

Simultaneous reduction to normal forms of commuting singular vector fields with linear parts having Jordan blocks

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April 18, 2005

Abstract

We study a simultaneous linearizability of d -actions (and the corresponding d -dimensional Lie algebras) defined by commuting singular vector fields in \mathbb{C}^n fixing the origin with a nontrivial Jordan block in the linear parts. We prove the analytic convergence of a formal linearizing transformation under a certain invariant geometric condition (cone condition) for the spectrum of d vector fields generating a Lie algebra. (cf. Example 1.4.) If the condition fails, then we show the existence of divergent solutions of an overdetermined system of linearized homological equations. In a smooth category, the situation is

*Supported by NATO grant PST.CLG.979347, and GNAMPA-INDAM, Italy.

†Supported by Grant-in-Aid for Scientific Research (No. 14340042), Ministry of Education, Science and Culture, Japan and GNAMPA-INDAM, Italy.

completely different. We will show Sternberg's theorem for a commuting system of vector fields with a Jordan block although they do not satisfy a cone condition.

Key words: singular vector field, linearization, Jordan block, homological equation, Diophantine conditions, Gevrey spaces, decomposition

MSC classification: 37C05, 37F50, 37G05

1 Simultaneous linearization

Let \mathbb{K} be $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, and $B = \infty$, $B = \omega$ or $B = k$ for some $k > 0$. Let \mathcal{G}_B^n denotes a d -dimensional Lie algebra of germs at $0 \in \mathbb{K}^n$ of C^B vector fields vanishing at 0. Let ρ be a germ of singular infinitesimal \mathbb{K}^d ($d \geq 2$) actions of class C^B

$$\rho : \mathbb{K}^d \longrightarrow \mathcal{G}_B^n. \quad (1.1)$$

We denote by $Act^B(\mathbb{K}^d : \mathbb{K}^n)$ the set of germs of singular infinitesimal \mathbb{K}^d actions of class C^B in $0 \in \mathbb{K}^n$. By choosing a basis $e_1, \dots, e_d \in \mathbb{K}^n$, the infinitesimal action can be identified with a d -tuple of germs at 0 of commuting vector fields $X^j = \rho(e_j)$, $j = 1, \dots, d$ (cf. [10], [21]). We can define, in view of the commutativity relation, the action

$$\tilde{\rho} : \mathbb{K}^d \times \mathbb{K}^n \longrightarrow \mathbb{K}^n, \quad (1.2)$$

$$\tilde{\rho}(s; z) = X_{s_1}^1 \circ \dots \circ X_{s_d}^d(z) = X_{s_{\sigma_1}}^{\sigma_1} \circ \dots \circ X_{s_{\sigma_d}}^{\sigma_d}(z), \quad s = (s_1, \dots, s_d), \quad (1.3)$$

for all permutations $\sigma = (\sigma_1, \dots, \sigma_d)$ of $\{1, \dots, d\}$, where X_t^j denotes the flow of X^j . We denote by ρ_{lin} the linear action formed by the linear parts of the vector fields defining ρ .

We shall investigate the necessary and sufficient condition for the linearization of ρ , namely, whether there exists a C^B diffeomorphism g preserving 0 such that g conjugates $\tilde{\rho}$ and ρ_{lin}

$$\tilde{\rho}(s; g(z)) = g(\widetilde{\rho_{lin}}(s, z)), \quad (s, z) \in \mathbb{K}^d \times \mathbb{K}^n. \quad (1.4)$$

We recall that in [10], [17], [28] the linear parts were supposed to be diagonalizable, while in [32] the existence of $n - d$ analytic first integrals was

required. (See also [1], [16]). Following Katok's argument in [21], we take a positive integer $m \leq n$ such that \mathbb{K}^n is decomposed into a direct sum of m linear subspaces invariant under all $A^\ell = \nabla X_\ell(0)$ ($\ell = 1, \dots, d$):

$$\begin{aligned} \mathbb{K}^n &= \mathbb{I}^{s_1} + \dots + \mathbb{I}^{s_m}, & \dim \mathbb{I}^{s_j} &= s_j, \quad j = 1, \dots, m, \\ s_1 + \dots + s_m &= n. \end{aligned} \quad (1.5)$$

The matrices A^1, \dots, A^d can be simultaneously brought in an upper triangular form, and we write again A^ℓ for the matrices,

$$A^\ell = \begin{pmatrix} A_1^\ell & 0_{s_1 \times s_2} & \dots & 0_{s_1 \times s_m} \\ 0_{s_2 \times s_1} & A_2^\ell & \dots & 0_{s_2 \times s_m} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{s_m \times s_1} & 0_{s_m \times s_2} & \dots & A_m^\ell \end{pmatrix}, \quad \ell = 1, \dots, d. \quad (1.6)$$

If $\mathbb{K} = \mathbb{C}$, the matrix A_j^ℓ is given by

$$A_j^\ell = \begin{pmatrix} \lambda_j^\ell & A_{j,12}^\ell & \dots & A_{j,1s_j}^\ell \\ 0 & \lambda_j^\ell & \dots & A_{j,2s_j}^\ell \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_j^\ell \end{pmatrix}, \quad \ell = 1, \dots, d, \quad j = 1, \dots, m, \quad (1.7)$$

with $\lambda_j^\ell, A_{j,\nu\mu}^\ell \in \mathbb{C}$. On the other hand, if $\mathbb{K} = \mathbb{R}$, then we have, for every $1 \leq j \leq m$ two possibilities: firstly, all A_j^ℓ ($\ell = 1, \dots, d$) are given by (1.7) with $\lambda_j^\ell \in \mathbb{R}$. Secondly, $s_j = 2\tilde{s}_j$ is even and A_j^ℓ is a $\tilde{s}_j \times \tilde{s}_j$ square block matrix given by

$$A_j^\ell = \begin{pmatrix} R_2(\lambda_j^\ell, \mu_j^\ell) & A_{\ell,j}^{12} & \dots & A_{\ell,j}^{1\tilde{s}_j} \\ 0 & R_2(\lambda_j^\ell, \mu_j^\ell) & \dots & A_{\ell,j}^{2\tilde{s}_j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_2(\lambda_j^\ell, \mu_j^\ell) \end{pmatrix}, \quad \ell = 1, \dots, d, \quad (1.8)$$

where

$$R_2(\lambda, \mu) := \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}, \quad (1.9)$$

and $A_{\ell,j}^{rs}$ are appropriate real matrices.

Following the decomposition (1.7) (respectively, (1.8)) we define $\tilde{\lambda}^j$ by

$$\tilde{\lambda}^k = (\lambda_1^k, \dots, \lambda_m^k) \in \mathbb{K}^m, \quad k = 1, \dots, d. \quad (1.10)$$

Then we assume

$$\tilde{\lambda}^1, \dots, \tilde{\lambda}^d \text{ are linearly independent in } \mathbb{K}^m. \quad (1.11)$$

One can easily see that (1.11) is invariantly defined.

We introduce a cone condition. By (1.6) we define

$$\vec{\lambda}_j = {}^t(\lambda_j^1, \dots, \lambda_j^d) \in \mathbb{K}^d, \quad j = 1, \dots, m, \quad (1.12)$$

and

$$\Lambda_m := \{\vec{\lambda}_1, \dots, \vec{\lambda}_m\}. \quad (1.13)$$

We define the cone $\Gamma[\Lambda_m]$ by

$$\Gamma[\Lambda_m] = \left\{ \sum_{j=1}^m t_j \vec{\lambda}_j \in \mathbb{K}^d; t_j \geq 0, j = 1, \dots, m, \sum_{j=1}^m t_j^2 \neq 0 \right\}. \quad (1.14)$$

Definition 1.1 *We say that the \mathbb{K}^d -action ρ satisfies a cone condition if there exists a base $\Lambda_m \subset \mathbb{K}^m$ such that $\Gamma[\Lambda_m]$ is a proper cone in \mathbb{K}^d , namely it does not contain a straight real line. If the condition is not satisfied, then, we say that the \mathbb{K}^d action is in a Siegel domain.*

Note that the definition is invariant under the choice of the basis Λ_m .

Next, we introduce the notion of simultaneous resonances. For $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{K}^m$, $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{K}^m$, we set $\langle \alpha, \beta \rangle = \sum_{\nu=1}^m \alpha_\nu \beta_\nu$. For a positive integer k we define $\mathbb{Z}_+^m(k) = \{\alpha \in \mathbb{Z}_+^m; |\alpha| \geq k\}$. Put

$$\omega_j(\alpha) = \sum_{\nu=1}^d |\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu|, \quad j = 1, \dots, m, \quad (1.15)$$

$$\omega(\alpha) = \min\{\omega_1(\alpha), \dots, \omega_m(\alpha)\}. \quad (1.16)$$

Definition 1.2 *We say that Λ_m is simultaneously nonresonant (or, in short ρ is simultaneously nonresonant), if*

$$\omega(\alpha) \neq 0, \quad \forall \alpha \in \mathbb{Z}_+^m(2). \quad (1.17)$$

If (1.17) does not hold, then we say that Λ_m is simultaneously resonant.

Clearly, the simultaneously nonresonant condition (1.17) is invariant under the change of the basis Λ_m . We state the first main result of our paper

Theorem 1.3 *Let ρ be a \mathbb{K}^d analytic action which is simultaneously nonresonant and satisfies the cone condition. Then ρ is linearizable by an analytic change of the variables.*

Example 1.4. We compare our theorem with the results of Stolovitch [28] and Zung [32]. Let ρ be a \mathbb{R}^2 action in \mathbb{R}^n , $n \geq 4$ with $m = 3$. We choose a basis Λ_2 of \mathbb{R}^3 such that

$$\Lambda_2 = \{ {}^t(1, 1, \nu), {}^t(0, 1, \mu) \}, \quad \nu, \mu \in \mathbb{R}. \quad (1.18)$$

(cf. [DGY] for similar and more general reductions of commuting vector fields on the torus).

We will characterize the set of $(\nu, \mu) \in \mathbb{R}^2$ satisfying the cone condition, and determine the simultaneous resonances. By (1.14), $\Gamma[\Lambda_2]$ is generated by the set of vectors $\{(1, 0), (1, 1), (\nu, \mu)\}$. Hence the cone condition holds if and only if these vectors generate a proper cone, namely (ν, μ) is not in the set $\{(\nu, \mu) \in \mathbb{R}^2; \nu \leq \mu \leq 0\}$. We note that the interesting case is $\mu < \nu \leq 0$, where every generator in (1.18) is in a Siegel domain. Theorem 1.3 can be applied to such a case. In §3 we will show that if a cone condition is violated, i.e., $\nu < \mu < 0$, the linearized overdetermined system of two homological equations has a divergent solution.

Next we will determine (ν, μ) so that a simultaneous resonance exists. If $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{Z}_+^3(2)$ is a simultaneous resonance, we have the following set of equations:

$$\begin{aligned} (1) \quad & \eta_1 + \eta_2 + \nu\eta_3 = 1, \quad \eta_2 + \mu\eta_3 = 0, \\ (2) \quad & \eta_1 + \eta_2 + \nu\eta_3 = 1, \quad \eta_2 + \mu\eta_3 = 1, \\ (3) \quad & \eta_1 + \eta_2 + \nu\eta_3 = \nu, \quad \eta_2 + \mu\eta_3 = \mu. \end{aligned}$$

By elementary computations, in order that one of these equations has a solution η the (ν, μ) satisfies the following:

- a) Case $\nu \leq \mu \leq 0$. The resonance exists iff $(\nu, \mu) \in \mathbb{Q}_- \times \mathbb{Q}_-$, where \mathbb{Q}_- is the set of nonpositive rational numbers. The resonance is given by $(1 + (\mu - \nu)k, -\mu k, k)$ and $((\mu - \nu)k, 1 - k\mu, k)$ where $k \geq 1/(1 - \nu)$, $k \in \mathbb{Z}_+$, and $((\nu - \mu)(1 - k), \mu(1 - k), k)$, where $k \geq (2 - \nu)(1 - \nu)$, $k \in \mathbb{Z}_+$.
- b) Case $\nu > \mu$ and $\mu \leq 0$. The resonance is given by $(0, -\mu/(\nu - \mu), 1/(\nu - \mu))$,

where $-\mu/(\nu - \mu) \in \mathbb{Z}_+$, $1/(\nu - \mu) \in \mathbb{Z}_+$ and $2\nu - \mu \leq 1$.

c) Case $\mu > 0$, $\nu \leq \mu$. The resonance is given by $(0, 0, 1/\nu)$, when $\nu = \mu$, $\nu \leq 1/2$, $\nu^{-1} \in \mathbb{Z}_+$, $(0, \nu, 0)$, when $\nu = \mu \geq 2$, $\nu \in \mathbb{Z}_+$, $((\mu - \nu)/\mu, 0, 1/\mu)$, if otherwise, where $(\mu - \nu)/\mu \in \mathbb{Z}_+$, $1/\mu \in \mathbb{Z}_+$ and $\nu + \mu \leq 1$.

d) Case $\nu > \mu$, $\mu \geq 0$. The resonance is given by $(\nu - \mu, \mu, 0)$, where $\nu - \mu \in \mathbb{Z}_+$, $\mu \in \mathbb{Z}_+$ and $\nu \geq 2$.

Let ν be a negative rational number, $\nu = -k_1/k_2$, $k_1, k_2 \in \mathbb{Z}_+$, $k_2 \neq 0$. Let μ be a rational number and satisfy $\mu < \nu$. Assume that the nonlinear part of X^2 is zero. If the nonlinear part of X^1 consists of the resonant terms of X^2 , then we have $[X^1, X^2] = 0$. We can easily see that the linearizability of X^1 holds provided $\mu \neq \nu - 1/k_2 = -(k_1 + 1)/k_2$.

2 A cone condition

We start by showing equivalent forms of the cone condition.

Proposition 2.1 *The cone condition is equivalent to each of the following conditions*

i) *there exist a positive constant C and an integer k_0 such that*

$$\sum_{k=1}^d \left| \sum_{j=1}^m \lambda_j^k \alpha_j \right| \geq C_1 |\alpha|, \quad \forall \alpha \in \mathbb{Z}_+^m(k_0). \quad (2.1)$$

ii) *there exists a nonzero vector $c = (c_1, \dots, c_d) \in \mathbb{C}^d$ if $\mathbb{K} = \mathbb{C}$ (respectively, $c = (c_1, \dots, c_d) \in \mathbb{R}^d$ if $\mathbb{K} = \mathbb{R}$) such that*

$$c_1 \tilde{\lambda}^1 + \dots + c_d \tilde{\lambda}^d \text{ is in a Poincaré domain,} \quad (2.2)$$

namely, the convex hull of the set $\{\sum_{j=1}^d c_j \lambda_k^j; k = 1, \dots, m\}$ in \mathbb{C} does not contain $0 \in \mathbb{C}$ (respectively,

$$\text{the real parts of } c_1 \lambda_j^1 + \dots + c_d \lambda_j^d, j = 1, \dots, m, \text{ are positive.}) \quad (2.3)$$

Proof. First we show (2.1). Suppose that (2.1) does not hold. Then there exists a sequence $\alpha^\ell \in \mathbb{Z}_+^m$, $\ell \in \mathbb{N}$ such that $|\alpha^\ell| \rightarrow \infty$ ($\ell \rightarrow \infty$) and

$$\sum_{k=1}^d \left| \sum_{j=1}^m \lambda_j^k \alpha_j^\ell \right| \leq \frac{|\alpha^\ell|}{\ell}, \quad \ell \in \mathbb{N}. \quad (2.4)$$

By taking a subsequence, if necessary, we may assume that $\alpha^\ell/|\alpha^\ell| \rightarrow t^0 = (t_1^0, \dots, t_m^0) \in S_{\ell^1}^1 \cap \mathbb{R}_+^m$ when $\ell \rightarrow \infty$, where $S_{\ell^1}^1 := \{x \in \mathbb{K}^m; \|x\|_{\ell^1} = \sum_{j=1}^m |x_j| = 1\}$ stands for the ℓ^1 unit sphere. By letting $\ell \rightarrow \infty$ in (2.4) we get

$$\sum_{k=1}^d \left| \sum_{j=1}^m \lambda_j^k t_j^0 \right| = 0.$$

It follows that $\sum_{j=1}^m t_j^0 \vec{\lambda}_j = 0$. Let $J \subset \{1, \dots, m\}$ be such that $\sum_{j \in J} t_j^0 \vec{\lambda}_j \neq 0$. Such a set J exists by (1.11). It follows that

$$0 \neq \sum_{j \in J} t_j^0 \vec{\lambda}_j = - \sum_{j \in \{1, \dots, m\} \setminus J} t_j^0 \vec{\lambda}_j.$$

Hence $\Gamma[\Lambda_m]$ contains a straight line generated by $\sum_{j \in J} t_j^0 \vec{\lambda}_j \neq 0$. This contradicts the assumption that $\Gamma[\Lambda_m]$ is a proper cone.

Conversely, suppose that (2.1) is satisfied. We shall show that $\Gamma[\Lambda_m]$ is proper. Indeed, if otherwise, we can find $t^0 = (t_1^0, \dots, t_m^0) \in S_{\ell^1}^1 \cap \mathbb{R}_+^m \setminus 0$ such that

$$\sum_{j=1}^m t_j^0 \lambda_j^k = 0, \quad k = 1, \dots, d. \quad (2.5)$$

Because the set $\{\alpha/|\alpha|; \alpha \in \mathbb{Z}_+^m(2)\}$ is dense in $S_{\ell^1}^1 \cap \mathbb{R}_+^m$, there exists a sequence $\alpha^\ell \in \mathbb{Z}_+^m$, $\ell \in \mathbb{N}$ such that $|\alpha^\ell| \rightarrow \infty$ ($\ell \rightarrow \infty$) and $\lim_{\ell \rightarrow \infty} \alpha^\ell/|\alpha^\ell| = t^0$. Therefore, in view of (2.5), we get

$$\lim_{\ell \rightarrow \infty} \left(\frac{1}{|\alpha^\ell|} \sum_{k=1}^d \left| \sum_{j=1}^m \lambda_j^k \alpha_j^\ell \right| \right) = 0,$$

which contradicts (2.1)

Next, we show ii). Suppose that $\Gamma[\Lambda_m]$ be a proper cone in \mathbb{K}^d . Then we can find $c = (c_1, \dots, c_d) \in \mathbb{C}^d$ such that $\Gamma[\Lambda_m]$ is contained in the real half-space $P_c := \{z \in \mathbb{K}^d, \operatorname{Re}(\sum_{k=1}^d c_k z_k) > 0\}$. Therefore

$$0 < \operatorname{Re} \left(\sum_{k=1}^d c_k \sum_{j=1}^m t_j \lambda_j^k \right) = \sum_{j=1}^m t_j \operatorname{Re} \left(\sum_{k=1}^d c_k \lambda_j^k \right) \quad (2.6)$$

for all $t \in \mathbb{R}_+^m \setminus 0$, which yields $\operatorname{Re}(\sum_{k=1}^d c_k \lambda_j^k) > 0$ for $j = 1, \dots, m$. We note that, if $\mathbb{K} = \mathbb{R}$, then the use of the real part in the definition of the half-space is superfluous. Finally, we readily see, from (2.2) that, if $\mathbb{K} = \mathbb{C}$ (respectively, (2.3) if $\mathbb{K} = \mathbb{R}$), then the cone $\Gamma[\Lambda_m]$ is contained in P_c . Hence $\Gamma[\Lambda_m]$ is proper. The proof is complete.

The proof of Theorem 1.3 follows from

Proposition 2.2 *Let the action ρ satisfy the cone condition. Then we can find a vector field in the corresponding Lie algebra which is nonresonant and is in the Poincaré domain.*

Proof. By ii) of Proposition 2.1 we can find a Poincaré vector field in the Lie algebra as a linear combination of a base corresponding to (2.2). Let c_ν be the number in (2.2), and define $\tilde{\lambda}^0 := (\lambda_1^0, \dots, \lambda_m^0) = \sum_{\nu=1}^d c_\nu \tilde{\lambda}^\nu$. Consider

$$\langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = \sum_{\nu=1}^d c_\nu \left(\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu \right).$$

Because $\sum_{\nu=1}^d |\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda_j^\nu| \neq 0$ for $\forall \alpha \in \mathbb{Z}_+^m(2)$ by the simultaneous nonresonant condition it follows that the set $\langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = 0$ in $c = (c_1, \dots, c_d) \in \mathbb{C}^d$ is a hyperplane. It follows that the set

$$\{c = (c_1, \dots, c_d) \in \mathbb{C}^d; \langle \tilde{\lambda}^0, \alpha \rangle - \lambda_j^0 = 0, \exists j, 1 \leq j \leq m, \exists \alpha \in \mathbb{Z}_+^m(2)\}$$

is a countable union of nowhere dense closed set. Therefore we can find $c = (c_1, \dots, c_d)$ for which $\sum_{\nu=1}^d c_\nu \tilde{\lambda}^\nu$ satisfies a nonresonant condition and the cone condition. This proves Proposition 2.2.

We propose a geometric expression of a cone condition

Definition 2.3 *Let $r > 0$ and g be a Riemannian metric on \mathbb{R}^n . We denote by $\langle \cdot, \cdot \rangle_g$ and $\| \cdot \|_g$ the inner product and the norm with respect to g , respectively. We say that $\mathcal{X}_\nu := \sum_{j=1}^n X_j^\nu(x) \partial_{x_j}$ ($\nu = 1, \dots, d$) are simultaneously transversal to the sphere $\|x\|_g = r$ if, the vectors $X^\nu := (X_1^\nu, \dots, X_n^\nu)$ ($\nu = 1, \dots, d$) satisfy*

$$\sum_{\nu=1}^d |\langle X^\nu, x \rangle_g| \neq 0, \quad \forall x, \quad \|x\|_g = r. \quad (2.7)$$

Theorem 2.4 *Let $r > 0$. Suppose that $\mathcal{B}_\nu := \sum_{j=1}^n (xA^\nu)_j \partial_{x_j}$ ($\nu = 1, \dots, d$) be a commuting system of semi-simple linear real vector fields in \mathbb{R}^n . We choose a real nonsingular matrix P such that $\Lambda^\nu = P^{-1}A^\nu P$ is a block diagonal matrix given by $\Lambda^\nu = \text{diag} \{R_2(\xi_1^\nu, \eta_1^\nu), \dots, R_2(\xi_{n_1}^\nu, \eta_{n_1}^\nu), \lambda_{n_1+1}^\nu, \dots, \lambda_n^\nu\}$ for some integer $n_1 \leq n$. Let g be a Riemannian metric defined by $P^t P$. Then the following conditions are equivalent.*

- (a) \mathcal{B}_ν ($\nu = 1, \dots, d$) are simultaneously transversal to the sphere $\|x\|_g = r$.
- (b) \mathcal{B}_ν ($\nu = 1, \dots, d$) satisfy a cone condition.
- (c) There exist real numbers c_ν ($\nu = 1, \dots, d$) such that $\sum_{\nu=1}^d c_\nu \mathcal{B}_\nu$ is transversal to the sphere $\|x\|_g = r$.

Proof. We note that $\langle x, y \rangle_g = \langle Px, Py \rangle$ and $\|x\|_g = \|Px\|$. By inserting the relation $A^\nu = P\Lambda^\nu P^{-1}$ into (2.7) we can easily see that the simultaneous transversality condition is equivalent to

$$\sum_{\nu=1}^d |\langle y \Lambda^\nu, y \rangle| \neq 0, \quad \forall y = (y_1, \dots, y_n), \quad \|y\| = 1. \quad (2.8)$$

By definition, (2.8) can be written in

$$\sum_{\nu=1}^d \left| \sum_{j=1}^{n_1} \xi_j^\nu (y_{2j-1}^2 + y_{2j}^2) + \sum_{j=n_1+1}^n y_j^2 \lambda_j^\nu \right| \neq 0, \quad \forall y, \quad \|y\| = 1. \quad (2.9)$$

We define $t = (t_1, \dots, t_n)$, $t \in \mathbb{R}_+^n$, $|t| = 1$ by $t_j = (y_{2j-1}^2 + y_{2j}^2)/2$ if $j \leq n_1$ and $t_j = y_j^2$ if $j > 2n$. Noting that $\xi^n u_j (y_{2j-1}^2 + y_{2j}^2) = 2t_j \xi_j^\nu = t_j (\xi_j^\nu + i\eta_j^\nu + \xi_j^\nu - i\eta_j^\nu)$ we see that (2.9) is written in $\sum_{\nu=1}^d |\sum_{j=1}^n t_j \lambda_j^\nu| \neq 0$ for every $t \in \mathbb{R}_+^n$ and $|t| = 1$. This is equivalent to the cone condition by definition. Hence we have proved the equivalence of (a) and (b).

By Proposition 2.1 the condition (b) is equivalent to the existence of real numbers c_ν ($\nu = 1, \dots, d$) such that $\sum_{\nu=1}^d c_\nu \mathcal{B}_\nu$ is a Poincaré vector field. By what we have proved in the above ($d = 1$) this is equivalent to say that $\sum_{\nu=1}^d c_\nu \mathcal{B}_\nu$ is transversal to the sphere $\|x\|_g = r$. Hence we have proved Theorem 2.3.

3 Divergent solutions of overdetermined systems of linearized homological equations

We now study the action ρ_{in} which is in a Siegel domain and admits a Jordan block. We assume that the action is formally (simultaneously) linearizable and that does not satisfy the cone condition. We shall show that the unique formal solution of a linearized homological equation diverges.

Let $\mathbb{C}_2^n\{x\}$ be the set of n vector functions of convergent power series of x without constant and linear terms. We examine the system of the linearized homology equation

$$L_A v = (L_1 v, \dots, L_d v) = f, \quad f := (f_1, \dots, f_d) \in (\mathbb{C}_2^n\{x\})^d, \quad (3.1)$$

where

$$L_j v = \langle A_j x, \partial_x \rangle v - A_j v, \quad j = 1, \dots, d,$$

under the compatibility conditions

$$L_j f_k = L_k f_j, \quad j, k = 1, \dots, d. \quad (3.2)$$

First we consider a 2- \mathbb{C} action studied in Example 1.4. We assume that there exists a vector field in the two-dimensional Lie algebra which is not semisimple. In view of Example 1.4 we can choose a base X_1, X_2 with linear parts $A_j \in GL(4; \mathbb{C})$ satisfying $\text{spec}(A_1) = \{1, 1, \nu, \nu\}$ and $\text{spec}(A_2) = \{0, 1, \mu, \mu\}$, respectively, where $\nu \leq \mu \leq 0$, $(\nu, \mu) \notin \mathbb{Q} \times \mathbb{Q}$, and

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \nu & \varepsilon \\ 0 & 0 & 0 & \nu \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & \varepsilon_0 \varepsilon \\ 0 & 0 & 0 & \mu \end{pmatrix}, \quad (3.3)$$

where $\varepsilon \neq 0$ and $\varepsilon_0 \in \mathbb{C}$. We can make $|\varepsilon| > 0$ arbitrarily small by an appropriate linear change of variables.

Let $\omega(\alpha)$ be defined by (1.16). We say that the simultaneous Diophantine order of $\{\text{spec}(A_1), \text{spec}(A_2)\}$ is τ_0 , if, for every $\tau > \tau_0$ there exists $C = C_\tau > 0$ such that

$$\omega(\alpha) \geq C|\alpha|^{-\tau}, \quad \forall \alpha \in \mathbb{Z}_+^4(2), \quad (3.4)$$

while, for every $\tau < \tau_0$ there exists a subsequence $\alpha_\ell \in \mathbb{Z}_+^4(2)$ ($\ell = 1, 2, \dots$) such that

$$\omega(\alpha) \leq |\alpha_\ell|^{-\tau}, \quad \ell \in \mathbb{N}. \quad (3.5)$$

By a classical result in the number theory we have $\tau_0 \geq 1/2$ (cf. [26] or [22]). We say that α and β are simultaneously Liouville, if (3.5) holds for every $\tau > 0$.

Let $\sigma \geq 1$. We say that a formal power series $f(x) = \sum_\alpha f_\alpha x^\alpha$ is in a Gevrey space $G_2^\sigma(\mathbb{C}^4)$ if $f_\alpha = 0$ for $|\alpha| \leq 1$ and, there exist $C > 0$ and $R > 0$ such that

$$|f_\alpha| \leq CR^{|\alpha|} |\alpha|!^\sigma, \quad \forall \alpha \in \mathbb{Z}_+^4.$$

We consider the following equation

$$L_A v := (L_1 v, L_2 v) = f, \quad f = (f_1, f_2) \in (\mathbb{C}_2^4\{x\})^2, \quad x \in \mathbb{C}^4, \quad (3.6)$$

where (f_1, f_2) satisfies the compatibility condition $L_1 f_2 = L_2 f_1$. Then we have

Theorem 3.1 *Assume that $\nu < \mu \leq 0$ and $\varepsilon_0 \neq 0$. Then there exists $f = (f_1, f_2) \in (\mathbb{C}_2^4\{x\})^2$ such that $L_1 f_2 = L_2 f_1$ and the equation (3.6) has a formal power series solution $v \notin \bigcup_{1 \leq \sigma < 5/2} G_2^\sigma(\mathbb{C}^4)$.*

Furthermore, suppose that $(\nu, \mu) \notin \mathbb{Q} \times \mathbb{Q}$, (3.4), (3.5) and $\tau_0 < +\infty$ hold. Then (3.6) has a unique solution $v \in \bigcap_{\sigma > 3+2\tau_0} G_2^\sigma(\mathbb{C}^4)$ for every $(f_1, f_2) \in (\mathbb{C}_2^4\{x\})^2$ satisfying $L_1 f_2 = L_2 f_1$.

The proof of this theorem follows from the following propositions.

Proposition 3.2 *Suppose that $\nu < \mu \leq 0$, $\varepsilon_0 \neq 0$, and $(\nu, \mu) \notin \mathbb{Q} \times \mathbb{Q}$. Then there exist real numbers c_1 and c_2 such that for any $g \in \mathbb{C}_2^4\{x\}$ there exist $f_j \in \mathbb{C}_2^4\{x\}$ ($j = 1, 2$) such that $L_1 f_2 = L_2 f_1$ and $g = \sum_{j=1}^2 c_j f_j$. Moreover, $B := c_1 A_1 + c_2 A_2$ is nonresonant, and ω defined by (1.16) for $\text{spec}(B)$ satisfies (3.5).*

Proposition 3.3 *Suppose that $\nu < \mu \leq 0$, $\varepsilon_0 \neq 0$ and $(\nu, \mu) \notin \mathbb{Q} \times \mathbb{Q}$. Let c_1 and c_2 be the numbers given by Proposition 3.2. Then there exists $g \in \mathbb{C}_2^4\{x\}$ such that the homology equation $L_B v = g$ with $B = c_1 A_1 + c_2 A_2$ has a unique formal power series solution v which is not contained in $\bigcup_{1 \leq \sigma < 5/2} G_2^\sigma(\mathbb{C}^4)$.*

Remark 3.4 *Our divergence results imply in the case of a single holomorphic vector field, that generically vector fields obtained by nonlinear holomorphic perturbations are nonlinearizable (see R. Pérez Marco [25] and [20] for more details). As to the case of smooth C^∞ hyperbolic \mathbb{R}^2 actions we refer [10]. We point out that the divergence in Gevrey classes of formal solutions of overdetermined systems of linear homological equations generalize those for the single vector fields in the presence of nontrivial Jordan blocks (see [16], [18]).*

First we will show Theorem 3.1, assuming Propositions 3.2 and 3.3.

Proof of Theorem 3.1. We will prove the former half. By the result of Example 1.4 of the case (a), we know that if $(\nu, \mu) \in \mathbb{Q} \times \mathbb{Q}$, then (3.6) has an infinite resonance. It follows that (3.6) with $f = 0$ has a formal power series solution $v \notin \bigcup_{1 \leq \sigma < 5/2} G_2^\sigma(\mathbb{C}^4)$, because (3.6) is a linear equation. Next we assume $(\nu, \mu) \notin \mathbb{Q} \times \mathbb{Q}$. By Proposition 3.3 we can choose $g \in \mathbb{C}_2^4\{x\}$ such that the unique solution v of $L_B v = g$ with $B = c_1 A_1 + c_2 A_2$ is not contained in $\bigcup_{1 \leq \sigma < 5/2} G_2^\sigma(\mathbb{C}^4)$. By Proposition 3.2 we choose $f_j \in \mathbb{C}_2^4\{x\}$ ($j = 1, 2$) such that $L_1 f_2 = L_2 f_1$ and $g = \sum_{j=1}^2 c_j f_j$. Because the solution v of the system of equations $L_A v = f$ is a unique solution of a single equation $L_B v = g$ we see that v is not contained in $\bigcup_{1 \leq \sigma < 5/2} G_2^\sigma(\mathbb{C}^4)$. This proves the former half of the theorem.

We will prove the latter half. We consider the system of equations $L_j v = f_j$ ($j = 1, 2$), where $L_1 f_2 = L_2 f_1$. For the sake of simplicity, we assume that B is put in a Jordan normal form with the diagonal part $B^0 := \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_3\}$. The off diagonal element of B is denoted by ε_1 . The homology operator corresponding to B is given by

$$L_B v = \langle B^0 x, \partial_x \rangle v + \varepsilon_1 R[v] - Bv, \quad v \in \mathbb{C}_2^4\{x\}, \quad (3.7)$$

$$\langle B^0 x, \partial_x \rangle v = \sum_{|\alpha| \geq 2} (\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (\alpha_3 + \alpha_4)) v_\alpha x^\alpha, \quad (3.8)$$

where $v(x) = \sum_{|\alpha| \geq 2} v_\alpha x^\alpha$ and

$$R[v] = \sum_{|\alpha| \geq 2} (\alpha_3 + 1) v_{(\alpha_1, \alpha_2, \alpha_3+1, \alpha_4-1)} x^\alpha. \quad (3.9)$$

For $g(x) = (g_1, g_2, g_3, g_4)^t \in \mathbb{C}_2^4\{x\}$ we expand $g_k(x)$ in the Taylor series $g_k(x) = \sum_\alpha g_{\alpha; k} x^\alpha$. For nonnegative integers N , α_1 and α_2 we define V_k and

G_k by

$$V_k := \{v_{(\alpha_1, \alpha_2, N-\ell, \ell); k}\}_{\ell=0}^N, \quad G_k := \{g_{(\alpha_1, \alpha_2, N-\ell, \ell); k}\}_{\ell=0}^N \quad (k = 1, 2, 3, 4).$$

In view of (3.9), the equation $L_B v = g$ is equivalent to

$$(\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 N - \lambda_1) V_1 + \varepsilon_1 \mathcal{M}_N V_1 = G_1, \quad (3.10)$$

$$(\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 N - \lambda_2) V_2 + \varepsilon_1 \mathcal{M}_N V_2 = G_2, \quad (3.11)$$

$$(\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (N-1)) V_3 + \varepsilon_1 \mathcal{M}_N V_3 = G_3 + \varepsilon_1 V_4, \quad (3.12)$$

$$(\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (N-1)) V_4 + \varepsilon_1 \mathcal{M}_N V_4 = G_4, \quad (3.13)$$

where \mathcal{M}_N is given by

$$\mathcal{M}_N = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ N & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & N-1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & N-2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \quad N \geq 1, \quad (3.14)$$

and $\mathcal{M}_0 = 0$. We note that, for the equation $L_1 v = f_1$ we have $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \nu$, $\varepsilon_1 = \varepsilon$, while for $L_2 v = f_2$ we have $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = \mu$, $\varepsilon_1 = \varepsilon_0 \varepsilon$.

Let $f^j(x) = (f_1^j(x), \dots, f_4^j(x))$ and let $f_k^j(x) = \sum_{\alpha} f_{\alpha; k}^j x^{\alpha}$ ($j = 1, 2; k = 1, \dots, 4$) be the Taylor expansion of $f_k^j(x)$. We substitute the expansions of v and f^j into the equations $L_j v = f^j$. For every $(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and $N \in \mathbb{Z}_+$ such that $\alpha_1 + \alpha_2 + N \geq 2$ we compare the coefficients of x^{α} ($\alpha_3 + \alpha_4 = N$) with homogeneous degree $\alpha_1 + \alpha_2 + N$. If we set

$$F^j = (F_1^j, \dots, F_4^j), \quad F_k^j = \{f_{(\alpha_1, \alpha_2, N-r, r); k}^j\}_{r=0}^N, \quad j = 1, 2; k = 1, \dots, 4, \quad (3.15)$$

and $V = (V_1, \dots, V_4)$, $V_k := \{v_{(\alpha_1, \alpha_2, N-\ell, \ell); k}\}_{\ell=0}^N$ ($k = 1, 2, 3, 4$), $H = (0, 0, V_4, 0)$, then we can write the system of equations $L_j v = f^j$ ($j = 1, 2$) in the following form

$$\mathcal{A}V = F^1 + \varepsilon H, \quad \mathcal{B}V = F^2 + \varepsilon \varepsilon_0 H, \quad (3.16)$$

where the matrices \mathcal{A} and \mathcal{B} are the block diagonal matrices given by

$$\mathcal{A} := \text{diag}\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} = \text{diag} \begin{pmatrix} (\alpha_1 + \alpha_2 + \nu N - 1)Id + \varepsilon M_N \\ (\alpha_1 + \alpha_2 + \nu N - 1)Id + \varepsilon M_N \\ (\alpha_1 + \alpha_2 + \nu N - \nu)Id + \varepsilon M_N \\ (\alpha_1 + \alpha_2 + \nu N - \nu)Id + \varepsilon M_N \end{pmatrix}, \quad (3.17)$$

$$\mathcal{B} := \text{diag}\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\} = \text{diag} \begin{pmatrix} (\alpha_2 + \mu N)Id + \varepsilon_0 \varepsilon M_N \\ (\alpha_2 + \mu N - 1)Id + \varepsilon_0 \varepsilon M_N \\ (\alpha_2 + \mu N - \mu)Id + \varepsilon_0 \varepsilon M_N \\ (\alpha_2 + \mu N - \mu)Id + \varepsilon_0 \varepsilon M_N \end{pmatrix}. \quad (3.18)$$

We will construct a formal power series solution V from (3.16). Because $(\nu, \mu) \notin \mathbb{Q} \times \mathbb{Q}$ either ν or μ is an irrational number. Suppose that ν is an irrational number. We want to show that for each $k = 1, \dots, 4$ either \mathcal{A}_k or \mathcal{B}_k is nonsingular. In order to see this, suppose that $|\alpha| = \alpha_1 + \alpha_2 + N \geq 2$. If $N \neq 0, 1$, then by the irrationality of ν , the matrices \mathcal{A}_k ($k = 1, \dots, 4$) are nonsingular. If $N = 0$ or $N = 1$, then by the condition $\alpha_1 + \alpha_2 + N \geq 2$, \mathcal{A}_k ($k = 1, \dots, 4$) are nonsingular. Similarly, if μ is an irrational number, then we can show that either \mathcal{A}_k or \mathcal{B}_k is nonsingular for each $k = 1, \dots, 4$.

First we will determine V_4 . By inductive arguments and $L_1 f^2 = L_2 f^1$ we get

$$\begin{aligned} v_{(\alpha_1, \alpha_2, N-\ell, \ell); 4} &= \sum_{r=0}^{\ell} \frac{(-\varepsilon_1)^r}{(\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (N-1))^{r+1}} \\ &\times \frac{(N-\ell+r)!}{(N-\ell)!} g_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r); 4} \end{aligned} \quad (3.19)$$

for $\ell = 0, 1, \dots, N$, provided $\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (N-1) \neq 0$. Note that, if \mathcal{A}_4 is nonsingular, then (3.19) is valid for $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \nu$, $\varepsilon_1 = \varepsilon$, $g_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r); 4} = f_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r); 4}^1$, while if \mathcal{B}_4 is nonsingular, then (3.19) is valid for $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = \mu$, $\varepsilon_1 = \varepsilon_0 \varepsilon$, $g_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r); 4} = f_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r); 4}^2$. Similar explicit formulas are derived for $v_{\alpha_1, \alpha_2, N-\ell, \ell; k}$, $k = 1, 2$. As to the term $v_{(\alpha_1, \alpha_2, N-\ell, \ell); 3}$, there appears the term $\varepsilon_1 V_4^N$ in the right-hand side of (3.12).

By (3.4) we have

$$|\alpha_1 + \alpha_2 + \nu N - \nu| + |\alpha_2 + \mu N - \mu| \geq C |\alpha_1 + \alpha_2 + N|^{-\tau} \quad (3.20)$$

for some $C > 0$. It follows that either $|\alpha_1 + \alpha_2 + \nu N - \nu| \geq C|\alpha_1 + \alpha_2 + N|^{-\tau}/2$ or $|\alpha_2 + \mu N - \mu| \geq C|\alpha_1 + \alpha_2 + N|^{-\tau}/2$ holds. Suppose that the former estimate holds. We have the same estimate in case the latter inequality holds. Let τ be such that $\tau > \tau_0$. Then we have

$$\begin{aligned} |\alpha_1 + \alpha_2 + \nu(N-1)|^{r+1} &\geq (C/2)^{r+1} |\alpha_1 + \alpha_2 + N|^{-\tau(r+1)} \\ &\geq |\alpha_1 + \alpha_2 + N|^{-\tau(N+1)}. \end{aligned} \quad (3.21)$$

Noting that $(N - \ell + r)! / (N - \ell)! \leq N!$, we see from (3.19) that if $g_{(\alpha_1, \alpha_2, N - \ell + r, \ell - r); 4}$ has a G^s estimate, then $v_{(\alpha_1, \alpha_2, N - \ell, \ell); 4}$ has a $G^{s+\tau+1}$ estimate. Especially, $v_{(\alpha_1, \alpha_2, N - \ell, \ell); 4}$ has the $G^{\tau+2}$ estimate. Similarly, we can easily see that $v_{(\alpha_1, \alpha_2, N - \ell, \ell); j}$ ($j = 1, 2, 4$) have the $G^{\tau+2}$ estimate. Next we determine $v_{(\alpha_1, \alpha_2, N - \ell, \ell); 3}$ by a similar relation like (3.19). We can easily see that there appears $v_{(\alpha_1, \alpha_2, N - \ell, \ell); 4}$ in the right-hand side of the recurrence relation. Hence the right-hand side has the $G^{\tau+2}$ estimate. It follows that $v_{(\alpha_1, \alpha_2, N - \ell, \ell); 3}$ has the $G^{2\tau+3}$ estimate. Since $\tau > \tau_0$ is arbitrary, $v_{(\alpha_1, \alpha_2, N - \ell, \ell); 3}$ has the G^σ estimate for $\sigma > 3 + 2\tau_0$. This ends the proof of Theorem 3.1.

Proof of Proposition 3.2. The eigenvalues of $B := c_1 A_1 + c_2 A_2$ is given by $c_1, c_1 + c_2, c_1 \nu + c_2 \mu$ with multiplicity. We shall show that there exists a set $E \subset \mathbb{R}^2$ with Lebesgue measure zero such that if $(c_1, c_2) \notin E$, then B is non-resonant. For every $\alpha = (\alpha_1, \dots, \alpha_4) \in \mathbb{Z}_+^4$, $|\alpha| \geq 2$, the resonance relations are given by

$$c_1 \alpha_1 + (c_1 + c_2) \alpha_2 + (c_1 \nu + c_2 \mu) \alpha_3 = c_1, \quad (3.22)$$

and the ones with c_1 in the right-hand side replaced by $c_1 + c_2$ and $c_1 \nu + c_2 \mu$, respectively. Because the argument is similar, we consider the first relation. It follows from (3.22) that

$$c_1(\alpha_1 + \alpha_2 + \nu(\alpha_3 + \alpha_4) - 1) + c_2(\alpha_2 + \mu(\alpha_3 + \alpha_4)) = 0.$$

Because $(\nu, \mu) \notin \mathbb{Q} \times \mathbb{Q}$ and $|\alpha| \geq 2$, we can easily see that either $\alpha_1 + \alpha_2 + \nu(\alpha_3 + \alpha_4) - 1 \neq 0$ or $\alpha_2 + \mu(\alpha_3 + \alpha_4) \neq 0$ holds. Hence the set of $(c_1, c_2) \in \mathbb{R}^2$ satisfying (3.22) is a straight line. Therefore the set E of all (c_1, c_2) satisfying a resonance relations has Lebesgue measure zero.

In order to see that $\text{spec}(B)$ satisfies (3.5), let $\tilde{\omega}_j(\alpha)$ ($\alpha \in \mathbb{Z}_+^4$) be defined by (1.15) for B . Then there exists $K > 0$ such that $\tilde{\omega}_j(\alpha) \leq K \omega_j(\alpha)$ for $j = 1, \dots, 4$ and all $\alpha \in \mathbb{Z}_+^4$. It follows that (3.5) holds for $\tilde{\omega}_j(\alpha)$.

Let $(c_1, c_2) \notin E$ and $g \in \mathbb{C}_2^4$ be given. We want to solve the system of equations

$$L_1 f^2 = L_2 f^1, \quad c_1 f^1 + c_2 f^2 = g. \quad (3.23)$$

By expanding $f^j(x) = (f_1^j, \dots, f_4^j)$ into the Taylor series we define F^j by (3.15). We similarly define $G = (G_1, \dots, G_4)$, $G_k = \{g_{(\alpha_1, \alpha_2, N-r, r); k}\}_{r=0}^N$, where $g(x) = (g_1, \dots, g_4)$, $g_k(x) = \sum_{\alpha} g_{\alpha; k} x^{\alpha}$. We set $H^1 := (0, 0, F_4^1, 0)$ and $H^2 := (0, 0, F_4^2, 0)$. We substitute the expansions of f^j and g into (3.23). For every $(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and $N \in \mathbb{Z}_+$ such that $\alpha_1 + \alpha_2 + N \geq 2$ we compare the coefficients of x^{α} of homogeneous degree $\alpha_1 + \alpha_2 + N$. Then we can write (3.23) in the following form

$$\mathcal{A}F^2 - \mathcal{B}F^1 + \varepsilon H^2 - \varepsilon \varepsilon_0 H^1 = 0, \quad c_1 F^1 + c_2 F^2 = G, \quad (3.24)$$

where \mathcal{A} and \mathcal{B} are given by (3.17) and (3.18).

First we will construct a formal power series solution F^j ($j = 1, 2$) of (3.24) for a given G . Because we know that (cf. the proof of Theorem 3.1) either \mathcal{A}_k or \mathcal{B}_k is nonsingular for each $k = 1, \dots, 4$, it follows from (3.24) that

$$\mathcal{A}_k F_k^2 - \mathcal{B}_k F_k^1 = 0, \quad c_1 F_k^1 + c_2 F_k^2 = G_k, \quad k = 1, 2, 4.$$

Assuming that \mathcal{A}_k is nonsingular we obtain $F_k^2 = \mathcal{A}_k^{-1} \mathcal{B}_k F_k^1$, and hence $c_1 F_k^1 + c_2 \mathcal{A}_k^{-1} \mathcal{B}_k F_k^1 = G_k$. It follows that

$$F_k^1 = (c_1 + c_2 \mathcal{A}_k^{-1} \mathcal{B}_k)^{-1} G_k = (c_1 \mathcal{A}_k + c_2 \mathcal{B}_k)^{-1} \mathcal{A}_k G_k, \quad (3.25)$$

if $c_1 \mathcal{A}_k + c_2 \mathcal{B}_k$ is nonsingular. The last condition holds if (c_1, c_2) is not contained in a set of Lebesgue measure zero in \mathbb{R}^2 , which may depend on α_1, α_2, N . We have similar relations if \mathcal{B}_k is nonsingular.

In case $k = 3$, we obtain $\mathcal{A}_3 F_3^2 - \mathcal{B}_3 F_3^1 = -\varepsilon(F_4^2 - \varepsilon_0 F_4^1)$ instead of $\mathcal{A}_k F_k^2 - \mathcal{B}_k F_k^1 = 0$. A simple computation yields that

$$F_3^1 = (c_1 \mathcal{A}_3 + c_2 \mathcal{B}_3)^{-1} \mathcal{A}_3 G_3 + \varepsilon c_2 (c_1 \mathcal{A}_3 + c_2 \mathcal{B}_3)^{-1} (F_4^2 - \varepsilon_0 F_4^1).$$

By taking the union of all exceptional sets of (c_1, c_2) with α_1, α_2 and N in the set of nonnegative integers such that $\alpha_1 + \alpha_2 + N \geq 2$, we see that there exists a unique formal power series solution $f^j(x)$ ($j = 1, 2$) of (3.23), provided (c_1, c_2) is not in an exceptional set of Lebesgue measure zero.

We will show the convergence of $f^j(x)$ ($j = 1, 2$). It is sufficient to show the convergence of $f^1(x)$. Let $c_1 \mathcal{A}_k^0 + c_2 \mathcal{B}_k^0$ be the diagonal part of

$c_1\mathcal{A}_k + c_2\mathcal{B}_k$, ($k = 1, \dots, 4$). In view of (3.25) we will show the following estimate: if (c_1, c_2) is not contained in a set of Lebesgue measure zero such that $c_1 < 0, c_2 < 0$, then there exist K_1 and $K_2 > 0$ such that

$$\begin{aligned} \|c_1\mathcal{A}_k^0 + c_2\mathcal{B}_k^0\| &\geq K_1 + K_2(\alpha_1 + \alpha_2 + N), \\ \forall(\alpha_1, \alpha_2, N) &\in \mathbb{Z}_+^3(2), k = 1, 2, 3, 4. \end{aligned} \quad (3.26)$$

Since the proof is similar we consider the case $k = 1$. we have

$$c_1\mathcal{A}_1^0 + c_2\mathcal{B}_1^0 = (c_1\alpha_1 - c_1 + (c_1 + c_2)\alpha_2 + N(c_1\nu + c_2\mu)) Id. \quad (3.27)$$

Clearly the right-hand side has the estimate (3.26). If we have the Poincaré type estimate (3.26), then it is easy to see that the formal solution $f^1(x)$ converges in some neighborhood of the origin. This ends the proof.

Proof of Proposition 3.3 : Let g be the convergent power series defined by $g_{\alpha;k} = 0$ for $k = 1, 2, 3$ and all $\alpha \in \mathbb{Z}_+^4(2)$; $g_{\alpha_1, \alpha_2, \alpha_3, \alpha_4; 4} = 0$ if $\alpha_4 \geq 1$; $g_{(\alpha_1, \alpha_2, N, 0); 4} = 1$ for $(\alpha_1, \alpha_2, N, 0) \in \mathbb{Z}_+^4(2)$. We want to solve $L_B v = g$. Let λ_j be the eigenvalues of B . By the same argument as in the proof of Theorem 3.1 we have the formula (3.19). Then we have

$$v_{(\alpha_1, \alpha_2, 0, N); 4} = (-\varepsilon_1)^N (\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3(N-1))^{-N-1} N!, \quad (3.28)$$

for all $(\alpha_1, \alpha_2, N) \in \mathbb{Z}_+^3(2)$. The condition (3.5) implies that, for every $\tau < \tau_0$ we can find a constant $C > 0$ and subsequence $\{(\alpha_{1,k}, \alpha_{2,k}, N_k)\}_{k=1}^\infty$ such that

$$|(\lambda_1\alpha_{1,k} + \lambda_2\alpha_{2,k} + \lambda_3(N_k - 1))^{-1}| \geq CN_k^\tau, \quad \forall k \in \mathbb{N}.$$

Therefore, by (3.28)

$$|v_{(\alpha_{1,k}, \alpha_{2,k}, 0, N_k); 4}| \geq (C|\varepsilon_1|)^{N_k} N_k^{N_k\tau} N_k!, \quad k \in \mathbb{N}, \alpha_1 \in \mathbb{Z}_+(2). \quad (3.29)$$

Because $\varepsilon_1 \neq 0$, $\tau < \tau_0$ and $1/2 \leq \tau_0$, (3.29) and Stirling's formula, $N! \geq C^N N^N, \forall N \in \mathbb{Z}_+$ lead to the assertion. This ends the proof of Proposition 3.3.

We will briefly mention the general case $d - actions$. We suppose that there exist j , $1 \leq j \leq m$ and ℓ_0 , $1 \leq \ell_0 \leq d$ such that $A_j^{\ell_0}$ in (1.7) admits only one dimensional eigenspace, i.e., the geometric multiplicity of $\lambda_j^{\ell_0}$ is one.

For a positive integer r we define the r square nilpotent matrix N_r by

$$N_r = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 \dots & 0 & 1 \\ 0 & 0 & 0 \dots & 0 & 0 \end{pmatrix}. \quad (3.30)$$

By assumption we have

$$A_j^{\ell_0} = \lambda_j^{\ell_0} Id + \varepsilon N_{s_j}, \quad \varepsilon \neq 0. \quad (3.31)$$

By the explicit description of the centralizers of matrices (cf. [15]) all other matrices have the following form

$$A_j^\ell = \lambda_j^\ell Id + \sum_{k=1}^{s_j-1} \varepsilon_k^{\ell_j} (N_{s_j})^k \quad \varepsilon_k^{\ell_j} \in \mathbb{C}, \quad k = 1, \dots, s_j - 1. \quad (3.32)$$

We also suppose that there exists a sequence $\alpha^\ell \in \mathbb{Z}_+^m(2)$, $\ell \in \mathbb{N}$ and a positive number c_0 such that $|\alpha^\ell| \rightarrow \infty$ ($\ell \rightarrow \infty$) and

$$0 < \omega(\alpha^\ell) \leq c_0, \quad \ell \in \mathbb{N}. \quad (3.33)$$

We have

Theorem 3.5 *Assume that (3.31) and (3.33) hold. Moreover, suppose that simultaneously nonresonant condition (1.17) is satisfied. Then, one can find an $(f_1, \dots, f_d) \in (\mathbb{C}_2^n\{x\})^d$ satisfying (3.2) such that $v = L_A^{-1}f$ is not contained in the set $\bigcup_{1 \leq \sigma < 2} G_2^\sigma(\mathbb{C}^n)$.*

4 Sternberg's theorem for commuting vector fields

The results in section 2 imply that, if a cone condition holds, then simultaneous linearization of a system with a Jordan block is reduced essentially to the Poincaré–Dulac theorem for a single vector field in an analytic category. On the other hand, in view of the results in section 3, the reduction seems impossible if the cone condition is violated.

In this section we shall illustrate that the situation is completely different in a smooth category. We consider two commuting vector field in \mathbb{R}^4 which are in a Siegle domain and only one of the two has a linear part with nontrivial Jordan block. Obviously, the system does not satisfy a cone condition. We will show that they are simultaneously linearizable in C^k for every $k \geq 1$.

Let $X(y)$ and $Y(y)$ be commuting C^∞ vector fields with the common singular point at the origin $0 \in \mathbb{R}^4$. Suppose that $\nabla X(0) = A$, $\nabla Y(0) = B$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\nu & 0 \\ 0 & 0 & 0 & -\nu \end{pmatrix}, \quad (4.34)$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mu & \varepsilon \\ 0 & 0 & 0 & -\mu \end{pmatrix}, \quad \varepsilon \neq 0. \quad (4.35)$$

We assume that the cone condition is violated, namely, (cf. Example 4.1)

$$\nu > \mu > 0, \nu \in \mathbb{R} \setminus \mathbb{Q}. \quad (4.36)$$

We also note that the irrationality of ν implies that X , and hence the pair (X, Y) is nonresonant. Then we have

Theorem 4.1 *Suppose that the conditions (4.34), (4.35) and (4.36) are verified. Let $m \geq 1$ be an integer. Then there exists a C^m change of the variables $y = u(x) = x + v(x)$, $v(0) = 0$, $\nabla v(0) = 0$ near the origin which transforms both X and Y to their linear parts.*

We need to prepare lemmas in order to prove our theorem. In view of Sternberg's theorem we assume, without loss of generality, that X is linear, i.e.

$$Xv(y) = \langle \nabla v(y), Ay \rangle. \quad (4.37)$$

Let $R(y) = (R_1(y), R_2(y), R_3(y), R_4(y))$ be the nonlinear part of Y

$$Yf(y) = \langle \nabla f(y), By + R(y) \rangle. \quad (4.38)$$

Suppose that the change of variables $y = u(x) = x + v(x)$, $v(0) = 0$, $\nabla v(0) = 0$ linearizes both the vector fields X and Y . Then we can easily see that $v(x)$ satisfies the system of homology equations

$$\langle \nabla v(x), Bx \rangle - Bv = R(x + v(x)), \quad (4.39)$$

and

$$\langle \nabla v(x), Ax \rangle - Av = 0. \quad (4.40)$$

We write $x = (x_1, x_2, x'')$ and $z = (z_1, z')$. Let $c_1 > 0$ and $0 < c_2 \leq 1$ be constants. Then we define

$$\Omega = \{x' = (x_2, x_3, x_4) = (x_2, x'') \in \mathbb{R}^3; |x_2|^\nu |x''| < c_1, |x''| < c_2\}, \quad (4.41)$$

$$\Omega_1 = \{x_1 \in \mathbb{R}; |x_1| < 1\} \times \Omega. \quad (4.42)$$

Then we have

Lemma 4.2 *Let $k = \infty$ or $k \geq 1$ be an integer. Let L be given by*

$$L = \sum_{j=1}^2 x_j \partial_{x_j} - \nu \sum_{k=3}^4 x_k \partial_{x_k}.$$

Then a C^k solution of

$$Lf(x) - f(x) = 0, \quad x = (x_1, x_2, x_3, x_4) \in \Omega_1, \quad (4.43)$$

(respectively,

$$Lw(x) + \nu w(x) = 0 \quad x = (x_1, x_2, x_3, x_4) \in \Omega_1) \quad (4.44)$$

is given by

$$f(x) = x_1 \varphi_\pm \left(\frac{x_2}{x_1}, x_3 |x_1|^\nu, x_4 |x_1|^\nu \right), \quad \text{for } \pm x_1 > 0, \quad (4.45)$$

or

$$f(x) = x_2 \varphi_\pm \left(\frac{x_2}{x_1}, x_3 |x_1|^\nu, x_4 |x_1|^\nu \right), \quad \text{for } \pm x_1 > 0, \quad (4.46)$$

(respectively, by

$$w(x) = |x_1|^{-\nu} \psi_\pm \left(\frac{x_2}{x_1}, x_3 |x_1|^\nu, x_4 |x_1|^\nu \right), \quad \text{for } \pm x_1 > 0, \quad (4.47)$$

where $\varphi_\pm(z) \in C^k(\Omega)$ (respectively $\psi_\pm(z) \in C^k(\Omega)$.)

Proof. We apply the method of characteristics to the equations (4.43) and (4.44). Then we obtain the representations (4.45) and (4.47) for some φ_{\pm} in $\pm x_1 > 0$. \square

By the commutativity we see that every component of $v = R(x) = (R_1, \dots, R_4)$ satisfies either (4.43) or (4.44). Hence, by Lemma 4.1 we have

$$R_j(x) = x_j \Psi_{\pm}^j\left(\frac{x_2}{x_1}, x_3|x_1|^{\nu}, x_4|x_1|^{\nu}\right), \quad \text{for } \pm x_1 > 0, \quad j = 1, 2, \quad (4.48)$$

$$R_j(x) = |x_1|^{-\nu} \Psi_{\pm}^j\left(\frac{x_2}{x_1}, x_3|x_1|^{\nu}, x_4|x_1|^{\nu}\right), \quad \text{for } \pm x_1 > 0, \quad j = 3, 4 \quad (4.49)$$

for some functions Ψ_{\pm}^j . In the following we will cut off $R_j(x)$ with a smooth function being identically equal to 1 in some neighborhood of the origin and with support contained in a small neighborhood of the origin, which we give in the proof of Theorem 4.1. For the sake of simplicity, we denote the modified $R_j(x)$ with the same letter. We set

$$z_1 = x_2/x_1, \quad z_2 = x_3|x_1|^{\nu}, \quad z_3 = x_4|x_1|^{\nu}. \quad (4.50)$$

For every $x_1 \neq 0$, we define $\Psi_{\pm}^j(z)$ by (4.48) and (4.49), namely

$$\Psi_{\pm}^j(z) = x_j^{-1} R_j(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3), \quad \text{for } \pm x_1 > 0, \quad j = 1, 2, \quad (4.51)$$

$$\Psi_{\pm}^j(z) = |x_1|^{\nu} R_j(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3), \quad \text{for } \pm x_1 > 0, \quad j = 3, 4. \quad (4.52)$$

We can easily see that $\Psi_{\pm}^j \in C^{\infty}(\mathbb{R}_z^3)$ ($j = 1, 2, 3, 4$).

By (4.40) and simple computations we see that every component of $v(x) = (v_1(x), \dots, v_4(x))$ satisfies either (4.43) or (4.44). It follows from Lemma 4.1 that every component of v has an expression

$$v_j(x) = x_j \varphi_{\pm}^j\left(\frac{x_2}{x_1}, x_3|x_1|^{\nu}, x_4|x_1|^{\nu}\right), \quad \text{for } \pm x_1 > 0, \quad j = 1, 2, \quad (4.53)$$

and

$$v_j(x) = |x_1|^{-\nu} \varphi_{\pm}^j\left(\frac{x_2}{x_1}, x_3|x_1|^{\nu}, x_4|x_1|^{\nu}\right), \quad \text{for } \pm x_1 > 0, \quad j = 3, 4, \quad (4.54)$$

for some φ_{\pm}^j .

We substitute the transformation (4.50) and (4.53), (4.54) into (4.39), and we rewrite (4.39) as an equation of z for the unknown functions $\varphi_{\pm}^j(z)$ with a parameter x_1 . Recalling that $v_j = x_j \varphi_{\pm}^j$ and $v_j = |x_1|^{-\nu} \varphi_{\pm}^j$ we obtain

$$x_2 \partial_{x_2} v_1 = x_1 z_1 \partial_{z_1} \varphi_{\pm}^1(z), \quad x_3 \partial_{x_3} v_1 = x_1 z_2 \partial_{z_2} \varphi_{\pm}^1(z), \quad (4.55)$$

$$x_4 \partial_{x_4} v_1 = x_1 z_3 \partial_{z_3} \varphi_{\pm}^1(z), \quad x_4 \partial_{x_3} v_1 = x_1 z_3 \partial_{z_2} \varphi_{\pm}^1(z), \quad (4.56)$$

and we have similar relations for $v_2 = x_2 \varphi_{\pm}^2(x)$ and $v_j = |x_1|^{-\nu} \varphi_{\pm}^j(x)$. In fact we have

$$\langle \nabla v_1(x), Bx \rangle = x_1 \mathcal{L} \varphi_{\pm}^1(z), \quad \text{for } \pm x_1 > 0, \quad (4.57)$$

$$\langle \nabla v_2(x), Bx \rangle - v_2(x) = x_2 \mathcal{L} \varphi_{\pm}^2(z), \quad \text{for } \pm x_1 > 0, \quad (4.58)$$

$$\langle \nabla v_j(x), Bx \rangle = |x_1|^{-\nu} \mathcal{L} \varphi_{\pm}^j(z), \quad \text{for } \pm x_1 > 0, \quad j = 3, 4, \quad (4.59)$$

where

$$\mathcal{L}f(z) = z_1 \partial_{z_1} f(z) - (\mu z_2 - \varepsilon z_3) \partial_{z_2} f(z) - \mu z_3 \partial_{z_3} f(z). \quad (4.60)$$

We define $\varphi_{\pm}(z) = {}^{tr}(\varphi_{\pm}^1(z), \varphi_{\pm}^2(z), \varphi_{\pm}^3(z), \varphi_{\pm}^4(z))$.

Lemma 4.3 *We have the expression*

$$R_j(x + v(x)) = x_j E_{\pm}^j(z, \varphi_{\pm}(z)), \quad \text{for } \pm x_1 > 0, \quad j = 1, 2, \quad (4.61)$$

where $E_{\pm}^j(z, w)$ is given by

$$E_{\pm}^j(z, w) = (1 + w_j) \Psi_{\pm}^j \left(z_1 \frac{1 + w_2}{1 + w_1}, (z_2 + w_3) |1 + w_1|^{\nu}, (z_3 + w_4) |1 + w_1|^{\nu} \right) \quad (4.62)$$

and

$$R_j(x + v(x)) = |x_1|^{-\nu} E_{\pm}^j(z, \varphi_{\pm}(z)) \quad \text{for } \pm x_1 > 0, \quad j = 3, 4, \quad (4.63)$$

with

$$E_{\pm}^j(z, w) = |1 + w_1|^{-\nu} \Psi_{\pm}^j \left(z_1 \frac{1 + w_2}{1 + w_1}, (z_2 + w_3) |1 + w_1|^{\nu}, (z_3 + w_4) |1 + w_1|^{\nu} \right). \quad (4.64)$$

Proof. We have

$$\begin{aligned}\frac{x_2 + v_2(x)}{x_1 + v_1(x)} &= \frac{x_2(1 + \varphi_{\pm}^2(z))}{x_1(1 + \varphi_{\pm}^1(z))} \\ &= \frac{x_2}{x_1} \frac{1 + \varphi_{\pm}^2(z)}{1 + \varphi_{\pm}^1(z)} = z_1 \frac{1 + \varphi_{\pm}^2(z)}{1 + \varphi_{\pm}^1(z)}.\end{aligned}\quad (4.65)$$

$$\begin{aligned}(x_3 + v_3(x))|x_1 + v_1|^{\nu} &= (x_3 + |x_1|^{-\nu} \varphi_{\pm}^3(z))|x_1|^{\nu} |1 + \varphi_{\pm}^1(z)|^{\nu} \\ &= (x_3 |x_1|^{\nu} + \varphi_{\pm}^3(z)) |1 + \varphi_{\pm}^1(z)|^{\nu} \\ &= (z_2 + \varphi_{\pm}^3(z)) |1 + \varphi_{\pm}^1(z)|^{\nu}.\end{aligned}\quad (4.66)$$

$$\begin{aligned}(x_4 + v_4(x))|x_1 + v_1|^{\nu} &= (x_4 + |x_1|^{-\nu} \varphi_{\pm}^4(z))|x_1|^{\nu} |1 + \varphi_{\pm}^1(z)|^{\nu} \\ &= (x_4 |x_1|^{\nu} + \varphi_{\pm}^4(z)) |1 + \varphi_{\pm}^1(z)|^{\nu} \\ &= (z_3 + \varphi_{\pm}^4(z)) |1 + \varphi_{\pm}^1(z)|^{\nu}.\end{aligned}\quad (4.67)$$

Hence, if $j = 1, 2$, we get

$$\begin{aligned}&R_j(x + v(x)) = (x_j + v_j(x)) \\ &\times \Psi_{\pm}^j\left(\frac{x_2 + v_2(x)}{x_1 + v_1(x)}, (x_3 + v_3(x))|x_1 + v_1|^{\nu}, (x_4 + v_4)|x_1 + v_1(x)|^{\nu}\right) \\ &= x_j(1 + \varphi_{\pm}^j) \Psi_{\pm}^j\left(z_1 \frac{1 + \varphi_{\pm}^2}{1 + \varphi_{\pm}^1}, (z_2 + \varphi_{\pm}^3) |1 + \varphi_{\pm}^1|^{\nu}, (z_3 + \varphi_{\pm}^4) |1 + \varphi_{\pm}^1|^{\nu}\right),\end{aligned}\quad (4.68)$$

which yields (4.61). Similarly, we can readily prove (4.63). The proof of the lemma is complete. \square

Now we are ready to write explicitly the reduction of the overdetermined system for v : $(X_A - A)v = 0$, $(X_b - B)v = R(x + v(x))$ into a 4×4 system of equations for $\varphi_{\pm}(z)$ in $z \in \Omega$ with a parameter x_1 . Then the new system of semilinear homological equations for φ_{\pm} is written as follows

$$(\mathcal{L} - \tilde{B})(\varphi_{\pm}) = E_{\pm}(z, \varphi_{\pm}(z)), \quad E_{\pm}(z, w) = (E_{\pm}^1(z, w), \dots, E_{\pm}^4(z, w)),\quad (4.69)$$

where $E_{\pm}^j(z, w)$ are given by (4.62) and (4.64) and

$$\tilde{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & \varepsilon \\ 0 & 0 & 0 & -\mu \end{pmatrix}.\quad (4.70)$$

We prepare a lemma.

Lemma 4.4 *Let $\nu > 0$ be an irrational number. Let $f(x)$ and $w(x)$ be smooth solutions of (4.43) and (4.44) in Ω_1 , respectively satisfying that*

$$f(0) = w(0) = 0 \quad (4.71)$$

$$\nabla f(0) = \nabla w(0) = 0. \quad (4.72)$$

We cut off $f(x)$ and $w(x)$ with a smooth function being identically equal to 1 in some neighborhood of the origin and with support contained in a small neighborhood of the origin. For the sake of simplicity we denote the modified functions with the same letter. Let $\varphi_{\pm}(z)$ and $\psi_{\pm}(z)$ be defined by (4.45), (4.46) and (4.47), respectively by the same way as (4.51) and (4.52). Then, for every $\alpha \in \mathbb{Z}_+^3$, we have

$$\partial_z^\alpha \Theta(z_1, 0) = 0, \quad \forall z = (z_1, 0) \in \Omega, \quad (4.73)$$

with $\Theta = \varphi_{\pm}$ and $\Theta = \psi_{\pm}$.

Proof. Because ν is an irrational number we can easily see, from (4.43) and (4.44) that every $f(x)$ and $w(x)$ satisfying (4.71) and (4.72) are flat at the origin, namely all derivatives $\partial_x^\alpha f(x)$, $\partial_x^\alpha w(x)$ ($\alpha \in \mathbb{N}^4$) vanish at the origin $x = 0$. Let $\Theta(z) = \varphi_{\pm}(z)$, and set $f(x) = x_1 \varphi_{\pm}(x_2/x_1, x_3|x_1|^\nu, x_4|x_1|^\nu)$, $x_1 \neq 0$. Then we have

$$\partial_x^{\alpha'} (x_1^{-1} f(x)) = \partial_x^{\alpha'} \varphi_{\pm}(x_2/x_1, x_3|x_1|^\nu, x_4|x_1|^\nu) \quad (4.74)$$

$$= x_1^{-\alpha_2} |x_1|^{\nu(\alpha_3 + \alpha_4)} \partial_{z_1}^{\alpha_2} \partial_{z_2}^{\alpha_3} \partial_{z_3}^{\alpha_4} \varphi_{\pm}(z) \Big|_{z_1=x_2/x_1, z_2=x_3|x_1|^\nu, z_3=x_4|x_1|^\nu}. \quad (4.75)$$

We let x tend to zero so as to satisfy $x_2/x_1 = z_1$, $z_2 = x_3|x_1|^\nu = 0$ and $z_3 = x_4|x_1|^\nu = 0$. Then we have

$$\partial_{z_1}^{\alpha_2} \partial_{z_2}^{\alpha_3} \partial_{z_3}^{\alpha_4} \varphi_{\pm}(z_1, 0, 0) = \lim_{x \rightarrow 0} x_1^{\alpha_2} |x_1|^{-\nu(\alpha_3 + \alpha_4)} \partial_x^{\alpha'} (x_1^{-1} f(x_1, x_2, 0, 0)) = 0, \quad (4.76)$$

because $f(x)$ is flat at the origin. The other cases will be proved similarly.

□

Remark. Let $\varphi_{\pm}(z) \in C^k(\Omega)$ be given. Assume that (4.73) is satisfied for $\Theta = \varphi_{\pm}$ up to some finite $|\alpha|$. Then the function $f(x)$ defined by (4.45) gives a finitely smooth solution of (4.43) if ν is an irrational number. Indeed, the finite smoothness at $x_1 = 0$ follows from the argument of Lemma 4.4.

In order to solve (4.69) we introduce a function space. Let $N \geq 1$ and $k \leq N$ be integers. Let $0 < c'_2 < c_2 \leq 1$ be a constant. Then we define

$$\begin{aligned} \|V\|_{k;N} &= \sup_{z \in \mathbb{R}^3, 0 < |z'| \leq c'_2} \sum_{|\alpha| \leq k} |z'|^{|\alpha|} |\partial_z^\alpha (|z'|^{-N} V(z))|, \\ |V(z)| &= \left(\sum_{j=1}^3 |V_j(z)|^2 \right)^{1/2}, \quad V(z) = (V_1(z), V_2(z), V_3(z)). \end{aligned} \quad (4.77)$$

The set of all C^k functions $V(z)$ such that $\|V\|_{k;N} < \infty$ is a Banach space $B_{k;N}$ with the norm $\|\cdot\|_{k;N}$. Then we have

Lemma 4.5 *i) For any integers $k \geq 0$ and $0 \leq \ell \leq N$, there exists a constant $C_{k,N} > 0$ such that*

$$\|u\|_{k;\ell} \leq C_{k,N} \|u\|_{k;N}, \quad \forall u \in B_{k;N}. \quad (4.78)$$

ii) For every $f, g \in B_{k;N}$ we have $fg \in B_{k;N}$ and there exists a constant $C_{k,N} > 0$ such that

$$\|fg\|_{k;N} \leq C_{k,N} \|f\|_{k;N} \|g\|_{k;N}, \quad \forall f, g \in B_{k;N}. \quad (4.79)$$

Proof. Because $|z'| \leq 1$, we have, for $|\alpha| \leq k$

$$\begin{aligned} &|z'|^{|\alpha|} \partial^\alpha (|z'|^{-\ell} u(z)) = |z'|^{|\alpha|} \partial^\alpha (|z'|^{N-\ell} |z'|^{-N} u(z)) \\ &= |z'|^{|\alpha|} \sum_{\beta+\gamma=\alpha} \partial^\beta |z'|^{N-\ell} \partial^\gamma (|z'|^{-N} u(z)) \leq C_1 \sup |z'|^{|\gamma|} |\partial^\gamma (|z'|^{-N} u(z))| \end{aligned}$$

for some $C_1 > 0$. This proves i).

In order to prove ii) we have, for $|\alpha| \leq k$

$$\begin{aligned} &|z'|^{|\alpha|} |\partial^\alpha (|z'|^{-N} fg)| \leq \sum_{\beta+\gamma=\alpha} |z'|^{|\beta|} |\partial^\beta (|z'|^{-N} f)| |z'|^{|\gamma|} |\partial^\gamma g| \\ &\leq C_2 \|f\|_{k;N} \|g\|_{k,0} \leq C_3 \|f\|_{k;N} \|g\|_{k;N}. \end{aligned} \quad (4.80)$$

Here $C_2 > 0$ and $C_3 > 0$ are constants. This proves ii). \square

Let C be given by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mu & \varepsilon \\ 0 & 0 & -\mu \end{pmatrix}. \quad (4.81)$$

Then we define the operator Q by

$$QV = - \int_0^\infty e^{-t\tilde{B}} V(e^{tC} z) dt, \quad V = (V_1, \dots, V_4) = (\varphi_\pm^1, \varphi_\pm^2, \varphi_\pm^3, \varphi_\pm^4). \quad (4.82)$$

We can easily see that $U = QV$ gives the solution of $(\mathcal{L} - \tilde{B})U = V$. Then we have

Lemma 4.6 *Let the integers k and N satisfy that $0 \leq k < N - \mu$ and $\mu(k + 1 - N) + k < 0$. Then there exists $C_{k,N}(\Omega) > 0$ such that*

$$\|QV\|_{k;N} \leq C_{k,N}(\Omega) \|V\|_{k;N}, \quad \forall V \in B_{k;N}. \quad (4.83)$$

Proof. First we note that

$$e^{tC} z = (e^t z_1, e^{-\mu t} z_2 + e^{-\mu t} \varepsilon t z_3, e^{-\mu t} z_3), \quad (4.84)$$

$$e^{-t\tilde{B}} V = (V_1, V_2, e^{\mu t} (V_3 - \varepsilon t V_4), e^{\mu t} V_4). \quad (4.85)$$

Hence we have

$$\begin{aligned} V(e^{tC} z) &= V(e^t z_1, e^{-\mu t} (z_2 + \varepsilon t z_3), e^{-\mu t} z_3) \\ &= e^{-\mu N t} ((z_2 + \varepsilon t z_3)^2 + z_3^2)^{N/2} \tilde{V}(e^{tC} z), \end{aligned} \quad (4.86)$$

where $\tilde{V}(\zeta) = V(\zeta)/|\zeta|^N$. It follows that the right-hand side integral of (4.82) converges, because the growing term $e^{\mu t}$ in $e^{-t\tilde{B}}$ can be absorbed by $e^{-\mu N t}$, ($\mu > 0$). First we consider the case $k = 0$. By (4.85) and (4.86) we have

$$\begin{aligned} \|QV\|_{0;N} &= \sup_{z \in \mathbb{R}^3, 0 < |z'| \leq c'_2} \left(\frac{1}{|z'|^N} \int_0^\infty |e^{-t\tilde{B}} V(e^{tC} z)| dt \right) \\ &\leq \sup_{z \in \mathbb{R}^3, 0 < |z'| \leq c'_2} \left(\frac{1}{|z'|^N} \int_0^\infty (1 + |\varepsilon|t) e^{\mu t} |V(e^{tC} z)| dt \right) \\ &\leq \sup \left(\frac{1}{|z'|^N} \int_0^\infty (1 + |\varepsilon|t) e^{\mu(1-N)t} ((z_2 + |\varepsilon|t z_3)^2 + z_3^2)^{N/2} |\tilde{V}(e^{tC} z)| dt \right). \end{aligned} \quad (4.87)$$

On the other hand we note that

$$|z'|^{-N} ((z_2 + |\varepsilon|t z_3)^2 + z_3^2)^{N/2} \leq |z'|^{-N} (|z'| + |\varepsilon|t |z_3|)^N \leq (1 + |\varepsilon|t)^N. \quad (4.88)$$

In order to estimate $\tilde{V}(e^{tC}z)$ we note the following inequality

$$e^{-\mu t}(|z_2 + \varepsilon t z_3|^2 + z_3^2)^{1/2} \leq |z'| (1 + |\varepsilon|t) e^{-\mu t} \leq |z'| \leq c'_2, \quad (4.89)$$

because we have $|\varepsilon| < \mu$. It follows that

$$|\tilde{V}(e^{tC}z)| \leq \sup_{z \in \mathbb{R}^3, 0 < |z'| \leq c'_2} |\tilde{V}(z)|. \quad (4.90)$$

It follows that the right-hand side of (4.87) is estimated in the following way

$$\leq \sup_{z \in \mathbb{R}^3, 0 < |z'| \leq c'_2} |\tilde{V}(z)| \int_0^\infty (1 + |\varepsilon|t)^{N+1} e^{\mu(1-N)t} dt \leq C \|V\|_{0;N} \quad (4.91)$$

for some $C > 0$ independent of V . It follows that $\|QV\|_{0;N} \leq C \|V\|_{0;N}$ for some $C > 0$.

Next we will estimate the derivative $|z'|^{|\alpha|} \partial_z^\alpha (|z'|^{-N} QV)$. By Leibnitz rule it is sufficient to estimate the term $|z'|^{|\alpha|} \partial^\gamma |z'|^{-N} \partial^{\alpha-\gamma} (QV)$, where $\alpha \geq \gamma$. By simple computations, we have $|z'|^{|\alpha|} \partial^\gamma |z'|^{-N} \leq C_1 |z'|^{-N+|\alpha|-\gamma}$ for some $C_1 > 0$ independent of z' . On the other hand, we have

$$\begin{aligned} \partial^{\alpha-\gamma} (QV) &= -\partial^{\alpha-\gamma} \int_0^\infty e^{-t\tilde{B}} ((z_2 + \varepsilon t z_3)^2 + z_3^2)^{N/2} e^{-\mu N t} \tilde{V}(e^{tC}z) dt \\ &= - \sum_{\beta \leq \alpha-\gamma} \int e^{-t\tilde{B}-\mu N t} \partial_z^\beta ((z_2 + \varepsilon t z_3)^2 + z_3^2)^{N/2} \partial^{\alpha-\gamma-\beta} \tilde{V}(e^{tC}z) dt. \end{aligned} \quad (4.92)$$

We can easily see

$$|\partial_z^\beta ((z_2 + \varepsilon t z_3)^2 + z_3^2)^{N/2}| \leq C_2 (1 + |\varepsilon|t)^N |z'|^{N-|\beta|} \quad (4.93)$$

for some $C_2 > 0$. If we set $\alpha - \beta - \gamma = \delta$, $\delta = (\delta_1, \delta_2, \delta_3)$, then we have

$$\partial^{\alpha-\beta-\gamma} \tilde{V}(e^{tC}z) = e^{t\delta_1 - \mu(\delta_2 + \delta_3)t} (\partial_1^{\delta_1} \partial_2^{\delta_2} (\varepsilon t \partial_2 + \partial_3)^{\delta_3} \tilde{V})(e^{tC}z). \quad (4.94)$$

It follows that

$$\begin{aligned}
& |z'|^{|\alpha|} \partial^\gamma |z'|^{-N} |\partial^{\alpha-\gamma}(QV)| \\
& \leq C_3 |z'|^{-N+|\alpha|-|\gamma|} \sum_{\beta} \int_0^\infty e^{\mu t - \mu N t} |\partial_z^\beta ((z_2 + \varepsilon t z_3)^2 + z_3^2)^{N/2}| \\
& \times |\partial^{\alpha-\gamma-\beta} \tilde{V}(e^{tC} z)| dt \\
& \leq C_3 |z'|^{-N+|\alpha|-|\gamma|} \sum_{\beta} \int_0^\infty e^{\mu t - \mu N t} (1 + |\varepsilon|t)^{N+1} |z'|^{N-|\beta|} |\partial^{\alpha-\gamma-\beta} \tilde{V}(e^{tC} z)| dt \\
& \leq C_4 \int_0^\infty \sum_{|\xi|=|\alpha-\beta-\gamma|\leq k} |z'|^{|\xi|} |(\partial_z^\xi \tilde{V})(e^{tC} z)| (1 + |\varepsilon|t)^{N+1+|\xi|} e^{\mu t - \mu N t + |\alpha|t} dt.
\end{aligned} \tag{4.95}$$

In order to estimate $|z'|^{|\xi|} |(\partial_z^\xi \tilde{V})(e^{tC} z)|$, we set $\zeta = e^{tC} z$. Then we have

$$\begin{aligned}
& |z'|^{|\xi|} |(\partial_z^\xi \tilde{V})(e^{tC} z)| = |(e^{-tC} \zeta)'|^{|\xi|} |(\partial^\xi \tilde{V})(\zeta)| \\
& \leq e^{\mu|\xi|t} |(\partial^\xi \tilde{V})(\zeta)| ((\zeta_2 + \varepsilon t \zeta_3)^2 + \zeta_3^2)^{|\xi|/2} \\
& \leq e^{\mu k t} (1 + |\varepsilon|t)^k |\zeta'|^{|\xi|} |(\partial^\xi \tilde{V})(\zeta)| \leq \|V\|_{k;N} e^{\mu k t} (1 + |\varepsilon|t)^k.
\end{aligned} \tag{4.96}$$

By assumption we have $(1 + k - N)\mu + |\alpha| \leq (1 + k - N)\mu + k < 0$. Hence the right-hand side integral in (4.95) converges. Therefore we see that the right-hand side of (4.95) can be estimated by $C_5 \|V\|_{k;N}$. \square

Proof of Theorem 4.1. By setting $\varphi_\pm = QV$, (4.69) is equivalent to

$$V = E_\pm(z, QV). \tag{4.97}$$

We define the sequence V_\pm^j ($j = 0, 1, 2, \dots$) by

$$V_\pm^0 = E_\pm(z, 0), \quad V_\pm^1 = E_\pm(z, QV_\pm^0) - E_\pm(z, 0), \tag{4.98}$$

and

$$V_\pm^{j+1} = E_\pm(z, V_\pm^0 + \dots + V_\pm^j) - E_\pm(z, V_\pm^0 + \dots + V_\pm^{j-1}), \quad j = 1, 2, \dots \tag{4.99}$$

Let the integers k and N satisfy that $0 \leq k < N - \mu$ and $\mu(k + 1 - N) + k < 0$. We will show the convergence of $\sum_{j=0}^\infty V_\pm^j$. By definition we have $V_\pm^0 = E_\pm(z, 0) = \Psi_\pm(z)$. Next we have

$$V_\pm^1 = E_\pm(z, QV_\pm^0) - E_\pm(z, 0) = QV_\pm^0 \int_0^1 \nabla_w E_\pm(z, \tau QV_\pm^0) d\tau. \tag{4.100}$$

Let $\varepsilon' > 0$ be a small constant chosen later, and suppose that

$$\|\Psi_{\pm}\|_{k;N} < \varepsilon', \quad \|\nabla\Psi_{\pm}\|_{k;N} < \varepsilon'. \quad (4.101)$$

Then, by Lemma 4.5 and the definition of V_{\pm}^0 we have

$$\|\tau QV_{\pm}^0\|_{k;N} \leq c_1\|V_{\pm}^0\|_{k;N} = c_1\|\Psi_{\pm}\|_{k;N} < c_1\varepsilon' \quad (4.102)$$

for some $c_1 > 0$ independent of Ψ_{\pm} .

In order to estimate $\|\nabla_w E_{\pm}(\cdot, \tau QV_{\pm}^0)\|_{k;N}$, we set $w = (w_1, \dots, w_4) = \tau QV_{\pm}^0$ and

$$\zeta = (\zeta_1, \zeta') = \left(z_1 \frac{1+w_2}{1+w_1}, (z_2+w_3)|1+w_1|^{\nu}, (z_3+w_4)|1+w_1|^{\nu} \right).$$

The differentiation $\partial_z^{\alpha}(\nabla_w E_{\pm}(z, \tau QV_{\pm}^0))$ consists of terms which are product of $\partial^{\beta}\nabla\Psi_{\pm}(\zeta)$ ($\alpha \geq \beta$) and the differentiations of w . First, the product of differentiations of w is bounded by a constant in view of (4.102). On the other hand, in order to estimate

$$|z'|^{|\beta|}|\partial^{\beta}\nabla\Psi_{\pm}(\zeta)| \leq |z'|^{|\beta|}|\zeta'|^{-|\beta|}|\zeta'|^{|\beta|}|\partial^{\beta}\nabla\Psi_{\pm}(\zeta)| \leq |z'|^{|\beta|}|\zeta'|^{-|\beta|}\|\nabla\Psi_{\pm}\|_{k;N},$$

we consider $|z'|^{|\beta|}|\zeta'|^{-|\beta|}$. By Lemma 4.2 we see that w_3 and w_4 can be divisible by $z_2^2 + z_3^2$, respectively. By the smallness of w , the term $|z'|^{|\beta|}|\zeta'|^{-|\beta|}$ can be bounded by a constant. Hence, if $\varepsilon' > 0$ is sufficiently small, then we obtain, by the definition of $E_{\pm}(z, w)$ in (4.69), (4.62) and (4.64),

$$\|\nabla_w E_{\pm}(\cdot, \tau QV_{\pm}^0)\|_{k;N} \leq c_2\|\nabla\Psi_{\pm}\|_{k;N} < c_2\varepsilon' \quad (4.103)$$

for some $c_2 > 0$ independent of ε' and Ψ_{\pm} .

It follows from (4.100) that

$$\|V_{\pm}^1\|_{k;N} \leq \|QV_{\pm}^0\|_{k;N} \int_0^1 \|\nabla_w E_{\pm}(z, \tau QV_{\pm}^0)\|_{k;N} d\tau \leq c_1 c_2 \varepsilon'^2.$$

In order to show the general case, we assume that $\|V_{\pm}^j\|_{k;N} \leq c_1^j c_2^j \varepsilon'^{j+1}$ for $j = 0, 1, 2, \dots, k$. Then we have

$$\left\| \sum_{j=0}^k V_{\pm}^j \right\|_{k;N} \leq \sum_{j=0}^k c_1^j c_2^j \varepsilon'^{j+1} \leq \frac{\varepsilon'}{1 - c_1 c_2 \varepsilon'}. \quad (4.104)$$

By definition we have

$$\begin{aligned} V_{\pm}^{k+1} &= E_{\pm}(z, Q(V_{\pm}^0 + \cdots + V_{\pm}^k) - E_{\pm}(z, Q(V_{\pm}^0 + \cdots + V_{\pm}^{k-1})) \\ &= QV_{\pm}^k \int_0^1 \nabla_w E_{\pm}(z, Q(V_{\pm}^0 + \cdots + V_{\pm}^{k-1}) + \tau QV_{\pm}^k) d\tau. \end{aligned} \quad (4.105)$$

By the apriori estimate (4.104) and Lemma 4.4 the substitution in the right-hand side of (4.105) is well defined. Moreover, by the same argument as in the proof of (4.103) we see that

$$\|\nabla_w E_{\pm}(z, Q(V_{\pm}^0 + \cdots + V_{\pm}^{k-1}) + \tau QV_{\pm}^k)\|_{k;N} \leq c_2 \varepsilon'.$$

It follows from (4.105) that

$$\|V_{\pm}^{k+1}\|_{k;N} \leq \|QV_{\pm}^k\|_{k;N} c_2 \varepsilon' \int_0^1 d\tau \leq c_1^{k+1} c_2^{k+1} \varepsilon'^{k+2}.$$

Hence we have the estimate of V_{\pm}^j for $j = k + 1$. It follows that the series $V_{\pm} := \sum_{j=0}^{\infty} V_{\pm}^j$ converges in $B_{k;N}$ and V_{\pm} is a solution of (4.97). We note that, by (4.104) V_{\pm} satisfies the estimate $\|V_{\pm}\|_{k;N} \leq \varepsilon'(1 - c_1 c_2 \varepsilon')^{-1}$, and V_{\pm} is divisible by $|z'|^2$.

Next we verify the smallness assumption (4.101) uniformly with respect to $x_1 \neq 0$ in some neighborhood of $x_1 = 0$. Because the argument is similar we consider the condition $\|\Psi_{\pm}\|_{k;N} < \varepsilon'$. In view of the definition of Ψ_{\pm} in (4.51) and (4.52), we estimate $x_j^{-1} R_j(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3)$, ($j = 1, 2$) and $|x_1|^{\nu} R_j(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu})$, ($j = 3, 4$) with $x_1 \neq 0$ close to 0. Because the argument is similar, we consider the case $j = 1$. We have

$$|z'|^{|\alpha|} |\partial_z^{\alpha}(|z'|^{-N} \Psi_{\pm}^1(z))| = x_1^{-1} |z'|^{|\alpha|} |\partial_z^{\alpha}(|z'|^{-N} R_1(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3))|. \quad (4.106)$$

By Lemma 4.4 we have that, for every positive integer p , the term

$$R_1(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3) |z'|^{-p}$$

is smooth at $z = 0$. Because

$$|z'|^p = (|x_1|^{\nu} |x_1|^{-\nu} |z'|)^p = (|x_1|^{\nu} |x''|)^p, \quad x'' = (x_3, x_4),$$

and $|x''|$ is bounded by the support condition of R_j , the negative power $|z'|^{-N}$ in the right-hand side of (4.106) is absorbed by $|z'|^p$ if p is sufficiently large. On the other hand, if the differentiation ∂_z^{α} is applied to

$R_1(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3)$, then the negative power of $|x_1|$ appears. These terms are also uniformly bounded when $x_1 \rightarrow 0$, because there appears positive power of $|x_1|$ from $|z'|^p$. Because all derivatives of $R(x)$ at the origin vanish, we see that the right-hand side of (4.106) can be made arbitrarily small if we cut off $R(x)$ in a sufficiently small neighborhood of the origin. This proves that we have (4.101).

We set $\varphi_{\pm} = QV_{\pm} \in B_{k;N}$, and $\varphi_{\pm}(z) = (\varphi_{\pm}^1(z), \varphi_{\pm}^2(z), \varphi_{\pm}^3(z), \varphi_{\pm}^4(z))$. The function φ_{\pm} is a solution of (4.69). Then we define $v^j(x)$ ($j = 1, 2, 3, 4$) by (4.53) and (4.54). For a given integer m , we can easily see that $v^j(x)$ is a C^m function if we take k and N in $B_{k;N}$ sufficiently large. If we rewrite (4.69) with the variable x , then we see that v is a solution of (4.39), where the nonlinear part R is modified by a cutoff function. In order to show that v is a solution of the original (4.39) we will show the a priori estimate of v . In fact, if $|x + v| < \varepsilon''$ for sufficiently small ε'' , then v is a solution of (4.39). By Lemma 4.6 and the uniform estimate of V_{\pm} in x_1 we know that $\phi_{\pm}^1(z)$ is uniformly bounded in z and x_1 . It follows that $v_1(x) = x_1 \phi_{\pm}^1$ is arbitrarily small if x_1 is sufficiently small. Similarly we can show that $v_2(x) = x_2 \phi^2$ is small by the estimate of V_{\pm} . On the other hand, we have $x_3 + v_3(x) = x_3 + |x_1|^{-\nu} \phi_{\pm}^3(z)$. Because ϕ_{\pm}^3 is divisible by $|z'|^2$ and $|z'| = |x_1|^{\nu} |x''|$, by Lemma 4.4 we see that $|x_3 + v_3(x)| < \varepsilon''$ uniformly in x_1 . Similarly we can show the same estimate for $x_4 + v_4$. Therefore we see that v is a solution of (4.39). This completes the proof.

Acknowledgements. This work was done when the first author stayed in RIMS, Kyoto University and the second author stayed in the University of Cagliari. The first author thanks Prof. T. Kawai for the invitation to RIMS, Kyoto University and for the excellent conditions for the research on the subject of the paper. The second author expresses the sincere gratitude to the University of Cagliari for the support in the beginning of this research. The authors also acknowledge useful discussions on problems related to the paper with S. Marmi, R. Perez Marco and L. Stolovitch.

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