) = Re\, c\sigma'(\xi) + Re\, c\sigma''(z,\xi) > d - K_0, \quad \forall z \in \mathbf{T}_R^n, \forall \xi \in \mathbf{Z}_+^n,$$

for some $K_0 > 0$ such that $d > K_0$. Hence, if $\varepsilon > 0$ is sufficiently small we have

$$\|1 - \varepsilon c\sigma(\cdot,\xi)\|_{L^\infty} < 1 - \varepsilon d + \varepsilon K_0, \quad \forall \xi \in \mathbf{Z}_+^n. \tag{5.6}$$

For $u \in W_R(\mathbf{T}^n_R)$ we have

$$\|(I - \varepsilon c\pi\sigma)u\|_R = \|\pi(1 - \varepsilon c\sigma)u\|_R \leq \|(1 - \varepsilon c\sigma)u\|_{\ell^1_R}, \tag{5.7}$$

where we used the boundedness of $\pi : \ell^1_R \to \ell^1_{R,+}$. If we can show that $\|(1 - \varepsilon c\sigma)u\|_{\ell^1_R} < \|u\|_{\ell^1_R} = \|u\|_R$ we have that $\|(I - \varepsilon c\pi\sigma)u\|_R < \|u\|_R$. This proves that $\varepsilon c\pi\sigma$ is invertible on $W_R(\mathbf{T}^n_R)$. Hence $\pi\sigma$ is invertible.

We set $Q(\theta, \xi) = 1 - \varepsilon c\sigma(Re^{i\theta}, \xi)$ and we expand $Q(\cdot, \xi)$ into Fourier series, $Q(\theta, \xi) = \sum_\nu \hat{b}_\nu(\xi) R^\nu e^{i\nu\theta}$. The sum is convergent by the smoothness of Q. In the following we denote by $\mathcal{Q}$ the pseudodifferential operator with symbol given by $Q(\theta, \xi)$. For $u = \sum_\xi \hat{u}(\xi) R^\xi e^{i\theta\xi} \in W_R(\mathbf{T}^n_R)$, we have

$$(\mathcal{Q}u)(\theta) = \sum_\xi Q(\theta, \xi)\hat{u}(\xi) R^\xi e^{i\theta\xi} = \sum_\xi \sum_\nu e^{i\nu\theta} \hat{b}_\nu(\xi)\hat{u}(\xi) R^{\xi+\nu} e^{i\theta\xi}. \tag{5.8}$$

By changing the summation, the right-hand side is equal to

$$\sum_\nu \sum_\xi \hat{b}_\nu(\xi)\hat{u}(\xi) R^{\xi+\nu} e^{i\theta(\xi+\nu)} = \sum_\nu T^\nu u, \tag{5.9}$$

where

$$(T^\nu u)(\theta) := \sum_\xi \hat{b}_\nu(\xi)\hat{u}(\xi) R^{\xi+\nu} e^{i\theta(\xi+\nu)} = \sum_\xi \hat{b}_\nu(\xi - \nu)\hat{u}(\xi - \nu) R^\xi e^{i\theta\xi}. \tag{5.10}$$

Hence we have

$$\begin{aligned} \|T^\nu u\|_R &= \sum_\xi |\hat{b}_\nu(\xi - \nu)||\hat{u}(\xi - \nu)| R^\xi \\ &\leq \sup_\xi |\hat{b}_\nu(\xi) R^\nu| \sum_\xi |\hat{u}(\xi - \nu)| R^{\xi - \nu} = \|u\|_R \sup_\xi |\hat{b}_\nu(\xi)| R^\nu. \end{aligned} \tag{5.11}$$

Let $\sigma(Re^{i\theta}, \xi) = \sum_\nu \hat{\sigma}_\nu(\xi) R^\nu e^{i\nu\theta}$ be the Fourier expansion of $\sigma(Re^{i\theta}, \xi)$. Then we have

$$Q(\theta, \xi) = 1 - \varepsilon c\sigma(Re^{i\theta}, \xi) = 1 - \varepsilon c\hat{\sigma}_0(\xi) - \varepsilon c \sum_{\nu \neq 0} \hat{\sigma}_\nu(\xi) R^\nu e^{i\nu\theta}. \tag{5.12}$$

Suppose that we have the estimate

$$\sum_\nu \sup_\xi |\hat{b}_\nu(\xi)| R^\nu = \sup_\xi |1 - \varepsilon c\hat{\sigma}_0(\xi)| + \varepsilon \sum_{\nu \neq 0} \sup_\xi |\hat{\sigma}_\nu(\xi)| R^\nu < 1. \tag{5.13}$$

Then we get, from (5.11) that

$$\|(1 - \varepsilon c\sigma)u\|_{\ell^1_R} \leq \sum_\nu \|T^\nu u\|_R \leq \|u\|_R \sum_\nu \sup_\xi |\hat{b}_\nu(\xi)| R^\nu < \|u\|_R. \tag{5.14}$$

In order to prove (5.13) we note that (5.13) is clearly satisfied for sufficiently small ε if the summation in ν is finite. (See also (5.6). In the general case we note

$$R^\nu \hat{\sigma}_\nu(\xi) = (2\pi)^{-n} \int_{\mathbf{T}^n} \sigma''(Re^{i\theta}, \xi) e^{-i\nu\theta} d\theta. \tag{5.15}$$

Hence by the partial integration we have

$$R^\nu \nu^\alpha \hat{\sigma}_\nu(\xi) = (2\pi)^{-n} \int_{\mathbf{T}^n} D_\theta^\alpha \sigma''(Re^{i\theta}, \xi) e^{-i\nu\theta} d\theta. \tag{5.16}$$

It follows that $\sup_\xi |\nu^\alpha \hat{\sigma}_\nu(\xi)| R^\nu \le \sup_{\xi,\theta} |D_\theta^\alpha \sigma''(Re^{i\theta}, \xi)| =: K_\alpha$. Therefore the quantity $(1+|\nu|)^{n+1} R^\nu |\hat{\sigma}_\nu(\xi)|$ ($|\nu| = |\nu_1| + \cdots + |\nu_n|$) is bounded by some constant K depending only on R, σ'' and n. By (5.4) the left-hand side of (5.13) is bounded by

$$\sup_\xi |1 - \varepsilon c \sigma'(\xi)| + \varepsilon \sum_{\nu \ne 0} \sup_\xi |\hat{\sigma}_\nu(\xi)| R^\nu \le 1 - \varepsilon d + \varepsilon \sum_{\nu \ne 0} (1+|\nu|)^{-n-1} \le 1 - \varepsilon d + \varepsilon K', \tag{5.17}$$

for some $K' > 0$ depending on K. If d is sufficiently large we have that $1 - \varepsilon d + \varepsilon K' < 1$. $\square$

Proof of Theorem 5.1. Let $\tilde{\Omega}$ be a real representation of Ω. By assumption Ω' is covered by a finite number of D_R's with $R = (R_1, \ldots, R_n) \in \tilde{\Omega}$ such that $R_j \ge \rho_j > 0$. In each D_R we will solve (5.1). Indeed, by Lemma 5.2 and the argument of the proof of Theorem 2.2 (5.1) has a unique holomorphic solution if d in (5.4) is sufficiently large and $\sup_\Omega |f|$ is sufficiently small. The intersection of two disks D_R and $D_{R'}$ is equal to $D_{R''}$ with $R'' = (R_1'', \ldots, R_n'')$, $R_j'' = \min(R_j, R_j') \ge \rho_j$. Because Ω is a complete Reinhardt domain it follows that $R'' \in \tilde{\Omega}$. Hence, by the unique solvability of the holomorphic solution in $D_{R''}$ the solutions in D_R and $D_{R'}$ coincide in $D_{R''}$ if d in (5.4) is sufficiently large and $\sup_\Omega |f|$ is sufficiently small. It follows that we can continue a unique holomorphic solution over a finite union of D_R's. Therefore we have a unique solution in Ω'. $\square$

We will apply the above method to Monge-Ampère type equations under a weaker condition than (5.4). With the same notations as above we consider

$$M(u) := \det(u_{x_j x_k}) + \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta} x^\alpha \partial_x^\beta, \tag{5.18}$$

where $a_{\alpha\beta} \in \mathbf{C}$ and $u_{x_j x_k} = \partial^2 u / \partial x_j \partial x_k$. Let u^0 be holomorphic in Ω and define $M(u^0) = f_0$. We want to solve the equation (5.1) for $g \in \mathcal{O}(\Omega)$ such that $\operatorname{ord} g$ is sufficiently large. Here the order of g is defined in the introduction. The Toeplitz symbol $\sigma(z, \xi)$ of the linearized operator P is given by

$$\sigma(z, \xi) = (z_1 \cdots z_n)^{-2} \det(\xi_j \xi_k + z_j z_k u^0_{x_j x_k}(z)) - f_0(z) + \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta} z^{\alpha-\beta} \xi^\beta. \tag{5.19}$$

We make the decomposition (5.3) and assume that there exist constants $c \in \mathbf{C}$, $|c| = 1$, $N \geq 1$ and $d > 0$ such that

$$Re\, c\sigma'(\xi) \geq d > 0 \quad \text{for all} \quad \forall \xi \in \mathbf{Z}_+^n, |\xi| \geq N. \tag{5.20}$$

The condition (5.20) is slightly weaker than (5.4). Then we have

Theorem 5.3 *Let $\Omega' \subset\subset \Omega$ be arbitrarily given. Suppose that (5.20) is satisfied. Then there exist d_0, $N_0 \geq 1$ and $\varepsilon > 0$ such that if $d \geq d_0$ in (5.20) the equation (5.1) in Ω' has a unique holomorphic solution v in Ω' for any g holomorphic in Ω such that $ord\, g \geq N_0$, $\sup_\Omega |g(x)| \leq \varepsilon$.*

The proof of Theorem 5.3 is almost the same as that of Theorem 5.1 except for that we work in the class of functions v such that $\mathrm{ord}\, v \geq N_0$. (See also the proof of Theorem 2.2.)

Example. We consider the case $n = 3$, $u^0 = x_1x_2x_3$ and the linear part is equal to $-2((x_1\partial_1)^2 + (x_2\partial_2)^2 + (x_3\partial_3)^2)$. We can easily verify the condition (5.20).

6 Monge-Ampère equations in an outer domain

Let Ω be a simply connected complete Reinhardt domain containing the infinity, $x_1 = \infty, \ldots, x_n = \infty$. We say that Ω is complete if for each $(a_1, \ldots, a_n) \in \Omega$ the disk at ∞, $D_a = \{|x_1| \geq |a_1|^{-1}\} \times \cdots \times \{|x_n| \geq |a_n|^{-1}\}$ is contained in Ω. Let $\tilde{\Omega}$ be the real representation of Ω. We say that a Reinhardt domain Ω' is compactly contained in Ω and denote it by $\Omega' \subset\subset \Omega$ if there exist $\rho_j > 0$ $(j = 1, \ldots, n)$ such that the real representation $\tilde{\Omega}'$ is contained in the set $\{\mathbf{R}_+^n; x_j \geq \rho_j, j = 1, \ldots, n\}$ and $\tilde{\Omega}' \subset\subset \tilde{\Omega}$.

We consider the operator given by the left-hand side of (5.18). By the transformation $x_j = y_j^{-1}$ $(j = 1, \ldots, n)$ we have that $u_{x_jx_k} = y_jy_k(\delta_j\delta_k u + \varepsilon_{jk}\delta_j u)$, where $\delta_j = y_j\partial_{y_j}$ and ε_{jk} is a Kronecker delta. Hence it follows that $\det(u_{x_jx_k}) = (y_1 \ldots y_n)^2 \det(\delta_j\delta_k u + \varepsilon_{jk}\delta_j u)$. Because $\det(\delta_j\delta_k u + \varepsilon_{jk}\delta_j u)$ is divisable by $y_1 \cdots y_n$ we set $\tilde{M} := (y_1 \cdots y_n)^{-1} \det(\delta_j\delta_k u + \varepsilon_{jk}\delta_j u)$ and we have $\det(u_{x_jx_k}) = (y_1 \cdots y_n)^3 \tilde{M}$. Hence, if $a_{\alpha\beta} = 0$ in (5.18), (5.18) is equivalent to $\tilde{M} = g$ for appropriately chosen g. In view of this we consider the following operator

$$M(u) := (x_1 \cdots x_n)^3 \det(u_{x_jx_k}) + \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}x^\alpha\partial_x^\beta = f(x). \tag{6.1}$$

Let u^0 be a holomorphic function in Ω such that $ord\, u^0 \geq 2$ and set $f_0 = M(u^0)$. Let P be the linearized operator of $M(u)$ at $u = u_0$. The Toeplitz symbol

$\sigma(z,\xi)$ is then given by

$$\sigma(z,\xi)=\frac{\det(\xi_j\xi_k+z_jz_ku^0_{x_jx_k}(z))}{z_1\cdots z_n}-h(z)+\sum_{|\alpha|=|\beta|=2}a_{\alpha\beta}(-1)^{|\beta|}z^{\beta-\alpha}\xi^\beta, \quad (6.2)$$

where

$$h(z)=(z_1\cdots z_n)^{-1}\det(\delta_j\delta_k u^0). \quad (6.3)$$

We have the following theorem.

Theorem 6.1 *Let $\Omega'\subset\subset\Omega$ be an arbitrarily Reinhardt domain. We make the decomposition (5.3) for $\sigma(z,\xi)$ in (6.2), and we assume (5.20). Then there exist d_0, $N_0\geq 1$ and $\varepsilon>0$ such that if $d\geq d_0$ in (5.20) the equation (6.1) with $u=u^0+v$ and $f=f_0+g$ in Ω' has a unique holomorphic solution v for any g holomorphic in Ω_ρ such that $\mathrm{ord}\, g\geq N_0$, $\sup_{\Omega_\rho}|g|<\varepsilon$.*

Proof. We make the change of variables $x_j=y_j^{-1}$ in (6.1) to obtain

$$\tilde{M}+Q=g(y),\quad g(y)=f(y^{-1}), \quad (6.4)$$

for some Q. By simple calculations the Topelitz symbol of the linearized operator of the left-hand side of (6.4) at $u=u^0$ is given by (6.2) and (6.3). Moreover, we may solve (6.4) in a bounded complete Reinhardt domain because $\Omega'\subset\subset\Omega_\rho$. By the similar argument as in Theorem 5.3, we can prove the existence of a solution for (6.4). □

In the above theorem the existence of a linear part is crucial in actual applications. In the following, we suppose that the linear part in (6.1) is zero, namely $a_{\alpha\beta}=0$. Furthermore, we assume $n=2$, and we shall study regular and singular solutions to the equation

$$u_{x_1x_1}u_{x_2x_2}-u^2_{x_1x_2}=f(x), \quad (6.5)$$

where $x=(x_1,x_2)\in\mathbf{R}^2$. Let $\alpha=(\alpha_1,\alpha_2)\in\mathbf{Z}^2$. Let X_j be the linear space of all formal power series in x_j given by $X_j=\{v; v=x_j^{-\alpha_j}\sum_{k=0}^\infty v_kx_j^{-k}\}$ $(j=1,2)$. Then we have

Theorem 6.2 *Suppose that $f=0$ and $|\alpha|=\alpha_1+\alpha_2\neq-1$. Then every solution of (6.5) of the form $u=x^{-\alpha}\sum_{\eta\in\mathbf{Z}^2_+}u_\eta x^{-\eta}$ is contained either in X_1 or in X_2. Hence they depend only on one variable.*

Remark 6.3 *Theorem 6.2 can be stated in a more general setting. Indeed, consider general Monge-Ampère equation*

$$Ar+Bs+Ct+D(rt-s^2)-E=0,\quad r=z_{x_1x_1}, s=z_{x_1x_2}, t=z_{x_2x_2},$$

where A, B, C, D and E are smooth functions of $x_1, x_2, z, p = \partial z/\partial x_1, q = \partial z/\partial x_2$. Let λ_1 and λ_2 be the roots of the equation $\lambda^2 + B\lambda + (AC + DE) = 0$ and consider the Monge system $\mathcal{M}$

$$\mathcal{M} : dz - pdx_1 - qdx_2 = Ddp + Cdx_1 + \lambda_1 dx_2 = Ddq + \lambda_2 dx_1 + Adx_2 = 0.$$

It is well-known that if $\mathcal{M}$ has three independent first integrals, then the above Monge-Ampère equation is mapped to (6.5) with $f = 0$ by contact transformations.

We first note that the change of variables $x_j = z_j^{-1}, (j = 1, 2)$ in (6.5) yields

$$M(u) \equiv (\delta_1^2 u + \delta_1 u)(\delta_2^2 u + \delta_2 u) - (\delta_1\delta_2 u)^2 = g(z), \tag{6.6}$$

where $\delta_j = z_j(\partial/\partial z_j)$ $(j = 1, 2)$ and $g(z) = z_1^{-2} z_2^{-2} f(z_1^{-1}, z_2^{-1})$.

Proof of Theorem 6.2. We set $|\alpha| = m$ and $g = 0$. By substituting the expansion $u = z^\alpha \sum_{\eta\in\mathbf{Z}_+^2} u_\eta z^\eta$ $(u_0 \neq 0)$ in (6.6) and by comparing the homogeneous part of degree $2m$ we obtain $(\alpha_1^2+\alpha_1)(\alpha_2^2+\alpha_2) - \alpha_1^2\alpha_2^2 = 0$. It follows from the condition $|\alpha| \neq -1$ that $\alpha_1\alpha_2 = 0$, which implies $\alpha = (m, 0)$ or $\alpha = (0, m)$.

Suppose that $m \leq -2$ and $\alpha = (m, 0)$. Let $u = z_1^m + u_1 + u_2 + \cdots$ be the homogeneous expansion of u with u_j being homogeneous degree $m + j$. By comparing $2m+1$ homogeneous part of (6.6) we have $(\delta_2^2+\delta_2)u_1 = 0$. Because u_1 contains only nonnegative powers of z_2 by assumption $\alpha = (m, 0)$ it follows that $u_1 = c_1 z_1^{-m+1}$ for some constant c_1. Suppose that $u_j = c_j z_1^{m+j}$ for $j \leq k$. By comparing the homogeneous part of degree $2m + k + 1$ in (6.6) we obtain $(\delta_2^2 + \delta_2)u_{k+1} = 0$, to yield $u_{k+1} = c_{k+1} z_1^{m+k+1}$. This proves that $u \in X_1$. We can similarly prove that $u \in X_2$ in case $\alpha = (0, m)$. In view of $|\alpha| \neq -1$, the other case $m \geq 0$ is contained in the following proposition. □

Proposition 6.4 *Let $u = \sum_{\nu=1}^{\infty} u_\nu$ be the homogeneous expansion of u in (6.6). Suppose that $u_j = 0$ for any $j < k$ and that $u_k \neq 0$. Then we have the followings*
a) Either $u_k = cz_1^k$ or $u_k = cz_2^k$ holds for some $c \neq 0$.
b)Either $u_\nu = c_\nu z_1^\nu$ $(\nu > k)$ or $u_\nu = c_\nu z_2^\nu$ $(\nu > k)$ holds for some c_ν.

Proof. If we prove a), b) follows by the same argument as in Theorem 6.2. In order to prove a) we note that the comparison of terms of homogeneous order $2k$ in (6.6) implies that u_k satisfies (6.6) with $g = 0$. We substitute the expansion of u_k, $u_k = \sum_{j=0}^{k} a_j z_1^j z_2^{k-j}$ into (6.6). Since the term z_1^{2k} does not appear we compare the coefficients of $z_1^{2k-1} z_2$, to obtain $a_k a_{k-1} = 0$.

If $a_k \neq 0$ it follows that $a_{k-1} = 0$. Next, by comparing the terms of $z_1^{2k-2} z_2^2$, we obtain $a_k a_{k-2} = 0$, that is $a_{k-2} = 0$. In the same way we can show that u_k depends only on z_1.

We consider the case $a_k = 0$. Because the assertion is trivial for $k = 1$ we assume $k \geq 2$. By comparing the coefficients of $z_1^{2k-2} z_2^2$ in $M(u_k) = 0$ we have that $a_{k-1}^2 = 0$. Suppose that $a_{k-\nu} = 0$ for $\nu \leq j \leq k-2$. It follows that u_k is of degree greater than $j+1$ in z_2. Hence, by comparing the coefficients of degree $2(j+1)$ in z_2, we obtain $a_{k-\nu-1}^2 = 0$. It follows that $a_k = \cdots = a_1 = 0$. Hence u_k depends only on z_2. □

We will study the solutions of the form $u = \sum_{\eta \in \mathbf{Z}_+^2} u_\eta x^{-\eta}$ of (6.5) or (6.6). By writing $g(z) = \sum_{j=2}^{\infty} g_j$ with g_j being homogeneous degree j, we seek $u = \sum_{j=1}^{\infty} u_j$ where $u_1 = az_1 + bz_2$ and u_j is homogeneous degree j. We assume $ab \neq 0$. If there exists a formal solution of (6.6), we can easily see that $g(z)$ satisfies the condition $g(z) = O(z_1 z_2)$. By comparing terms of homogeneous degree 2 in (6.6) we have $g_2(z) = 4abz_1z_2$. We assume these compatibility conditions. By the scale change of variables we may assume $a = b = 1/2$.

Theorem 6.5 *Let g be an arbitrarily power series satisfying $g_2(z) = z_1 z_2$, $g(z) = O(z_1 z_2)$. For any formal power series v such that $M(v) = 0$, $\operatorname{ord} v \geq 2$ there exists a unique formal power series ϕ such that $\operatorname{ord} \phi \geq 2$ and $u := u_1 + v + \phi$ satisfies (6.6). We write $\phi = Sv$. Conversely, let $u = u_1 + w$, $\operatorname{ord} w \geq 2$ be a solution of (6.6) and let $j = 1$ or $j = 2$. Then there exists a unique power series v of x_j such that, $M(v) = 0$, $\operatorname{ord} v \geq 2$ and $u = u_1 + v + Sv$. Moreover, Sv does not contain the powers of x_j only.*

Proof. We use the same notations as above. In order to determine u_2 we compare the terms of homogeneous degree 3 in (6.6). We then have $P_1 u_2 = g_3(z)$, where $P_1 = z_1(\delta_2^2 + \delta_2) + z_2(\delta_1^2 + \delta_1)$. If we substitute the expansions $u_2(z) = c_0 z_2^2 + c_1 z_1 z_2 + c_2 z_1^2$ and $g_3(z) = d_1 z_1 z_2^2 + d_2 z_1^2 z_2$ into the equation $P_1 u_2 = g_3(z)$ we obtain that $6c_0 + 2c_1 = d_1$ and $2c_1 + 6c_2 = d_2$. These equations have a unique solution once we give c_0 or c_2, which is a kernel element of $M(v) = 0$ in view of Theorem 6.2.

Suppose that we have determined u_j for $j < k$. By comparing $(k+1)$-th homogeneous part of (6.6) we see that u_k satisfies $P_1 u_k + (\cdots) = g_{k+1}(z)$, where the dots denotes the terms determined by u_j with $j < k$. Because we can easily see, from the definition of M that these term are divisable by $z_1 z_2$ the recurrence relations has the same structures as for u_2 and we can determine u_k if we assign the kernel element in u_k. The rest of Theorem 6.5 is clear. □

Corollary 6.6 *Let $u = u_1 + v + Sv$ be a formal power series solution of (6.6). If u converges, it follows that the kernel element v converges.*

Proof. If otherwise, u does not converge because Sv does not contain the powers of x_j only. □

Remark 6.7 *We note that the operator P_1 in the proof of Theorem 6.5 is degenerate elliptic-hyperbolic near the origin. Indeed, the principal symbol of*

P_1 is given by $-z_1z_2(z_2\xi_2^2 + z_1\xi_1^2)$. We remark that Corollary 2.5 cannot be applied since P_1 does not satisfy (A.1).

Concerning the convergence of formal solutions u in Theorem 6.5 we note that the existence of a nontrivial kernel element v, $M(v) = 0$ leads to the divergence in general.

We define the space W_R as in Theorem 2.2 with respect to the variable z. Now, we drop the assumption $ab \neq 0$ in Theorem 6.5. For simplicity we assume $u_1 = 0$ and we look for the solution of (6.6) in the form $u = \sum_{j=2}^{\infty} u_j$, where $u_2(z) = az_1^2 + bz_1z_2 + cz_2^2$, u_j being homogeneous degree j. We define $g_4 = M(u_2)$.

Theorem 6.8 *We define the Toeplitz symbol by $\sigma(t,\xi) := 3a\xi_2^2 t + b(\xi_1^2 + \xi_2^2 - \xi_1\xi_2) + 3c\xi_1^2 t^{-1}$. Suppose that*

$$\sigma(t,\xi) \neq 0 \quad \text{for } \forall t \in \mathbf{C}, \quad |t| = 1, \ \forall \xi \in \mathbf{R}_+^2, \ |\xi| = 1, \tag{6.7}$$

$$ind_{|t|=1}\,\sigma(t,\xi) \equiv \frac{1}{2\pi i}\oint_{|t|=1} d_t \log\sigma(t,\xi) = 0 \ \exists \xi \in \mathbf{R}_+^2, \ |\xi| = 1. \tag{6.8}$$

Then there exists $N \geq 3$ and $r > 0$ such that for every $g = g_4 + \hat{g} \in W_R$ such that $ord\,\hat{g} \geq N$ and $\|\hat{g}\|_R < r$ the equation (6.6) has a unique solution $u \in W_R$ such that $u = u_2 + v$, $ord\,v \geq N$.

Proof. Because both sides of (6.6) is divisable by z_1z_2 by definition we consider $z_1^{-1}z_2^{-1}M(u_2+v)/2 = z_1^{-1}z_2^{-1}g/2$. By simple computation we have that $M(u_2+v) = M(u_2) + 2z_1z_2Pv + M(v)$, where

$$P = 3az_1z_2^{-1}(\delta_2^2 + \delta_2) + b(\delta_1^2 + \delta_2^2 - \delta_1\delta_2 + \delta_1 + \delta_2) + 3cz_1^{-1}z_2(\delta_1^2 + \delta_1). \tag{6.9}$$

The conditions (A.1) and (A.2) in Theorem 2.2 are equivalent to (6.7) and (6.8), respectively. Hence Theorem 6.8 follows from Theorem 2.2. □

Corollary 6.9 *Let u_2 be as in the above. Suppose that $|b| > 6|ac|^{1/2}$. Then, there exists $r > 0$ such that for every $g \in W_R$ such that $ord\,g \geq 4$, $\|g\|_R < r$ and $g_4 = M(u_2)$ there exists a solution $u = u_2 + u_3 + \cdots \in W_R$ of the equation (6.6).*

Proof. First we note that M is invariant under the scale change of variables. If $ac = 0$, we may assume $c = 0$ without loss of generality. By the scale change of variables, one can make a arbitrarily small, while b remains unchanged. Then (6.7) and (6.8) are clearly satisfied.

If $ac \neq 0$ one may assume, by scale change that $|a| = |c| = |ac|^{1/2}$ in u_2. The Toeplitz symbol is equal to $\sigma(e^{i(\theta_1-\theta_2)}, \xi)$, with $\sigma(t,\xi)$ given in Theorem 6.8. Hence (A.1) holds if $|b|/(3|ac|^{1/2}) > (\xi_1^2 + \xi_2^2)(\xi_1^2 + \xi_2^2 - \xi_1\xi_2)^{-1}$. Because the maximum of the right-hand side is 2 the condition is easily verfied by the assumption $|b| > 6|ac|^{1/2}$. As to (A.2), by setting $\xi_1 = 0, \xi_2 = 1$ we consider $b + 3ce^{i(\theta_2-\theta_1)}$. The condition (A.2) is easily verified.

To end the proof, we will show that there is no finite -dimensional kernel for P in (6.9). The coefficient of z^η in Pu, $u = \sum u_\eta z^\eta$ is equal to

$$(\eta_2+1)^2 u_{\eta_1-1,\eta_2+1} + b(\eta_1^2+\eta_2^2-\eta_1\eta_2+\eta_1+\eta_2)u_\eta + (\eta_1+1)^2 u_{\eta_1+1,\eta_2-1}.$$

Because P preserves homogeneous polynomials, for a given $k \geq 3$ we write the system of equations $|\eta| = k$ in the matrix form. We apply Gershgorin's theorem to the matrices obtained from this recurrence relation. For $\eta_1 = \nu$, $\eta_2 = k - \nu$ Gerschgorin's condition implies that any eigenvalue λ satisfies

$$|b(\nu^2 + (k-\nu)^2 - \nu(k-\nu) + k) - \lambda| \leq 3|ac|^{1/2}(\nu^2 + (k-\nu)^2). \tag{6.10}$$

Hence $\lambda = 0$ is not an eigenvalue if $|b| > 6|ac|^{1/2}$. □

Remark 6.10 *(a) Theorem 6.8 and Corollary 6.9 hold in the case of local solvability. Namely, we take $R > 0$ sufficiently small instead of the smallness of g. Because the statement is similar to Proposition 2.4, we omit it. It is also possible to extend the above results to the case of a complete Reinhardt outer domain in $\mathbf{C}^2$ as in Theorem 6.1.*

(b) The operator P in (6.9) is of mixed type in general. Indeed, the principal symbol is given by $az_1z_2\xi_2^2 + cz_1z_2\xi_1^2 + b(z_1^2\xi_1^2 + z_2^2\xi_2^2 - z_1z_2\xi_1\xi_2)$. If $a = c = 0$ and $b \neq 0$, P is degenerate elliptic near the origin. If $c = 0$, $a = 1$ or $a = c = 1$, P is degenerate elliptic-hyperbolic near the origin.

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