Finitely smooth solutions of nonlinear singular partial differential equations

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We solve a Fuchsian system of singular nonlinear partial differential equations with resonances. These equations have no smooth solutions in general. We show the solvability in a class of finitely smooth functions. Typical examples are a homology equation for a vector field and a degenerate Monge-Ampère equation.

1 Introduction

This paper is concerned with the solvability of a Fuchsian system of a singular nonlinear partial differential equations in a bounded domain $\Omega \subset \mathbf{R}^n$ or in $\mathbf{R}^n$. These equations naturally appear when we solve a class of Monge-Ampère equations or when we linearize a singular vector field by a coordinate change. (See §2). As we can see from the simple example $\mathcal{L}u := (t\frac{d}{dt} - 1)u = t$, these equations do not have a smooth solution in general. Indeed, if $u(t) = c_0 + c_1 t + v(t)$, $v = O(t^2)$ is a solution, then the relations $\mathcal{L}(c_0 + c_1 t) = -c_0$ and $\mathcal{L}v = O(t^2)$ imply that u is not smooth at $t = 0$. In fact, if we allow a singular solution, then we see that $u = ct + t\log t$, (c, constant) gives a solution. Here we take the branch of the logarithm such that $\log 1 = 0$. If we restrict t to the real line, then u gives a Hölder continuous function on the real line. Similar property holds for $\mathcal{L}u = x_1$, where $\mathcal{L} = x_1\frac{\partial}{\partial x_1} + mx_2\frac{\partial}{\partial x_2} - 1$, $(m > 1)$. The equation $\mathcal{L}u = x_1$ has no smooth solution at the origin, while $u = cx_1 + x_1\log x_1$, (c, constant) is a singular solution. It gives a Hölder continuous function on the real line for an appropriate choice of the branch of $\log x_1$. We also note that this phenomenon is closely related with a Grobman-Hartman theorem. (cf. Remark 2.9). These examples are known as a so-called totally characteristic type partial differential equation.(cf. [3]). As to formal solutions of nonlinear first order totally characteristic type equations we refer [3], and as to singular solutions of nonlinear singular partial differential equations we refer [19]. We also remark a related work [11] concerning symbolic calculus on manifolds with edges.

The object of this paper is to solve this type of equations in a class of finitely smooth functions. For this purpose we employ a rapidly convergent iteration method in a class of non smooth functions, because the Fuchsian equations have a loss of regularity. We stress that the usual rapidly convergent iteration scheme is not useful in order to solve this type of equations, because one requires high regularity in the iterative scheme, while our solution does not have such smoothness in general. We introduce a partial smoothing operator which preserves the vanishing order of approximate solutions on every coordinate axis. This smoothing operator is useful in the iterative scheme because the Fuchsian partial differential operators which we study in this paper lose derivatives of the transversal direction of every coordinate axis, although they preserve the vanishing order. Concerning the loss of regularity of nonlinear equations (of multiple characteristics) we refer [5], [7] and [18].

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This paper is organized as follows. In §2 we state the main theorem and we give several consequences and applications. In §3 we prepare lemmas which are necessary for the proof of the main theorem. The proof of the main theorem is given in §4 by using a rapidly convergent iteration method.

2 Statement of results

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ be the variable in $\mathbb{R}^n$ $(n \geq 2)$. For a multiinteger $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We define

$$\partial_j = \partial/\partial x_j,\ \delta_j = x_j \partial_j\ (j = 1, \dots, n), \quad \delta^\alpha = \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n}.$$

Let $m \geq 1$, $m \geq s \geq 0$, $N \geq 1$ be integers, and let

$$p_j(\delta) = \sum_{|\alpha| \leq m} a_{\alpha j} \delta^\alpha, \quad (a_{\alpha j} \in \mathbb{R},\ j = 1, \dots, N)$$

be Fuchsian partial differential operators. Let

$$a_j(x, z), z = (z_\alpha)_{|\alpha| \leq s} \quad j = 1, \dots, N,$$

be real-valued C^∞ functions of $(x, z) \in \mathbb{R}^n \times \Omega$, where $\Omega \subset \mathbb{R}^{kN}$, $(k = \#\{\alpha \in \mathbb{Z}_+^n; |\alpha| \leq s\})$ is a neighborhood of the origin.

We study the solvability of the system of equations for $u = (u_1, \dots, u_N)$

$$G_j(u) := p_j(\delta) u_j + a_j(x, \delta^\alpha u; |\alpha| \leq s) = 0, \quad j = 1, \dots, N. \tag{2.1}$$

Let σ be a nonnegative number, and Γ be a domain of $\mathbb{R}^n$. We define $H_\sigma \equiv H_{\sigma,\Gamma}$ as the set of holomorphic (vector) funtions $v(\zeta) = (v_1(\zeta), \dots, v_N(\zeta))$ of $\zeta = \eta + i\xi \in \Gamma + i\mathbb{R}^n$ such that

$$|\!|\!| v |\!|\!|_{\sigma,\Gamma} := \sup_{\eta \in \Gamma} \int_{\mathbb{R}^n} \langle \zeta \rangle^\sigma |v(\zeta)| d\xi < \infty, \tag{2.2}$$

where $\langle \zeta \rangle = 1 + \sum_{j=1}^n |\zeta_j|$, and $|v(\zeta)| = (\sum_{j=1}^N |v_j(\zeta)|^2)^{1/2}$. The space $H_{\sigma,\Gamma}$ is a Banach space with the norm (2.2). The fundamental properties of $H_{\sigma,\Gamma}$ is given in Proposition 3.1 which follows.

Let $f(x)$ be an integrable N- vector function on $\mathbb{R}_+^n$, $\mathbb{R}_+ := \{t \in \mathbb{R}; t \geq 0\}$ and let $\hat{f}(\zeta)$ be the Mellin transform of f

$$\hat{f}(\zeta) \equiv M(f)(\zeta) = \int_{\mathbb{R}_+^n} f(x) x^{\zeta - e} dx, \quad e = (1, \dots, 1),\ \zeta = \eta + i\xi, \eta \in \Gamma, \xi \in \mathbb{R}^n, \tag{2.3}$$

where $x^{\zeta - e} = x_1^{\zeta_1 - 1} \cdots x_n^{\zeta_n - 1}$, $\zeta = (\zeta_1, \dots, \zeta_n)$. It is easy to see that $\hat{f}(\zeta)$ is analytic if the integral (2.3) absolutely converges. The inverse Mellin transform is given by

$$f(x) = M^{-1}(\hat{f})(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\eta + i\xi) x^{-\eta - i\xi} d\xi, \tag{2.4}$$

where $x_j > 0$ $(j = 1, \dots, n)$ and η is so taken that the integral converges. We note that these formulas follow from the corresponding ones of the Fourier transform by the change of variables $e^{\theta_j} \to x_j$.

We define $\mathcal{H}_{\sigma,\Gamma}$ as the inverse Mellin transform of $H_{\sigma,\Gamma}$. We note that the Mellin transform gives the one to one correspondence between the spaces $\mathcal{H}_{\sigma,\Gamma}$ and $H_{\sigma,\Gamma}$. For $u \in \mathcal{H}_{\sigma,\Gamma}$ we define the norm $\|u\|_{\sigma,\Gamma}$ of u by

$$\|u\|_{\sigma,\Gamma} := |\!|\!| M(u) |\!|\!|_{\sigma,\Gamma}.$$

For an integer $k \geq 1$ we denote by $(\mathcal{H}_{\sigma,\Gamma})^k$ the product of k copies of $\mathcal{H}_{\sigma,\Gamma}$. The norm in $(\mathcal{H}_{\sigma,\Gamma})^k$ is defined as the sum of the norm of each component. For simplicity, we denote the norm in $(\mathcal{H}_{\sigma,\Gamma})^k$ by $\|\cdot\|_{\sigma,\Gamma}$ if there is no fear of confusion.

Let $p_j(\zeta) = \sum_{|\alpha|\le m} a_{\alpha j}(-\zeta)^\alpha$ be the indicial polynomial associated with $p_j(\delta)$, where $\zeta = (\zeta_1, \dots, \zeta_n)$ is the covariable of x in the sense of the Mellin transform. We assume

$(A.1)$ There exists a constant $c > 0$ such that

$$|p_j(\eta + i\xi)| \ge c(|\eta| + |\xi|)^s, \qquad \forall \eta \in \Gamma, \forall \xi \in \mathbb{R}^n, \quad j = 1, \dots, N.$$

We set $a(x, z) = (a_1(x, z), \dots, a_N(x, z))$. Then we assume that $a(x, z) \in (C^\infty(\mathbb{R}^n \times \Omega))^N$ and

(A.2) $\forall \alpha \in \mathbb{Z}_+^n$, $\forall \beta \in \mathbb{Z}_+^{kN}$, $\exists C_{\alpha\beta} > 0$ such that

$$|(\partial/\partial z)^\beta \delta_x^\alpha a(x, z)| \le C_{\alpha\beta}, \quad \forall (x, z) \in \mathbb{R}^n \times \Omega.$$

Then our main theorem in this paper is the following

Theorem 2.1 *Let $\sigma \ge m$ be an integer. Suppose that $(A.1)$ hold for some bounded domain $\Gamma \subset \mathbb{R}^n$ containing the origin. Assume (A.2). Then there exist an integer $\nu = \nu(\sigma) \ge 0$ and an $\varepsilon = \varepsilon(\sigma) > 0$ depending on σ such that, if the following conditions are satisfied*

$$\|a(\cdot, 0)\|_{\nu,\Gamma} < \varepsilon, \qquad \|\nabla_z a(\cdot, 0)\|_{\nu,\Gamma} < \varepsilon,$$

then Eq. (2.1) has a solution $u \in (\mathcal{H}_{\sigma,\Gamma})^N$.

Next we study the local solvability. We say that $u \in (\mathcal{H}_{\nu,\Gamma})^N$ at the origin if there exists a $\psi \in C^\infty(\mathbb{R}^n)$ with compact support and being identically equal to one in some neighborhood of the origin such that $\psi u \in (\mathcal{H}_{\nu,\Gamma})^N$. For open sets $\Gamma_1 \subset \mathbb{R}^n$ and $\Gamma_2 \subset \mathbb{R}^n$ the relation $\Gamma_1 \subset\subset \Gamma_2$ means $\overline{\Gamma_1} \subset \Gamma_2$, where $\overline{\Gamma_1}$ is the closure of Γ_1. Then we have

Theorem 2.2 *Let $\sigma \ge m$ be an integer. Suppose that $(A.1)$ holds for some bounded domain Γ containing the origin. Then there exists an integer $\nu \ge 0$ such that, if*

$$a(x, 0) \in (\mathcal{H}_{\nu,\Gamma})^N \quad \textit{and} \quad \nabla_z a(x, 0) \in (\mathcal{H}_{\nu,\Gamma})^{kN} \quad \textit{at the origin},$$

then there exists a solution $u \in (\mathcal{H}_{\sigma,\Gamma'})^N$ of (2.1) in some neighborhood of the origin for every $\Gamma' \subset\subset \Gamma$.

Remark 2.3 a) Theorem 2.1 and Theorem 2.2 yield the solvability of (2.1) in some neighborhood of the origin in a class of finitely smooth functions. Indeed, we can solve (2.1) in the sectors $\{\varepsilon_j x_j \ge 0; j = 1, \dots, n\}$, $(\varepsilon_j = \pm 1)$, after the change of variables $x_j \mapsto \varepsilon_j x_j$, $(j = 1, \dots, n)$, because δ_j is invariant under the change of variables. By the assumption $0 \in \Gamma$ and the definition of $\mathcal{H}_{\nu,\Gamma}$, the solution u together with the derivatives $\delta^\alpha u$, $|\alpha| \le s$ vanishes (to a finite order) on the coordinate planes $x_j = 0$ $(j = 1, \dots, n)$. (See Proposition 3.1.) Hence, by patching the solutions in these sectors we obtain a finitely smooth solution in some neighborhood of the origin.

b) (Bifurcation from a resonance) The uniqueness of solutions in Theorem 2.1 and Corollary 2.2 does not always hold if there is a resonance. Indeed, we consider the equation

$$p(\delta)u + \lambda a(x, u) = 0, \quad a(x, u) = O(|u|^2),$$

where u is a scalar unknown function, λ is a real parameter, and where $p(\delta)$ is an Fuchisian partial differential operator similar to $p_j(\delta)$ in (2.1). We note that $u \equiv 0$ is a trivial solution of the equation. We assume (A.1) for some domain $\Gamma \ni 0$. Then we shall show that the above equation has a non trivial family of solutions $u = u_\lambda$, $u_\lambda = \lambda u_0 + v_\lambda$ for sufficiently small λ, where u_0 satisfies $p(\delta)u_0 = 0$.

First we note that there exists u_0 such that $p(\delta)u_0 = 0$ if there is a resonance. (See also Example 2.8 which follows.) If we set $v = v_\lambda$, then v satisfies

$$p(\delta)v + \lambda a(x, \lambda u_0 + v) = 0.$$

The conditions in Theorem 2.1 read:

$$\|\lambda a(\cdot, \lambda u_0)\|_{\nu,\Gamma} < \varepsilon \quad \text{and} \quad \|\lambda \nabla_u a(\cdot, \lambda u_0)\|_{\nu,\Gamma} < \varepsilon.$$

These conditions are satisfied for sufficiently small λ if $a(x, \lambda u_0) \in \mathcal{H}_{\nu,\Gamma}$ and $\nabla_u a(x, \lambda u_0) \in \mathcal{H}_{\nu,\Gamma}$ for all λ close to 0. For example, if the local solvability is concerned, these conditions are verified if $a(x, \lambda u_0)$ and $\nabla_u a(x, \lambda u_0)$ vanish to some order for all sufficiently small λ. (We also refer Lemma 4.2 which follows.)

It follows from Theorem 2.1 or Corollary 2.2 that there exists a solution v for sufficiently small λ. Moreover, by the constructions of an approximate sequence w_k in (4.12), we have $v = \lim_k w_k$ and

$$w_1 = S_0\rho_0, \quad L_0\rho_0 = g_0 = -\lambda a(x, \lambda u_0), \ldots$$

It follows from the assumption on a that the vanishing order of g_0 at the origin is greater than u_0. Therefore, we see that the vanishing order of w_1 at the origin is greater than that of u_0, because L_0^{-1} and S_0 preserve the vanishing order. Inductively, we can easily see that the vanishing order at the origin of the solution $v = \lim w_k$ is greater than that of u_0. It follows that $u = \lambda u_0 + v \neq 0$. Therefore, we have a family of solutions of our equation.

Remark 2.4 The smallness conditions in Theorem 2.1 for the nonlinear part $a(x, 0)$ and $\nabla_z a(x, 0)$ of the equation (2.1) are fulfilled if the following conditions are satisfied

$$a(x, 0) = 0, \quad \nabla_z a_j(x, 0) = 0, \quad j = 1, \ldots, n. \tag{2.5}$$

On the other hand, the condition (A.2) in Theorem 2.1 is fulfilled if $a(x, z)$ is independent of x or $a(x, z)$ has a compact support with respect to x.

Example 2.5 We give the example which satisfies (A.1). Let

$$p_2(\zeta) := \zeta_1^2 - \sum_{j=2}^{n} c_j \zeta_j^2, \quad c_j > 0.$$

Let $p_1(\zeta)$ be a linear function of ζ with real coefficients. We set $p(\zeta) = p_2(\zeta) + p_1(\zeta)$. We assume that

$$p_1(\xi) + \eta \cdot \nabla p_2(\xi) \neq 0 \quad \text{for } \forall\eta \in \Gamma, \text{ and } \forall\, \xi \in \mathbb{R}^n \text{ such that } p_2(\xi) \geq 0, |\xi| = 1.$$

We want to show that there exists real number K such that $p(\zeta) + K$ satisfies (A.1) with $s = 1$. We have

$$p(\eta + i\xi) + K = K - p_2(\xi) + p(\eta) + i(p_1(\xi) + \eta \cdot \nabla p_2(\xi)).$$

Because η moves in a bounded set it follows that if $K > 0$ is sufficiently large, the zero set of the polynomial of ξ, $\Re\, p(\eta + i\xi) + K$ is contained in the set $p_2(\xi) \geq 0$, $|\xi| \geq 1$, where $\Re p$ is the real part of p. On the other hand, by assumption and the homogeneity, the imaginary part $\Im\, p(\eta + i\xi)$ does not vanish on the set $p_2(\xi) \geq 0$, $|\xi| \geq 1$. It follows that $p(\eta + i\xi) + K \neq 0$ for all $\eta \in \Gamma$ and ξ.

In order to show (A.1) with $s = 1$ it is sufficient to consider ξ such that $|\xi| \geq N > 0$ for large N. If ξ is in a conical neighborhood of ξ_0 such that $p_2(\xi_0) \neq 0$, we have (A.1) with $s = 2$. If otherwise, the assumption implies that $p_1(\xi) + \eta \cdot \nabla p_2(\xi) \neq 0$. Hence we have

$$|p(\eta + i\xi)| \geq |\Im\, p(\eta + i\xi)| \geq c|\xi| \geq c'(|\xi| + |\eta|)$$

for some $c > 0$ and $c' > 0$. This proves (A.1) with $s = 1$.

Example 2.6 We write $x_1 = x$, $x_2 = y$, and we consider the Monge-Ampère operator

$$M(u) := u_{xx}u_{yy} - u_{xy}^2 + kxyu_{xy} + cu, \quad 4 < k < 12,\ c \in \mathbb{C}.$$

Let $u_0 = x^2y^2$ and set $f_0 = M(u_0) = (4k - 12 + c)x^2y^2$. We want to solve the equation

$$M(u_0 + v) = f_0(x, y) + g(x, y), \quad \text{in } \mathbb{R}^2,$$

where $g(x, y)$ is a given function. If we define

$$Q = 2x^2\partial_x^2 + 2y^2\partial_y^2 + (k - 8)xy\partial_x\partial_y + c,\ \tilde{M}(u) = M(u) - kxyu_{xy} - cu,$$

then the equation can be written in the form

$$Qv + \tilde{M}(v) = g.$$

In order to write the equation in the form (2.1) we introduce a new unknown function w by $v(x,y) = x^2y^2w(x,y)$. By simple computations we have

$$\begin{aligned}
& x^{-2}y^{-2}\tilde{M}(x^2y^2w) \\
= \;& (x^2w_{xx} + 4xw_x + 2w)(y^2w_{yy} + 4yw_y + 2w) - (xyw_{xy} + 2xw_x + 2yw_y + 4w)^2. \\
& x^{-2}y^{-2}Q(x^2y^2w) \\
= \;& 2(\delta_x^2 + 4\delta_x)w + 2(\delta_y^2 + 4\delta_y)w + (k-8)(\delta_x\delta_y + 2\delta_x + 2\delta_y)w + (4k - 24 + c)w,
\end{aligned}$$

where $\delta_x = x\partial/\partial x$ and $\delta_y = y\partial/\partial y$. This proves that our equation can be written in the form (2.1). We note that the condition (A.2) is fulfilled. (cf. Remark 2.4).

The indicial polynomial is given by

$$p(\zeta) := 2(\zeta_1^2 - 4\zeta_1) + 2(\zeta_2^2 - 4\zeta_2) + (k-8)(\zeta_1\zeta_2 - 2\zeta_1 - 2\zeta_2) + c + 4k - 24.$$

We will show (A.1) with $s = 2$ for some bounded domain $\Gamma \subset \mathbb{R}^n \setminus \{p(\eta) = 0\}$ containing the origin if $c = iK$, $K > 0$ is sufficiently large . We note that $p(\xi)$ is elliptic by the condition $4 < k < 12$. It follows that there exist $\xi_0 > 0$ and $\alpha > 0$ independent of K and η such that $\Re\, p(\eta + i\xi) \ge \alpha|\xi|^2$ if $|\xi| > \xi_0$ and $\eta \in \Gamma$. If $|\xi| \le \xi_0$, then $\Im\, p(\eta + i\xi)$ does not vanish if K is sufficiently large. Therefore we have (A.1) with $s = 2$.

Next we apply our argument to the normal form theory of a singular hyperbolic vector field $\chi = \sum_{j=1}^n X_j(x)\partial_j$, $\partial_j = \partial/\partial x_j$ on $\mathbb{R}^n$. We say that χ is singular if $X_j(0) = 0$ $(j = 1, \dots, n)$. We set $X = (X_1, \dots, X_n)$. For the sake of simplicity, we assume

$$X(x) = x\Lambda + R(x), \quad R(x) = (R_1(x), \dots, R_n(x)), \tag{2.6}$$

for a real-valued C^∞ function $R_j(x)$ such that $R_j(0) = 0$, $\nabla R_j(0) = 0$, and a diagonal matrix $\Lambda =$ diag $(\lambda_1, \cdots, \lambda_n)$, $\lambda_j \in \mathbb{R}$. We want to find a change of variables $y \mapsto x = y + v(y)$ which linearizes χ. It follows that v satisfies the so-called homology equation

$$X(y + v(y))(1 + \nabla v)^{-1} = y\Lambda,$$

or equivalently,

$$\mathcal{L}v = R(y + v(y)), \quad \mathcal{L}v := \sum_{j=1}^n \lambda_j\delta_j v - v\Lambda. \tag{2.7}$$

We define $p(\zeta) = -\sum_{j=1}^n \zeta_j\lambda_j I - \Lambda$, where I is an identity matrix. Then we have

Theorem 2.7 *Suppose that (A.1) is satisfied for $s = 0$ and some bounded domain Γ containing the origin. Assume (2.6). Let $\sigma \ge 1$ be an integer. Then there exists $\nu \ge 0$ such that, if the following conditions are satisfied*

$$R \in (\mathcal{H}_{\nu,\Gamma})^n,\ \nabla R_j \in (\mathcal{H}_{\nu,\Gamma})^{n^2} \quad \textit{at the origin} \quad (j = 1, \dots, n),$$

then Eq. (2.7) has a solution $v \in (\mathcal{H}_{\sigma,\Gamma'})^n$ for every $\Gamma' \subset\subset \Gamma$.

Example 2.8 We give examples which satisfy (A.1). Suppose that $\lambda_1 \cdots \lambda_n \ne 0$. By definition the k-th component of $\Re\, p(\zeta)$ $(\zeta = \eta + i\xi)$ is given by $-\sum_{j=1}^n \eta_j\lambda_j - \lambda_k$. Hence the set of η such that $\Re\, p(\zeta) = 0$ consists of n hyperplanes, $\sum_j \eta_j\lambda_j + \lambda_k = 0$ not passing through the origin. Therefore we have (A.1) with $s = 0$ for some open set Γ containing the origin. The followings are typical cases which satisfy (A.1).
(i) Poincaré case; i.e., $\lambda_j > 0$ $(j = 1, \dots, n)$.
(ii) Nonresonant Siegel case; namely, some λ_j are positive and others are negative, and $p(\zeta) = 0$ $(\zeta \in \mathbb{Z}_+^n, |\zeta| \ge 2)$ has no solution.
(iii) Infinite resonances case; that is, $p(\zeta) = 0$ $(\zeta \in \mathbb{Z}_+^n)$ has an infinitely many solutions.

The third case contains a volume preserving vector fields, namely $\sum_{j=1}^n \lambda_j = 0$. In the case (i) the set

$$\{\eta \in -\mathbb{R}^n_+; \Re\, p(\eta + i\xi) = 0 \ \text{ for some } \xi \in \mathbb{R}^n\}$$

is a compact set not containing the origin. Hence we can take Γ in (A.1) as a bounded domain in $\mathbb{R}^n_+ \setminus \{\Re\, p(\zeta) = 0\}$. In the case (ii) the intersection of the hyperplanes $\sum_j \eta_j\lambda_j + \lambda_k = 0$ and $-\mathbb{R}^n_+$ is noncompact. Hence the set Γ in (A.1) may be a smaller set. In the case (iii) there is an additional restriction to Γ due to an infinite resonances apart from the ones caused by a Siegel condition. We note that the larger the set Γ is, the more regular the solution is.

Remark 2.9 By Remark 2.3 and Theorem 2.7 we can construct a finitely smooth coordinate change which linearizes χ even in the case of resonances. It is natural to ask wether there exists a C^∞ coordinate change which linearizes χ. The answer to this question is not affirmative. Indeed, if the vector field has a resonance, $\mathcal{L}$ has a (infinite) kernel. It follows that if (2.7) has a C^∞ solution v, then the Taylor expansion of v at the origin gives a formal power series solution of (2.7). Hence the Taylor expansion of R satisfies a compatibility condition. Because we do not assume any compatibility condition a priori, the solution is not smooth in general. We stress that the regularity of the solution is related with the property of a resonance as we note in the preceeding example. If we assume the weaker condition $\lambda_1 \cdots \lambda_n \neq 0$, the solution is continuous. We remark that this fact was essentially noted as a Grobman-Hartman theorem for a vector field, which asserts the existence of a continuous solution of a homology equation (cf. [1], p.127 and p191).

Theorem 2.7 can be extended to a commuting system of hyperbolic singular vector fields on $\mathbb{R}^n$,

$$\chi = \{\chi^\mu; \mu = 1, \dots, d\}, \quad [\chi^\mu, \chi^\nu] = 0 \quad \text{for all } \nu \text{ and } \mu.$$

We write $\chi^\mu = \sum_{j=1}^n X_j^\mu(x)\partial_j$ and set $X^\mu = (X_1^\mu, \dots, X_n^\mu)$. For the sake of simplicity we assume that $X^\mu(x) = x\Lambda^\mu + R^\mu(x)$ for some real-valued C^∞ vector function R^μ such that

$$R^\mu(0) = 0, \quad \nabla R^\mu(0) = 0,$$

and diagonal matrices

$$\Lambda^\mu = \text{diag}\,(\lambda_1^\mu, \cdots, \lambda_n^\mu), \quad \lambda_j^\mu \in \mathbb{R}, \ \ \mu = 1, \dots, d.$$

We are interested in the simultaneous linearization of χ by the change of variables $y \mapsto x = y + v(y)$. It follows that v satisfies an overdetermined system of equations

$$\mathcal{L}^\mu v = R^\mu(x + v),$$

where $\mathcal{L}^\mu$ is similarly given by (2.7). Let $\mathcal{C}$ be a positive cone generated by the vectors $(\lambda_j^1, \dots, \lambda_j^d) \in \mathbb{R}^d$, $(j = 1, \dots, n)$, namely

$$\mathcal{C} := \{\sum_{j=1}^n t_j(\lambda_j^1, \dots, \lambda_j^d) \in \mathbb{R}^d; t_j \geq 0, (j = 1, \dots, n), t_1^2 + \cdots + t_n^2 \neq 0\}.$$

We say that χ satisfies a simultaneous Poincaré condition if the cone $\mathcal{C}$ does not contain the origin. In case $d = 1$, this condition is equivalent to that the quantity $t_1\lambda_1^1 + \cdots + t_n\lambda_n^1$ does not vanish for $t_j \geq 0$ such that $t_1^2 + \cdots + t_n^2 \neq 0$. The last condition is equivalent to say that $\lambda_1^1 > 0, \dots, \lambda_n^1 > 0$. This is a well-known Poincaré condition for a single vector field. We have

Theorem 2.10 *Let $\sigma \geq 1$. Suppose that the simultaneous Poincaré condition is satisfied. Then there exists $\nu \geq 0$ such that, if*

$$R^\mu \in (\mathcal{H}_{\nu,\Gamma})^n \quad \textit{and} \quad \nabla R^\mu \in (\mathcal{H}_{\nu,\Gamma})^{n^2} \ \textit{at the origin } \ \textit{for } \mu = 1, \dots, d,$$

then χ is simultaneously linearized in some neighborhood of the origin by the change of the variables $y \mapsto x = y + v(y)$, with $v \in (\mathcal{H}_{\sigma,\Gamma'})^n$, $\forall \Gamma' \subset\subset \Gamma$.

3 Some lemmas

In this section we will prepare lemmas which are necessary in the calculus of a class of pseudo-differential operators of totally characteristic type in a Mellin's sense. We cite [11] concerning symbolic calculus of operators on manifolds with edges.

Let Γ be an open set in $\mathbb{R}^n$. First we study fundamental properties of $H_{s,\Gamma}$ $(s \in \mathbb{R}_+)$ defined in §1.

Proposition 3.1 (1) *Let $s \geq 0$ be an integer and let $\hat{u} \in H_{s,\Gamma}$. Then the inverse Mellin transform $u(x) = M^{-1}(\hat{u})(x)$ of $\hat{u}$ is a bounded continuous function on $\mathbb{R}^n_+$ such that for every $\alpha, |\alpha| \leq s$ and $\eta \in \Gamma$, the function $x^\eta \delta^\alpha u(x)$ is continuous and satisfies*

$$x^\eta \delta^\alpha u(x) \quad \to \quad 0 \qquad \textit{as} \quad x_j \to 0, \qquad j = 1, \dots, n, \tag{3.1}$$

$$x^\eta \delta^\alpha u(x) \quad \to \quad 0 \qquad \textit{as} \quad x_j \to +\infty, \quad j = 1, \dots, n. \tag{3.2}$$

Moreover, for every $\Gamma' \subset\subset \Gamma$ there exists $c > 0$ independent of $\hat{u}$ such that

$$\sup_{x \in \mathbb{R}^n_+, |\alpha| \leq s, \eta \in \overline{\Gamma'}} |x^\eta \delta^\alpha u(x)| \leq c\|\hat{u}\|_{s,\Gamma}, \quad \forall \hat{u} \in H_{s,\Gamma}. \tag{3.3}$$

(2) *Let $s \geq 0$ be an integer and let $u(x)$ be any bounded continuous function on $\mathbb{R}^n_+$ satisfying (3.1) and (3.2) for every $\eta \in \Gamma$. Then the Mellin transform $\hat{u}(\zeta) = M(u)(\zeta)$ of u exists and $\hat{u}(\zeta)$ is holomorphic in $\Gamma + i\mathbb{R}^n$. Moreover, for every $\Gamma' \subset\subset \Gamma'' \subset\subset \Gamma$ there exist $C > 0$ such that*

$$\langle \zeta \rangle^s |\hat{u}(\zeta)| \leq C \sup_{x \in \mathbb{R}^n_+, |\alpha| \leq s, \eta \in \overline{\Gamma''}} |x^\eta \delta^\alpha u(x)|, \qquad \forall \zeta, \Re\, \zeta \in \Gamma' \tag{3.4}$$

where $\langle \zeta \rangle = 1 + \sum_{j=1}^n |\zeta_j|$.
(3) $H_{s,\Gamma}$ *is a Banach space with the norm (2.2).*

Proof. We will prove (1). The inverse Mellin transform of $\hat{u}$ exists because $\hat{u} \in H_{s,\Gamma}$. Moreover we have

$$x^\eta \delta^\alpha u(x) = (2\pi i)^{-n} \int_{\mathbb{R}^n} (-\zeta)^\alpha \hat{u}(\zeta) x^{\eta - \zeta} d\xi, \quad \eta \in \Gamma, \Re\, \zeta \in \Gamma. \tag{3.5}$$

We take η and $\Re\zeta$ in (3.5) such that $\eta_j - \Re\, \zeta_j > 0$ if $x_j < 1$, $\eta_j - \Re\, \zeta_j < 0$ if $x_j \geq 1$. We easily see that (3.1) and (3.2) hold. The estimate (3.3) follows from (3.5) because $|x^{\eta - \zeta}|$ is bounded by some constant.

We prove (2). The conditions (3.1) and (3.2) with $\alpha = 0$ imply that the Mellin transform $M(u)(\zeta)$ exists and it is holomorphic in $\Gamma + i\mathbb{R}^n$. In order to show (3.4), we first note that the right-hand side of (3.4) is finite by (3.1) and (3.2). It follows from (3.1) and (3.2) that, for $|\alpha| \leq s$

$$\zeta^\alpha \hat{u}(\zeta) = \int u(x) \zeta^\alpha x^{\zeta - e} dx = \int u(x)(\partial_x \cdot x)^\alpha x^{\zeta - e} dx = \int_{\mathbb{R}^n_+} \delta^\alpha u(x) x^{\zeta - e} dx. \tag{3.6}$$

Let τ_j $(j = 1, \dots, n)$ be such that $\tau_j = 1$ or $\tau_j = -1$ and define $\tau = (\tau_1, \dots, \tau_n)$. We define S_τ by

$$S_\tau = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n_+; 0 \leq x_j^{\tau_j} \leq 1\}.$$

By (3.6), there exists $C' > 0$ independent of ζ such that, if $\Re\, \zeta \in \Gamma'$

$$\langle \zeta \rangle^s |\hat{u}(\zeta)| \leq C' \sup_{|\alpha| \leq s} \left| \int_{\mathbb{R}^n_+} x^{\zeta - e} \delta^\alpha u(x) dx \right| \leq C' \sup_{|\alpha| \leq s} \sum_\tau \left| \int_{S_\tau} x^{\zeta - e} \delta^\alpha u(x) dx \right|. \tag{3.7}$$

By assumption, for each S_τ we take an $\eta = \eta(\tau) = (\eta_1, \dots, \eta_n)$ and a small $\varepsilon_1 > 0$ such that $\Re\, \zeta_j - \eta_j > \varepsilon_1 > 0$ if $\tau_j = 1$, and $\Re\, \zeta_j - \eta_j \leq -\varepsilon_1$ if $\tau_j = -1$. For a given Γ'', $\Gamma' \subset\subset \Gamma'' \subset\subset \Gamma$ we can choose ε_1 so small that $\eta \in \Gamma''$.

Therefore, there exists $C'' > 0$ independent of ζ such that

$$\left|\int_{S_\tau} x^{\zeta-e}\delta^\alpha u(x)dx\right| = \left|\int_{S_\tau} x^{\zeta-e-\eta}x^\eta\delta^\alpha u(x)dx\right|$$
$$\leq \sup_{x\in\mathbb{R}^n_+} |x^\eta\delta^\alpha u(x)| \left|\int_{S_\tau} x^{\Re\zeta-e-\eta}dx\right| \leq C'' \sup_{x\in\mathbb{R}^n_+} |x^\eta\delta^\alpha u(x)|. \quad (3.8)$$

Hence there exists $C > 0$ such that

$$\langle\zeta\rangle^s|\hat{u}(\zeta)| \leq C \sup_{x\in\mathbb{R}^n_+,\eta\in\overline{\Gamma''},|\alpha|\leq s} |x^\eta\delta^\alpha u(x)|.$$

This proves (3.4).

We will prove (3). In order to show that $H_{s,\Gamma}$ is complete, suppose that $\|\hat{w}_n - \hat{w}_m\|_{s,\Gamma} \to 0$ $(m, n \to \infty)$. It follows from (3.3) and (3.4) that $\{\hat{w}_n(\zeta)\}$ converges compactly uniformly in $\Re\zeta \in \Gamma$ to a function $w(\zeta)$ holomorphic in $\zeta \in \Gamma + i\mathbb{R}^n$. Let $\eta \in \Gamma$ be arbitrarily taken and fixed. By assumption, for every $\varepsilon > 0$ there exists $N \geq 1$ such that

$$\int \langle\zeta\rangle^s|\hat{w}_n(\zeta) - \hat{w}_m(\zeta)|d\xi < \varepsilon, \quad \forall n, m \geq N.$$

It follows that, for any compact set $K \subset \mathbb{R}^n$ we have

$$\int_K \langle\zeta\rangle^s|\hat{w}_n(\zeta) - \hat{w}_m(\zeta)|d\xi < \varepsilon, \quad \forall n, m \geq N.$$

We let $m \to \infty$. Then we have $\int_K \langle\zeta\rangle^s|\hat{w}_n(\zeta) - \hat{w}(\zeta)|d\xi \leq \varepsilon$ for all $n \geq N$. Letting $K \uparrow \mathbb{R}^n$ we obtain $\int_{\mathbb{R}^n} \langle\zeta\rangle^s|\hat{w}_n(\zeta) - \hat{w}(\zeta)|d\xi \leq \varepsilon$ for all $n \geq N$. By taking the supremum with respect to $\eta \in \Gamma$, we see that $\hat{w}_n - \hat{w} \in H_{s,\Gamma}$ and $\{\hat{w}_n\}$ converges to $\hat{w}$ in $H_{s,\Gamma}$. $\square$

Now we define a smoothing operator in $\mathcal{H}_{s,\Gamma}$. Let $\phi \in C^\infty(\mathbb{R}^n)$, $0 \leq \phi \leq 1$ be a smooth function with a compact support such that $\phi \equiv 1$ in some neighborhood of the origin $x = 0$ and $\int_{\mathbb{R}^n} \phi(\sigma)d\sigma = 1$. Let $N \geq 1$, $\ell \geq 1$ be integers and let τ be an odd integer, $2\tau \geq \ell$. We set $\psi_N(\zeta) := \exp(N^{-2\tau}\sum_{j=1}^n \zeta_j^{2\tau})$ and define

$$\chi_N^\ell(\zeta) := \int_{\mathbb{R}^n} \phi(\sigma)\left\{\psi_N(\zeta)\left(e^{-\sigma\zeta/N} - \sum_{\nu=1}^{\ell}\left(-\frac{\sigma\zeta}{N}\right)^\nu \frac{1}{\nu!}\right) + (1-\psi_N(\zeta))e^{-\sigma\zeta/N}\right\}d\sigma. \quad (3.9)$$

The function $\chi_N^\ell(\zeta)$ is an entire function of ζ in $\mathbb{C}^n$ such that $\overline{\chi_N^\ell(\zeta)} = \chi_N^\ell(\bar{\zeta})$. We define a smoothing operator S_N by

$$S_N v := M^{-1}(\chi_{N+1}^\ell(\zeta)\hat{v}(\zeta)), \qquad v \in \mathcal{H}_{s,\Gamma} \quad (3.10)$$

where $\hat{v}(\zeta)$ is the Mellin transform of v and M^{-1} denotes the inverse Mellin transform. Then we have

Proposition 3.2 *Let Γ be a bounded domain. Then S_N has the following properties.*

(1) *For every $0 \leq s \leq r$ such that $r - s$ is an integer, there exists $C_r > 0$ such that*

$$\|S_N v\|_{r,\Gamma} \leq C_r(N+1)^{r-s}\|v\|_{s,\Gamma}, \qquad v \in \mathcal{H}_{s,\Gamma}.$$

(2) *For every $0 \leq s \leq r$ such that $r - s \leq \ell$ is an integer, there exists $C_r > 0$ such that*

$$\|(I - S_N)v\|_{s,\Gamma} \leq C_r(N+1)^{s-r}\|v\|_{r,\Gamma}.$$

(3) *S_N maps a real-valued function to a real-valued function.*

Proof. Proof of (1). In view of the definition of the norm $\|S_N v\|_{r,\Gamma}$ we consider

$$\int \langle\zeta\rangle^r |\chi^\ell_{N+1}(\zeta)\hat{v}(\zeta)| d\xi, \quad \zeta = \eta + i\xi,\ \eta \in \Gamma. \tag{3.11}$$

Writing $\langle\zeta\rangle^r = \langle\zeta\rangle^{r-s}\langle\zeta\rangle^s$ and recalling that $r-s$ is a nonnegative integer we have that $\langle\zeta\rangle^{r-s} = (1+\sum|\zeta_j|)^{r-s}$ is a polynomial of $|\zeta_j|$. Hence we will estimate $|\zeta^\alpha \chi^\ell_{N+1}(\zeta)|$ $(|\alpha| \le r-s)$. In view of (3.9) we consider

$$\begin{aligned} & \int \phi(\sigma)(1-\psi_{N+1}(\zeta))\zeta^\alpha e^{-\sigma\zeta/(N+1)} d\sigma \\ = \ & \int \phi(\sigma)(1-\psi_{N+1}(\zeta))(-(N+1)\partial_\sigma)^\alpha e^{-\sigma\zeta/(N+1)} d\sigma \\ = \ & (N+1)^{|\alpha|} \int \partial^\alpha_\sigma \phi(\sigma)(1-\psi_{N+1}(\zeta)) e^{-\sigma\zeta/(N+1)} d\sigma. \end{aligned} \tag{3.12}$$

In order to estimate the right-hand side, we note

$$\psi_{N+1}(\zeta) = \exp\left(\frac{1}{(N+1)^{2\tau}} \sum_{j=1}^n (\eta_j^2 + 2i\eta_j\xi_j - \xi_j^2)^\tau\right).$$

Because τ is an odd integer, $\psi_{N+1}(\eta+i\xi)$ tends to zero for $N = 1, 2, \ldots$ when ξ tends to infinity for a bounded η. Similarly, $e^{-\sigma\zeta/(N+1)}$ is bounded for $N = 1, 2, \ldots$ when $\xi \to \infty$ and η is bounded. Hence the term (3.12) can be estimated by $C_r(N+1)^{|\alpha|} \le C_r(N+1)^{r-s}$ for some constant $C_r > 0$.

We consider the term

$$I := \int \phi(\sigma)\psi_{N+1}(\zeta)\left(e^{-\sigma\zeta/(N+1)} - \sum_{\nu=1}^{\ell}\left(-\frac{\sigma\zeta}{N+1}\right)^\nu \frac{1}{\nu!}\right)\zeta^\alpha d\sigma.$$

By setting $t = (t_1, \ldots, t_n) = \zeta/(N+1)$ we have

$$I = (N+1)^{|\alpha|} \int \phi(\sigma)\psi_{N+1}(t(N+1))(e^{-t\sigma} - \sum_{\nu=1}^{\ell}(-\sigma t)^\nu (\nu!)^{-1}) t^\alpha d\sigma.$$

Because $\psi_{N+1}(t(N+1))$ is exponentially decreasing to zero when $\Im t \to \infty$ for $N = 1, 2, \ldots$, the integrand is uniformly bounded for $\xi \in \mathbb{R}^n$ and $N = 0, 1, 2, \ldots$. Therefore we see that $|\zeta^\alpha \chi^\ell_{N+1}(\zeta)|$ $(|\alpha| \le r-s)$ is bounded by $C'_r(N+1)^{r-s}$ for some constant $C'_r > 0$ which is uniform in $\eta \in \Gamma$, $N = 0, 1, 2, \ldots$ and $\xi \in \mathbb{R}^n$, $|\xi| \to \infty$. It follows that $|\langle\zeta\rangle^{r-s}\chi^\ell_{N+1}(\zeta)|$ is bounded by $C_r(N+1)^{r-s}$ for some constant $C_r > 0$ which is uniform in $\eta \in \Gamma$, $\xi \in \mathbb{R}^n$ and $N = 0, 1, 2, \ldots$ By (3.11) we obtain (1).

Proof of (2). By (3.10) we have

$$\|(I - S_N)v\|_{s,\Gamma} = \|(I - \chi^\ell_{N+1})\hat{v}\|_{s,\Gamma}.$$

For the sake of simplicity we set $s-r=a\le 0$. Recalling that $\int\phi(\sigma)d\sigma=1$ and $r-s\le\ell$ we have

$$\begin{aligned}\chi_{N+1}^{\ell}(\zeta)-1 &= \int\phi(\sigma)\psi_{N+1}(\zeta)\left(e^{-\sigma\zeta/(N+1)}-\sum_{\nu=1}^{\ell}\left(-\frac{\sigma\zeta}{N+1}\right)^{\nu}\frac{1}{\nu!}\right)d\sigma \\ &+ \int\phi(\sigma)(1-\psi_{N+1}(\zeta))e^{-\sigma\zeta/(N+1)}d\sigma-\int\phi(\sigma)d\sigma \\ &= \int\phi(\sigma)\psi_{N+1}(\zeta)\left(e^{-\sigma\zeta/(N+1)}-\sum_{\nu=0}^{\ell}\left(-\frac{\sigma\zeta}{N+1}\right)^{\nu}\frac{1}{\nu!}\right)d\sigma \\ &+ \int\phi(\sigma)(1-\psi_{N+1}(\zeta))e^{-\sigma\zeta/(N+1)}d\sigma-\int\phi(\sigma)d\sigma+\int\phi(\sigma)\psi_{N+1}(\zeta)d\sigma \\ &= \int\phi(\sigma)\psi_{N+1}(\zeta)\left(e^{-\sigma\zeta/(N+1)}-\sum_{\nu=0}^{\ell}\left(-\frac{\sigma\zeta}{N+1}\right)^{\nu}\frac{1}{\nu!}\right)d\sigma \\ &+ \int\phi(\sigma)(1-\psi_{N+1}(\zeta))(e^{-\sigma\zeta/(N+1)}-1)d\sigma\equiv I_1+I_2.\end{aligned}\tag{3.13}$$

By the definition of the norm we consider

$$(\chi_{N+1}^{\ell}(\zeta)-1)\langle\zeta\rangle^{a}=\langle\zeta\rangle^{a}I_1+\langle\zeta\rangle^{a}I_2.$$

As to the term $\langle\zeta\rangle^{a}I_1$, we set $\zeta=t(N+1)$. Then the integrand is equal to

$$\phi(\sigma)\langle Nt+t\rangle^{a}\psi_{N+1}(tN+t)(e^{-t\sigma}-\sum_{\nu=0}^{\ell}(-\sigma t)^{\nu}(\nu!)^{-1}).$$

We note that

$$\langle Nt+t\rangle=1+(N+1)\sum_j|t_j|.$$

If $\sum_j|t_j|\ge\varepsilon>0$ for some ε, we have $\langle Nt+t\rangle\ge(N+1)\varepsilon$. Hence it follows that $\langle Nt+t\rangle^{a}\le(N+1)^{a}\varepsilon^{a}$. Because $\psi_{N+1}(tN+t)$ is an exponentially decreasing function of $(\Im t_j)^{2\tau}$ when $\Im t\to\infty$ the integrand is bounded by $C(N+1)^{a}$ for some $C>0$ independent of t.

Next we consider the case $\sum_j|t_j|<\varepsilon$. Because $\langle Nt+t\rangle\ge(N+1)\sum_j|t_j|$ we have

$$\langle Nt+t\rangle^{a}\le(N+1)^{a}(\sum_j|t_j|)^{a}.$$

Hence we have

$$(|t_1|+\cdots+|t_n|)^{a}(e^{-t\sigma}-\sum_{\nu=0}^{\ell}(-\sigma t)^{\nu}(\nu!)^{-1})=(|t_1|+\cdots+|t_n|)^{a}\sum_{\nu=\ell+1}^{\infty}(-\sigma t)^{\nu}(\nu!)^{-1}.\tag{3.14}$$

Noting that

$$|\sigma t|\le(|t_1|+\cdots+|t_n|)(|\sigma_1|+\cdots+|\sigma_n|)$$

and $-a\le\ell$, the right-hand side of (3.14) is bounded by some constant independent of t. Hence $\langle\zeta\rangle^{a}I_1$ is estimated by $C(N+1)^{a}$ for some $C>0$ independent of ζ.

We will estimate $\langle\zeta\rangle^{a}I_2$. By setting $\zeta=t(N+1)$ we consider the term

$$J\equiv(1-\exp(\sum t_j^{2\tau}))(e^{-t\sigma}-1).$$

If $\sum |t_j| > \varepsilon > 0$, we have $\langle Nt+t\rangle^a \le \varepsilon^a (N+1)^a$. Hence we see that $\langle Nt+t\rangle^a J$ is estimated by $C(N+1)^a$ for some $C>0$ independent of t. In case $\sum |t_j| \le \varepsilon$ we have

$$\langle Nt+t\rangle^a \le (N+1)^a (\sum |t_j|)^a.$$

Because J can be divided by $\sum t_j^{2\tau}$ and $-a \le \ell \le 2\tau$, it follows that $(\sum |t_j|)^a J$ is bounded by some constant independent of t. Hence $\langle\zeta\rangle^a I_2$ can be estimated by $C(N+1)^a$. Hence $(\chi^\ell_{N+1}-1)\langle\zeta\rangle^a$ is bounded by $C(N+1)^a$. Because

$$(1-\chi^\ell_{N+1})\hat{v}\langle\zeta\rangle^s = -(\chi^\ell_{N+1}-1)\langle\zeta\rangle^a \hat{v}\langle\zeta\rangle^r$$

we obtain (2).

Proof of (3). We note that f is real-valued if and only if $\overline{M(f)(\zeta)} = M(f)(\bar{\zeta})$. Hence it is sufficient to show that $\overline{\chi^\ell_{N+1}(\zeta)\hat{v}(\zeta)} = \chi^\ell_{N+1}(\bar{\zeta})\hat{v}(\bar{\zeta})$. The last relation follows from the definition of the smoothing operator and the assumption on v. □

Lemma 3.3 *Let s be a positive integer and let Γ_1, Γ_2 and Γ be open connected sets such that $\Gamma \subset \Gamma_1 + \Gamma_2$. Let $u \in \mathcal{H}_{s,\Gamma_1}$ and $v \in \mathcal{H}_{s,\Gamma_2}$. Then, it follows that $uv \in \mathcal{H}_{s,\Gamma}$ and the following estimate holds*

$$\|uv\|_{s,\Gamma} \le \|u\|_{s,\Gamma_1}\|v\|_{s,\Gamma_2}.$$

Proof. Suppose that $f(x)\prod_{j=1}^n x_j^{\eta_j-1}$ $(\eta \in \mathbb{R}^n)$ is an integrable function on $\mathbb{R}^n_+$. Let $M(f)(\eta+i\xi)$ be the Mellin transform of f

$$M(f)(\eta+i\xi) = \int_{\mathbb{R}^n_+} f(x)\prod_{j=1}^n x_j^{\eta_j-1+i\xi_j}dx = \int_{\mathbb{R}^n_+} f(x)\prod_{j=1}^n x_j^{\eta_j-1}\prod_{j=1}^n x_j^{i\xi_j}dx.$$

We set $t_j = \log x_j$, $t=(t_1,\dots,t_n)$. Noting that $dt = \prod_{j=1}^n x_j^{-1}dx$ we have the expression

$$M(f)(\eta+i\xi) = \int_{\mathbb{R}^n} e^{\eta\cdot t} f(e^{t_1},\dots,e^{t_n})e^{i\xi t}dt = \mathcal{F}^{-1}(e^{\eta\cdot}f(e^{\cdot}))(\xi),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Let $\hat{u}_j = \mathcal{F}^{-1}(u_j)$, $(j=1,2)$. We assume $\hat{u}_j \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Let

$$\hat{u}_1 * \hat{u}_2 := \int \hat{u}_1(\xi)\hat{u}_2(\eta-\xi)d\xi$$

be the convolution of $\hat{u}_1$ and $\hat{u}_2$. We can easily show that $\hat{u}_1 * \hat{u}_2 = \mathcal{F}^{-1}(u_1u_2)$.

Let $\gamma' \in \Gamma$. By assumption $\Gamma \subset \Gamma_1+\Gamma_2$, we have $\gamma' = \gamma_1+\gamma_2$, $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2$. If we set $\gamma_2 = \gamma$, we obtain $\gamma_1 = \gamma'-\gamma$. Hence we have the expression

$$\gamma' = \gamma + \gamma' - \gamma, \quad \gamma \in \Gamma_2, \quad \gamma'-\gamma \in \Gamma_1.$$

By the conditions $u \in \mathcal{H}_{s,\Gamma_1}$ and $v \in \mathcal{H}_{s,\Gamma_2}$ and Proposition 3.1 we have

$$e^{\gamma t}v(e^t) \in L^1 \cap L^2 \quad \text{and} \quad e^{(\gamma'-\gamma)t}u(e^t) \in L^1\cap L^2.$$

It follows that

$$\begin{aligned} & \mathcal{F}^{-1}(e^{\gamma\cdot}v(e^{\cdot})e^{(\gamma'-\gamma)\cdot}u(e^{\cdot})) = \mathcal{F}^{-1}(e^{\gamma'\cdot}v(e^{\cdot})u(e^{\cdot})) = M(vu)(\gamma'+i\xi) \\ = \; & \mathcal{F}^{-1}(e^{\gamma\cdot}v(e^{\cdot})) * \mathcal{F}^{-1}(e^{(\gamma'-\gamma)\cdot}u(e^{\cdot})) = (M(v)(\gamma+i\cdot) * M(u)(\gamma'-\gamma+i\cdot))(\xi). \end{aligned}$$

Hence we have

$$M(vu)(\gamma'+i\xi) = (M(v)(\gamma+i\cdot) * M(u)(\gamma'-\gamma+i\cdot))(\xi).$$

We note

$$\langle\zeta\rangle = 1 + \sum |\zeta_j| \le 1 + \sum(|\eta_j| + |\zeta_j - \eta_j|) \le \langle\eta\rangle\langle\zeta-\eta\rangle.$$

Let

$$\Re\,\zeta = \gamma' \in \Gamma, \quad \Re\,(\zeta-\eta) = \gamma'-\gamma \in \Gamma_1,\ \Re\,\eta = \gamma \in \Gamma_2,\ \Im\,\zeta = \xi,\ \Im\,\eta = \eta'.$$

Then, by the definition of the convolution we have

$$\begin{aligned}\|uv\|_{s,\Gamma} = [\![M(uv)]\!]_{s,\Gamma} &= \sup_{\Re\,\zeta=\gamma'\in\Gamma}\int \langle\zeta\rangle^s |M(uv)(\gamma'+i\xi)|d\xi \\ \le\ & \sup \int \langle\eta\rangle^s\langle\zeta-\eta\rangle^s \int |M(u)(\gamma'-\gamma+i(\xi-\eta'))|\,|M(v)(\gamma+i\eta')|d\eta' d\xi.\end{aligned}$$

By Fubini's theorem the right-hand side is estimated in the following way

$$\begin{aligned}\le\ & \sup \int \langle\zeta-\eta\rangle^s|M(u)(\zeta-\eta)|d\xi \int \langle\eta\rangle^s|M(v)(\eta)|d\eta' \\ \le\ & [\![M(u)]\!]_{s,\Gamma_1}[\![M(v)]\!]_{s,\Gamma_2} = \|u\|_{s,\Gamma_1}\|v\|_{s,\Gamma_2}.\square\end{aligned}$$

Let $\phi_0(t)$ be a smooth function on $\mathbb{R}$ such that $\phi_0 \equiv 1$ for $|t| \le 1/2$ and $\operatorname{supp}\phi_0 \subset \{|t| \le 1\}$. We define $\phi(x) := \phi_0(x_1)\cdots\phi_0(x_n)$. Then we have

Lemma 3.4 *Assume that* $0 \in \Gamma$. *Let* $\sigma \ge 0$ *and let* $g \in (\mathcal{H}_{\sigma+n+1,\Gamma})^N$. *We define* $\tilde{g}(x) := g(x)\phi(x/\lambda)$, *where* $\lambda > 0$. *Then for every* $\Gamma' \subset\subset \Gamma$ *we have* $\|\tilde{g}\|_{\sigma,\Gamma'} \to 0$ *when* $\lambda \to 0$.

Proof. We define $h(x)$ by $h(x) = g(x)x^\eta$, where $\eta \in \Gamma$. Then, by Proposition 3.1 and the Leibnitz rule $\delta^\beta h(x)$ is continuous for every β, $|\beta| \le \sigma+n+1$. For every $\zeta = (\zeta_1,\dots,\zeta_n)$ such that $\Re\,\zeta \in \Gamma'$ we take $\eta = (\eta_1,\dots,\eta_n) \in \Gamma$ such that $\eta_j < \Re\,\zeta_j$ for $j = 1,\dots,n$. Then we have

$$M(\tilde{g})(\zeta) = \int_{\mathbb{R}^n_+} \tilde{g}(x)x^{\zeta-e}dx = \int_{\mathbb{R}^n_+} h(x)\phi(x/\lambda)x^{\zeta-\eta-e}dx, \quad e = (1,\dots,1),\ \Re\,\zeta \in \Gamma'. \tag{3.15}$$

By the assumption $0 \in \Gamma$ and Proposition 3.1 we have $\delta^\beta g(x) = \delta^\beta(h(x)x^{-\eta})$ vanishes as $x_j \to 0$ $(j = 1,\dots,n)$. Because the support of $\tilde{g}$ is contained in $\{|x_j| \le \lambda, j = 1,\dots,n\}$ there exists $c > 0$ such that, for $|\alpha| = \sigma+n+1$

$$\begin{aligned}|\zeta^\alpha M(\tilde{g})(\zeta)| &= \left|\int h(x)\phi(x/\lambda)x^{-\eta}\zeta^\alpha x^{\zeta-e}dx\right| \qquad (3.16)\\ =\ & \left|\int h(x)\phi(x/\lambda)x^{-\eta}(\partial\cdot x)^\alpha x^{\zeta-e}dx\right| = \left|\int \delta^\alpha(h(x)\phi(x/\lambda)x^{-\eta})x^{\zeta-e}dx\right| \\ \le\ & c\max_{|\beta|\le\sigma+n+1,|x_j|\le\lambda}|\delta^\beta h(x)|\max_{|x_j|\le\lambda,|\beta|\le\sigma+n+1}|\delta^\beta\phi(x/\lambda)||\eta|^{\sigma+n+1}\lambda^{\mathrm{Re}\zeta_1+\cdots+\mathrm{Re}\zeta_n-\eta_1-\cdots-\eta_n},\end{aligned}$$

where we used

$$\int |x^{\zeta-\eta-e}|dx \le C\lambda^{\mathrm{Re}\zeta_1+\cdots+\mathrm{Re}\zeta_n-\eta_1-\cdots-\eta_n}$$

for some $C > 0$ independent of λ. Because $\delta^\beta h(x)$ is continuous on $\mathbb{R}^n_+$, we can easily see that the quantity $\max|\delta^\beta h(x)|$ is bounded in λ when $\lambda \le 1$. Similarly, the term $\max|\delta^\beta\phi(x/\lambda)|$ is uniformly bounded in λ when $\lambda \le 1$. Indeed, we have

$$\delta_j\phi_j(\frac{x_j}{\lambda}) = x_j\partial_{x_j}\phi_j(\frac{x_j}{\lambda}) = \frac{x_j}{\lambda}\phi_j'(\frac{x_j}{\lambda}).$$

This is uniformly bounded in λ by the condition of the support of ϕ_j. Therefore it follows from (3.16) that there exists $\varepsilon(\lambda)$, $\varepsilon(\lambda) \to 0$ $(\lambda \to 0)$ such that

$$\langle\zeta\rangle^{\sigma+n+1}|M(\tilde{g})(\zeta)| \le \varepsilon(\lambda).$$

It follows that $\|\tilde{g}\|_\sigma \to 0$ when $\lambda \to 0$. $\square$

Proposition 3.5 *Suppose that an open set Γ in $\mathbb{R}^n$ contains a sequence $\eta^i = (\eta_1^i, \dots, \eta_n^i)$ $(i = 1, 2, \dots)$ such that $\eta_j^i \to -\infty$ as $k \to \infty$ for $j = 1, \dots, n$. Moreover, suppose that $u \in \bigcap_{\nu=0}^{\infty} (\mathcal{H}_{\nu,\Gamma})^N$. Then, by extending u as 0 for $x \notin \mathbb{R}_+^n$ we have*

$$(x_1 \cdots x_n)^{-k} u \in C^\infty(\mathbb{R}^n) \quad \textit{for all} \quad k = 0, 1, 2, \dots$$

Proof. Let $\alpha \in \mathbb{Z}_+^n$ and an integer $k \geq 0$ be arbitrarily given. We take η such that $\eta \in \Gamma$ and $-\eta - (k, \dots, k) > \alpha$. By the assumption and Proposition 3.1 $x^\eta \delta^\alpha u(x)$ is continuous on $\mathbb{R}_+^n$. Because $x^\alpha \partial_x^\alpha u$ is a linear combination of $\delta^\beta u$ $(|\beta| \leq |\alpha|)$ $x^\eta x^\alpha \partial_x^\alpha u$ is continuous on $\mathbb{R}_+^n$. Because $\alpha + \eta < -(k, \dots, k)$ we see that $(x_1 \cdots x_n)^{-k} \partial_x^\alpha u$ is continuous on $\mathbb{R}_+^n$. □

Lemma 3.6 *(Interpolation estimate) Let $\sigma \geq \tau \geq 0$, $\sigma \neq 0$, and $r \geq 0$. Let Γ be an open set in $\mathbb{R}^n$ containing the origin. Suppose $u \in (\mathcal{H}_{\sigma+r,\Gamma})^N$. Then we have*

$$\|u\|_{\tau+r,\Gamma} \leq \|u\|_{\sigma+r,\Gamma}^{\tau/\sigma} \|u\|_{r,\Gamma}^{1-\tau/\sigma}.$$

Proof. Because the inequality is trivial in case $\sigma = \tau$ or $\tau = 0$ we assume that $\sigma > \tau > 0$. We set $p = \sigma/\tau$ and $q = (1 - p^{-1})^{-1}$. Then we have $p^{-1} + q^{-1} = 1$ and, by Hölder's inequality we obtain, for $\xi = \Im\zeta$

$$\begin{aligned} &\int \langle\zeta\rangle^{\tau+r} |\hat{u}(\zeta)| d\xi = \int \langle\zeta\rangle^{\tau+r} |\hat{u}(\zeta)|^{1/p+1/q} d\xi \qquad (3.17) \\ \leq \quad &\left(\int \langle\zeta\rangle^{\sigma+r} |\hat{u}(\zeta)| d\xi\right)^{1/p} \left(\int \langle\zeta\rangle^{r} |\hat{u}(\zeta)| d\xi\right)^{1/q} \leq [\![\hat{u}]\!]_{\sigma+r,\Gamma}^{\tau/\sigma} [\![\hat{u}]\!]_{r,\Gamma}^{1-\tau/\sigma} = \|u\|_{\sigma+r,\Gamma}^{\tau/\sigma} \|u\|_{r,\Gamma}^{1-\tau/\sigma}. \end{aligned}$$

4 Proof of Theorem 2.1

For the sake of simplicity we denote by $P(\delta)$ the diagonal matrix with diagonal elements $p_1(\delta), \dots, p_N(\delta)$ in this order. Then we can write (2.1) in the form

$$P(\delta)u + a(x, \delta^\alpha u) = 0.$$

Let $u \in (\mathcal{H}_{\sigma+s,\Gamma})^N$, and let L_u be the linearized operator of (2.1) at u. We consider

$$L_u v \equiv P(\delta)v + Y \cdot \nabla_z a(x, z)|_{z=(\delta^\alpha u), Y=(\delta^\alpha v)} = g. \qquad (4.1)$$

where $v = (v_1, \dots, v_N)$. Then we have

Proposition 4.1 *Let σ be a nonnegative integer. Assume (A.2) and suppose that (A.1) holds for some bounded domain Γ containing the origin. Moreover, assume that $\nabla_z a(x, 0) \in (\mathcal{H}_{\sigma,\Gamma})^{kN}$, where*

$$k = \#\{\alpha \in \mathbb{Z}_+^n; |\alpha| \leq s\}.$$

Then there exists $\varepsilon > 0$ such that, if

$$\|u\|_{\sigma+s+n+1,\Gamma} < \varepsilon \quad \textit{and} \quad \|\nabla_z a(\cdot, 0)\|_{\sigma,\Gamma} < \varepsilon,$$

then Eq. (4.1) has a solution $v \in (\mathcal{H}_{\sigma+s,\Gamma})^N$ for every $g \in (\mathcal{H}_{\sigma,\Gamma})^N$. Moreover, there exists $C > 0$ independent of g such that

$$\|v\|_{\sigma+s,\Gamma} \leq C\|g\|_{\sigma,\Gamma}, \quad \forall g \in (\mathcal{H}_{\sigma,\Gamma})^N. \qquad (4.2)$$

The solution v is real-valued if g and u are real-valued.

In order to prove Proposition 4.1 we prepare a lemma.

Lemma 4.2 *(Superposition estimate) Let σ be a nonnegative integer. Suppose that $0 \in \Gamma$ and (A.2) is fulfilled. Let $\gamma \in \mathbb{Z}_+^n$ and let the integer k be given in Proposition 4.1. Assume that $\nabla_z a(x,0) \in (\mathcal{H}_{\sigma+|\gamma|,\Gamma})^{kN}$. Then, for every neighborhood Γ_0 of the origin in $\mathbb{R}^n$, $\Gamma_0 \subset\subset \Gamma$ there exists $\varepsilon > 0$ such that , if $\|u\|_{s+n+1,\Gamma} < \varepsilon$ and $u \in (\mathcal{H}_{\sigma+s+n+1,\Gamma})^N$, we have*

$$(\nabla_z \delta_x^\gamma a)(\cdot, \delta^\alpha u) \in (\mathcal{H}_{\sigma,\Gamma_0})^{kN}.$$

Moreover there exists $C > 0$ independent of γ such that

$$\|(\nabla_z \delta_x^\gamma a)(\cdot, \delta^\alpha u)\|_{\sigma,\Gamma_0} \le \|\nabla_z a(\cdot,0)\|_{\sigma+|\gamma|,\Gamma_0} + C\|u\|_{\sigma+s+n+1,\Gamma}, \quad \forall u \in (\mathcal{H}_{\sigma+s+n+1,\Gamma})^N. \tag{4.3}$$

Proof. Because $0 \in \Gamma$ it follows from (1) of Proposition 3.1 that $\delta^\alpha u$ ($|\alpha| \le s$) are bounded and continuous on $\mathbb{R}_+^n$, and vanish at the origin. Moreover, by (3.3) $\sup |\delta^\alpha u|$ ($|\alpha| \le s$) are bounded by $\|u\|_{s,\Gamma} \le \|u\|_{s+n+1,\Gamma} < \varepsilon$. Hence, by taking $\varepsilon > 0$ sufficiently small, the function $(\nabla_z \delta_x^\gamma a)(x, \delta^\alpha u)$ is well-defined as a continuous function.

It follows that

$$\|(\nabla_z \delta_x^\gamma a)(\cdot, \delta^\alpha u)\|_{\sigma,\Gamma_0} \le \|(\nabla_z \delta_x^\gamma a)(\cdot, 0)\|_{\sigma,\Gamma_0} + \|(\nabla_z \delta_x^\gamma a)(\cdot, \delta^\alpha u) - (\nabla_z \delta_x^\gamma a)(\cdot, 0)\|_{\sigma,\Gamma_0}. \tag{4.4}$$

The first term in the right-hand side of (4.4) is bounded by $\|\nabla_z a(\cdot,0)\|_{\sigma+|\gamma|,\Gamma_0}$. Hence we will consider the second term. By Taylor's formula, it is equal to, with $z = (\delta^\alpha u)$,

$$\|\int_0^1 z \cdot (\nabla_z^2 \delta_x^\gamma a)(\cdot, t\delta^\alpha u) dt\|_{\sigma,\Gamma_0} \le \sup_t \|z \cdot (\nabla_z^2 \delta_x^\gamma a)(\cdot, t\delta^\alpha u)\|_{\sigma,\Gamma_0}. \tag{4.5}$$

In order to estimate the right-hand side it is sufficient to estimate

$$\|\delta^\beta (z \cdot \nabla_z^2 (\delta_x^\gamma a)(\cdot, t\delta^\alpha u))\|_{0,\Gamma_0} \quad \text{for } |\beta| \le \sigma.$$

We first consider the case $\beta = 0$. Let $\phi_0(t) \in C^\infty(\mathbb{R})$, $\phi_0(t) > 0$ be a smooth function such that $\phi_0 \equiv 1$ in some neighborhood of the origin $t = 0$ and that $\phi_0(t) = t^{-\tau}$ when $|t| \gg 1$, where $\tau > 0$ is a small constant to be chosen later. We define $\phi(x) := \prod_{j=1}^n \phi_0(x_j)$ and write

$$z \cdot \nabla_z^2 \delta_x^\gamma a(x, tz) = \phi^{-1}(x) z \cdot \phi \nabla_z^2 \delta_x^\gamma a(x, tz).$$

We take Γ_1 ($\Gamma_0 \subset\subset \Gamma_1 \subset\subset \Gamma$) and $\Gamma_2 \subset\subset \Gamma_2' := \{\eta > 0; 0 < \eta_j < \tau\}$ such that $\Gamma_0 \subset\subset \Gamma_1 + \Gamma_2' \subset\subset \Gamma$, by taking τ sufficiently small. In view of Lemma 3.3 we estimate $\|\phi^{-1} z\|_{0,\Gamma_1}$ and $\|\phi \nabla_z^2 \delta_x^\gamma a\|_{0,\Gamma_2}$.

We consider $\|\phi^{-1} z\|_{0,\Gamma_1}$. We set $s = n+1$ and $u = \phi^{-1} z$ in (3.4). By assumption $\Gamma_1 + \Gamma_2' \subset\subset \Gamma$ we take Γ'' such that $\Gamma_1 \subset\subset \Gamma'' \subset\subset \Gamma$, $\overline{\Gamma''} + \Gamma_2' \subset\subset \Gamma$. Then, by integration we have, for some $C > 0$,

$$\begin{aligned} \|\phi^{-1} z\|_{0,\Gamma_1} &= \left\| M(\phi^{-1} z) \right\|_{0,\Gamma_1} \\ &\le C \sup_{x, |\alpha| \le n+1, \eta \in \overline{\Gamma''}} |x^\eta \delta^\alpha (\phi^{-1} z)| \sup_{\Re \zeta \in \Gamma_1} \int \langle \zeta \rangle^{-n-1} d\xi, \quad \xi = \Im \zeta. \end{aligned}$$

Because $\phi^{-1}(x)$ has the growth x_j^τ near $x_j = \infty$ we have, for some $C_1 > 0$,

$$\sup_{x, |\alpha| \le n+1, \eta \in \overline{\Gamma''}} |x^\eta \delta^\alpha (\phi^{-1} z)| \le C_1 \sup_{x, |\alpha| \le n+1, \eta \in \overline{\Gamma''}} |x^{\eta + \tau e} \delta^\alpha z| \tag{4.6}$$

where $e = (1, \dots, 1)$. By Proposition 3.1, (3.3) with $\Gamma' \supset\supset \overline{\Gamma''} + \Gamma_2'$, $\Gamma' \subset\subset \Gamma$, the right-hand side term can be bounded by $C_2 \|z\|_{n+1,\Gamma}$ for some $C_2 > 0$. By the definition of z, the term $C_2 \|z\|_{n+1,\Gamma}$ is bounded by $C_3 \|u\|_{s+n+1,\Gamma}$ for some $C_3 > 0$.

Next we will estimate $\|\phi \nabla_z^2 \delta_x^\gamma a\|_{0,\Gamma_2}$ we note, from (A.2), that $\phi \nabla_z^2 \delta_x^\gamma a$ decays faster than or equal to $\prod x_j^{-\tau}$ when $x \to \infty$. By Proposition 3.1 we consider

$$\sup_{x \in \mathbb{R}_+^n, \eta \in \Gamma_2'', |\lambda| \le n+1} |x^\eta \delta_x^\lambda (\phi \nabla_z^2 \delta_x^\gamma a)|, \qquad \Gamma_2 \subset\subset \Gamma_2'' \subset\subset \Gamma_2' \subset\subset \Gamma. \tag{4.7}$$

We take Γ_2'' sufficiently small that $\eta_j - \tau < 0$ for every $\eta \in \Gamma_2''$. In the term $x^\eta \delta_x^\lambda(\phi \nabla_z^2 \delta_x^\gamma a)$, the terms which appear when the differentiation δ^λ is applied to ϕ are bounded, because the decay order of $x^\eta x_j \partial_{x_j} \phi$ when $x_j \to \infty$ is $x_j^{\eta_j - \tau}$. On the other hand the term $(\nabla_z^2 \delta_x^\gamma a)(x, t\delta^\alpha u)$ is bounded by some constant independent of γ because of (A.2). Hence the norm is bounded. Next we consider the case when the differentiation δ^λ is applied to the x variable of $\delta_x^\gamma a$. We can easily see that these terms are bounded by some constant independent of γ in view of (A.2). On the other hand, if the differentiation δ^λ is applied to $\delta^\alpha u$ in $(\nabla_z^2 \delta_x^\gamma a)(x, t\delta^\alpha u)$, these terms are bounded by some constant independent of γ by the assumption $\|u\|_{s+n+1,\Gamma} < \varepsilon$ and Lemma 3.3. It follows that $x^\eta \delta_x^\lambda(\phi \nabla_z^2 \delta_x^\gamma a)$ is bounded by some constant independent of γ. Therefore we see that $z \cdot \nabla_z^2(\delta_x^\gamma a) \in (\mathcal{H}_{0,\Gamma_0})^{kN}$, and satisfies the estimate

$$\|z \cdot \nabla_z^2(\delta_x^\gamma a)\|_{0,\Gamma_0} \le C\|u\|_{s+n+1,\Gamma}$$

for some $C > 0$ independent of γ. By (4.5) and (4.4) we see that $\nabla_z(\delta_x^\gamma a) \in (\mathcal{H}_{0,\Gamma_0})^{kN}$ and (4.3) holds.

In the general case $\beta \neq 0$, the term $\delta^\beta(z \cdot \nabla_z^2 \delta_x^\gamma a)(\cdot, tz))$ is the sum of products of the terms

$$\delta_x^\varepsilon((\partial_z^\gamma \nabla_z^2 \delta_x^\gamma a)\phi), \quad \prod_{j=1}^{\nu} \delta^{\alpha_j+\beta_j} u, \quad (|\alpha_j| \le s, \ \sum_1^\nu \beta_j \le \beta, \ \alpha_j, \beta_j \in \mathbb{Z}_+^n),$$

and the differentiations of ϕ^{-1} for some multiintegers γ and ε.

The term $\delta_x^\varepsilon((\partial_z^\gamma \nabla_z^2 \delta_x^\gamma a)\phi)$ can be estimated by (A.2) by the same argument as in the case $\beta = 0$. Concerning the products of the differentiations of ϕ^{-1} and $\prod \delta^{\alpha_j+\beta_j} u$, the argument in the case $\beta = 0$ implies that one may consider $\|\prod \delta^{\alpha_j+\beta_j} u\|_{n+1,\Gamma}$ when $u \in (\mathcal{H}_{\sigma+s+n+1,\Gamma_1})^N$ in view of the estimate of $\|\phi^{-1}\delta^{\alpha_j+\beta_j} u\|_{0,\Gamma_1}$. In the following we omit the suffix Γ of the norm for the sake of simplicity. By Lemma 3.3 and $|\alpha_j| \le s$ we have

$$\|\prod \delta^{\alpha_j+\beta_j} u\|_{n+1} \le \prod_j \|\delta^{\alpha_j+\beta_j} u\|_{n+1} \le \prod_j \|u\|_{|\alpha_j|+|\beta_j|+n+1} \le \prod_j \|u\|_{|\beta_j|+s+n+1}.$$

We set $a = \sum_j |\beta_j|$ $(a \le |\beta| \le \sigma)$. By Lemma 3.6 with $r = s+n+1$, $\tau = |\beta_j|$ we see that the right-hand side is estimated in the following way

$$\le \prod \|u\|_{a+s+n+1}^{|\beta_j|/a} \|u\|_{s+n+1}^{1-|\beta_j|/a} \le \|u\|_{a+s+n+1}^{\sum |\beta_j|/a} \|u\|_{s+n+1}^{1-\sum |\beta_j|/a} = \|u\|_{a+s+n+1} \le \|u\|_{\sigma+s+n+1}.$$

Summing up the above, the second term of the right-hand side of (4.4) can be bounded by $C\|u\|_{\sigma+s+n+1}$ for some $C > 0$ independent of u and γ. Therefore we see that $\nabla_z(\delta_x^\gamma a)(\cdot, \delta^\alpha u) \in (\mathcal{H}_{0,\Gamma_0})^{kN}$, and (4.3) holds. □

Proof of Proposition 4.1. We apply the Mellin transform to the equation $L_u v = g$. Then we have

$$P(\zeta)\hat{v}(\zeta) + \sum_\alpha \hat{q}_\alpha(\zeta) * (-\zeta)^\alpha \hat{v}(\zeta) = \hat{g}(\zeta), \quad \zeta = \eta + i\xi, \ \eta \in \Gamma, \ \xi \in \mathbb{R}^n, \tag{4.8}$$

where $*$ denotes the convolution and

$$\hat{q}_\alpha(\zeta) = M(q_\alpha)(\zeta), \quad q_\alpha(\zeta) = (\partial a/\partial z_\alpha)(x, \delta^\alpha u).$$

Because $a(x,z)$ is real-valued it follows that $\hat{q}_\alpha$ is real-valued if u is real-valued, namely $\overline{\hat{q}_\alpha(\zeta)} = \hat{q}_\alpha(\bar{\zeta})$. Because $P(\zeta)^{-1}$ exists for $\zeta = \eta + i\xi \in \Gamma + i\mathbb{R}^n$ by assumption it follows from (4.8) that

$$\hat{v}(\zeta) + P(\zeta)^{-1} \sum_\alpha \hat{q}_\alpha * ((-\zeta)^\alpha \hat{v}(\zeta)) = P(\zeta)^{-1}\hat{g}(\zeta). \tag{4.9}$$

We define the sequence $\{\hat{v}_k\}$ inductively, by

$$\begin{aligned} \hat{v}_0 &= P(\zeta)^{-1}\hat{g}, \ \hat{v}_1 = -A\hat{v}_0, \hat{v}_{k+1} = -A\hat{v}_k, \ k = 0, 1, 2, \dots, \\ A\hat{v}(\zeta) &= P(\zeta)^{-1} \sum_\alpha \hat{q}_\alpha(\zeta) * ((-\zeta)^\alpha \hat{v}(\zeta)). \end{aligned} \tag{4.10}$$

Indeed, by Lemma 4.2 we have $q_\alpha \in (\mathcal{H}_{\sigma,\Gamma_0})^N$ for all α, $|\alpha| \le s$ and some $\Gamma_0 \subset\subset \Gamma$. Because

$$\hat{q}_\alpha(\zeta) * ((-\zeta)^\alpha \hat{v}(\zeta)) = M(q_\alpha \delta^\alpha v)(\zeta),$$

we see, from Lemma 3.3, that $\hat{v}_k \in (H_{\sigma+s,\Gamma})^N$ for $k = 0, 1, 2, \ldots$. Moreover, $\hat{v}_k$ is real-valued if g and u are real-valued. If there exists a limit $\hat{v} = \sum_{k=0}^\infty \hat{v}_k \in (H_{\sigma+s,\Gamma})^N$, $\hat{v}$ is a real-valued solution of (4.8). Indeed, we have

$$\hat{v} + P^{-1} \sum_\alpha (\hat{q}_\alpha * (-\zeta)^\alpha \hat{v}) = \sum \hat{v}_k + \sum A\hat{v}_k = \sum (\hat{v}_k - \hat{v}_{k+1}) = \hat{v}_0 = P(\zeta)^{-1} \hat{g}(\zeta). \tag{4.11}$$

Therefore $v = M^{-1}(\hat{v})(\zeta)$ gives a solution of $L_u v = g$.

By Lemmas 3.3, 4.2 and the assumption we have, for some $C > 0$ and $C_1 > 0$

$$\begin{aligned} \|v_{k+1}\|_{\sigma+s,\Gamma} &= \left\| A\hat{v}_k \right\|_{\sigma+s,\Gamma} \le C \sum_\alpha \left\| \hat{q}_\alpha * (-\zeta)^\alpha \hat{v}_k \right\|_{\sigma,\Gamma} = C \sum_\alpha \|q_\alpha \delta^\alpha v_k\|_{\sigma,\Gamma} \\ &\le C \sum_\alpha \|q_\alpha\|_{\sigma,\Gamma_0} \|v_k\|_{\sigma+s,\Gamma} \\ &\le C \sum_\alpha (\|\nabla_z a(\cdot, 0)\|_{\sigma,\Gamma_0} + C_1 \|u\|_{\sigma+s+n+1,\Gamma}) \|v_k\|_{\sigma+s,\Gamma} \equiv K \|v_k\|_{\sigma+s,\Gamma}. \end{aligned}$$

We choose $\varepsilon > 0$ so small that $K \le 1/2$. It follows that the sum $\hat{v} = \sum \hat{v}_k$ converges in $(H_{\sigma+s,\Gamma})^N$. The estimate (4.2) easily follows from the same arguments as in (4.11). □

Proof of Theorem 2.1. We divide the proof into 5 steps.

Step 1. We define the iteration scheme. Let $1 < \tau < 2$. Let S_k $(k = 0, 1, 2, \ldots)$ be the smoothing operator defined by (3.10) with $N + 1 = \mu_k = d^{\tau^k}$, where $d > 1$. For a vector valued function $v = (v_1, \ldots, v_N)$ we define $S_k v := (S_k v_1, \ldots, S_k v_N)$. For the sake of simplicity we use the notation $S_k v$ for a vector valued function as well as for a scalar function. We set $G(u) = (G_1(u), \ldots, G_N(u))$ and define the sequences $\{w_k\}$ and $\{g_k\}$ by the following relations

$$w_0 = 0,\ w_{k+1} = w_k + S_k \rho_k,\ L_{w_k} \rho_k = g_k,\ g_k = -G(w_k),\ k = 0, 1, 2, \ldots, \tag{4.12}$$

where $g_0 = a(x, 0)$ and the linearized operator L_w is given by (4.1). We note that w_k is real-valued if g_k is real-valued. In the following we sometimes omit the suffix Γ of $\|\cdot\|_{\nu,\Gamma}$, and we denote it by $\|\cdot\|_\nu$ if there is no fear of confusion.

We choose ν and κ in the following way and we fix them.

$$\kappa > \max\{\sigma - s, 2(n+1)(2-\tau)^{-1}\}, \quad \nu > (1+\tau)\kappa + (\tau - 1)^{-1}(m + n + 2 - s + m\tau). \tag{4.13}$$

We want to show that there exists $C > 0$ independent of $d > 1$ and k such that

$$\|g_k\|_0 \le C \Lambda_\nu \mu_k^{-\kappa}, \quad k = 0, 1, 2, \ldots, \quad \Lambda_\nu = d^\kappa \|g_0\|_{\nu+1,\Gamma}. \tag{4.14}$$

The estimate (4.14) holds for $k = 0$. Indeed, we can take $C = 1$ because $d^\kappa \mu_0^{-\kappa} = 1$. We suppose that (4.14) holds up to k, and we shall show (4.14) for $k + 1$.

Step 2. For a nonnegative integer ℓ we shall show that there exists $C > 0$ independent of $d > 1$ and k such that, for $j = 1, \ldots, k + 1$,

$$\|w_j\|_\ell \le C\Lambda_\nu \text{ (if } \ell < \kappa + s), \quad \|w_j\|_\ell \le C\Lambda_\nu \mu_{j-1}^{\ell+1-\kappa-s} \text{ (if } \ell \ge \kappa + s). \tag{4.15}$$

In the following we denote constants independent of $d > 1$ and k by C, C_1, C_2 and so on. Let $0 \le j \le k$. By (4.12) we have $w_{j+1} = w_j + S_j \rho_j = \sum_{i=0}^j S_i \rho_i$. First we assume that $\ell \ge s$. Then, by (1) of Proposition 3.2, Proposition 4.1 and (4.14) with $k = i$ we have

$$\|w_{j+1}\|_\ell \le \sum_{i=0}^j \|S_i \rho_i\|_\ell \le C \sum_{i=0}^j \mu_i^{\ell-s} \|\rho_i\|_s \le C_1 \sum_{i=0}^j \mu_i^{\ell-s} \|g_i\|_0 \le C_2 \Lambda_\nu \sum_{i=0}^j \mu_i^{\ell-\kappa-s}. \tag{4.16}$$

In case $\ell < \kappa + s$, the sum $\sum_{i=0}^{j} \mu_i^{\ell-\kappa-s}$ is bounded by some constant independent of j and $d > d_0$ if $d_0 > 1$. If $\ell < s$, we have $\|w_{j+1}\|_\ell \le \|w_{j+1}\|_s$. Hence we are reduced to the case $\ell = s$. Therefore we have proved (4.15) when $\ell < \kappa + s$. In case $\ell \ge \kappa + s$ we have

$$\sum_{i=0}^{j} \mu_i^{\ell-s-\kappa} \le \mu_j^{\ell-s-\kappa} \sum_{i=0}^{j} d^{(\tau^i - \tau^j)(\ell-s-\kappa)}. \tag{4.17}$$

The sum in the right-hand side is bounded by some constant independent of $d > d_0$ and j if $\ell > \kappa + s$ and $d_0 > 1$. It follows that $\|w_{j+1}\|_\ell$ is bounded by $C_2(j+1)\Lambda_\nu$ (if $\ell = \kappa + s$) and $C_3 \Lambda_\nu \mu_j^{\ell-s-\kappa}$ (if $\ell > \kappa + s$) for some $C_3 > 0$. Because $j + 1 \le \mu_j$ for sufficiently large $d > 1$ we obtain (4.15).

Step 3. We want to show the estimate: there exists $C > 0$ independent of k and d such that

$$\|g_k\|_\nu \le C\Lambda_\nu (1 + \mu_k^{(\nu+m+n+2-\kappa-s)/\tau}). \tag{4.18}$$

By (4.12), (2.1) and (4.15) we have

$$\|g_k\|_\nu \le C\Lambda_\nu \mu_{k-1}^{\nu+m+1-\kappa-s} + \|a(\cdot, \delta^\alpha w_k) - g_0\|_\nu + \|g_0\|_\nu. \tag{4.19}$$

Recalling that $g_0 = a(x, 0)$ we obtain

$$\|a(\cdot, \delta^\alpha w_k) - g_0\|_\nu = \Vert M\left(a(\cdot, \delta^\alpha w_k) - a(\cdot, 0)\right)\Vert_\nu. \tag{4.20}$$

The right-hand side can be estimated by the argument given after (4.4). We replace $\nabla_z \delta_x^\gamma a(\cdot, \delta^\alpha w_k)$ with $a(\cdot, \delta^\alpha w_k)$. It follows that it is bounded by $C\|w_k\|_{s+\nu+n+1}$ for some $C > 0$ independent of k. By (4.15), it is estimated by $C\Lambda_\nu(1 + \mu_{k-1}^{\nu+n+2-\kappa})$ for possibly another constant $C > 0$ independent of k. Recalling that $\|g_0\|_\nu \le \Lambda_\nu$, $m \ge s$ and $\mu_k = \mu_{k-1}^\tau$ we get (4.18) from (4.19).

Step 4. We will show that there exists $C > 0$ independent of k and d such that

$$\|\rho_k\|_\nu \le C\Lambda_\nu(1 + \mu_k^{(\nu+m+n+2-s-\kappa)/\tau}). \tag{4.21}$$

It follows from (4.1) and (4.12) that ρ_k satisfies

$$P(\delta)\rho_k + \sum_\alpha \delta^\alpha \rho_k \cdot \nabla_{z_\alpha} a(x, \delta^\alpha w_k) = g_k. \tag{4.22}$$

Set $u_k = P\rho_k$. Then u_k satisfies

$$u_k + \sum_{|\alpha| \le s} \nabla_{z_\alpha} a(x, \delta^\alpha w_k) \cdot \delta^\alpha P^{-1} u_k \equiv u_k + B(w_k)u_k = g_k, \tag{4.23}$$

where $B(w_k) = \sum_\alpha \nabla_{z_\alpha} a(x, \delta^\alpha w_k) \cdot \delta^\alpha P^{-1}$. Hence u_k is given by

$$u_k = \sum_{j=0}^{\infty} (-B(w_k))^j g_k. \tag{4.24}$$

In view of the condition (A.2) we have

$$\|\rho_k\|_\nu = \|P^{-1}u_k\|_\nu \le C\|u_k\|_{\nu-s} \le C\|u_k\|_\nu,$$

for some $C > 0$ independent of ν, ρ_k. Hence we may consider $\|\delta^\beta u_k\|_0$ for $|\beta| = q \le \nu$.

We will estimate $\|\delta^\beta (B^j g_k)\|_0$ for $|\beta| = q$. We first consider the case $j = 1$, $\|\delta^\beta (Bg_k)\|_0$. In view of the definition of B and the Leibniz formula we have

$$\delta^\beta(Bg_k) = \sum_{\gamma+\varepsilon=\beta} \sum_\alpha \delta^\gamma(\nabla_{z_\alpha} a(x, \delta^\alpha w_k))\delta^\varepsilon \delta^\alpha P^{-1} g_k = \sum_{\alpha, \gamma+\varepsilon=\beta} \delta^\gamma(\nabla_{z_\alpha} a)\delta^{\varepsilon+\alpha} P^{-1} g_k$$
$$= \sum_{\alpha, \gamma+\varepsilon=\beta} \delta^\gamma(\nabla_{z_\alpha} a) P^{-1} \delta^{\alpha+\varepsilon} g_k = \sum_{\alpha, 0 \le \mu \le q} \quad \sum_{\gamma+\varepsilon=\beta, |\gamma|=q-\mu, |\varepsilon|=\mu} \delta^\gamma(\nabla_{z_\alpha} a) P^{-1} \delta^{\alpha+\varepsilon} g_k,$$

where we used the commutativity of P^{-1} and $\delta^{\varepsilon+\alpha}$. In view of the definition (A.1) and the assumption $|\alpha| \le s$ there exists a constant $C > 0$ such that

$$\|P^{-1}\delta^{\alpha+\varepsilon} g_k\|_0 \le C\|\delta^{\varepsilon} g_k\|_0 \le C\|g_k\|_{\mu}.$$

Next we study the term $\delta^{\gamma}(\nabla_{z_\alpha} a(x,z))$, $z = (\delta^{\alpha} w_k)_{\alpha}$. For the sake of simplicity we denote the j-th component of $z = (\delta^{\alpha} w_k)_{\alpha}$ by $z_j := \delta^{\alpha_j} w_k$, $(j = 1, \ldots, \ell, \ell \ge 1)$. By Leibnitz rule we have the expression

$$\delta^{\gamma}\nabla_{z_\alpha} a(x,z) = \sum{}^{*} (\delta_x^{\gamma_0}\nabla_{z_1}^{n_1}\nabla_{z_2}^{n_2}\cdots\nabla_{z_\ell}^{n_\ell}\nabla_{z_\alpha} a)(x,z) \prod_{j=1}^{n_1} \delta^{\gamma_{1,j}+\alpha_1} w_k \cdots \prod_{j=1}^{n_\ell} \delta^{\gamma_{\ell,j}+\alpha_\ell} w_k,$$

where the summation $\sum^{*}$ is taken for all pairs of indices

$$\gamma = \gamma_0 + \sum_{j=1}^{n_1} \gamma_{1,j} + \cdots + \sum_{j=1}^{n_\ell} \gamma_{\ell,j}; \quad \gamma_0, \gamma_{1,j}, \ldots, \gamma_{\ell,j} \in \mathbb{Z}_+^n; \quad n_1, \ldots, n_\ell \in \mathbb{Z}_+.$$

We first consider the term which appear when the differentiations are applied to the x variables of $\nabla a(x,z)$. Noting that $\delta_x^{\gamma}\nabla_{z_\alpha} a(x,z) = \nabla_{z_\alpha}(\delta_x^{\gamma} a)(x,z)$ we get, from (4.3) that

$$\|\delta^{\gamma}\nabla_z a(\cdot, \delta^{\alpha} w_k)\|_{0,\Gamma_0} \le \|\nabla_z a(\cdot, 0)\|_{|\gamma|,\Gamma_0} + C\|w_k\|_{s+n+1,\Gamma}. \tag{4.25}$$

The right-hand side can be made arbitrarily small, if $w_k \in (\mathcal{H}_{\nu+s+n+1})^N$ and

$$\|w_k\|_{s+n+1} < \varepsilon, \quad \|\nabla_z a(\cdot, 0)\|_{|\gamma|,\Gamma_0} \le \|\nabla_z a(\cdot, 0)\|_{\nu,\Gamma} < \varepsilon$$

for sufficiently small ε.

We next investigate the terms that appear when the differentiations are applied to $\delta^{\alpha} w_k$ in $\nabla a(x, \delta^{\alpha} w_k)$. For simplicity we consider the terms $\delta_x^{\gamma}\partial_z^{\theta}\nabla a(x,z)\delta^{\xi+\alpha} w_k$ for some multi integers ξ, θ, γ, $|\xi| \le q - \mu$. Let $\tau > 0$ be a small number, and let $\phi_0(t) \in C^{\infty}(\mathbb{R})$, $\phi_0(t) > 0$ be a smooth function such that $\phi_0(t) \equiv t^{\tau}$ in some neighborhood of the origin $t = 0$ and $\phi_0(t) = t^{-\tau}$ when $|t| \gg 1$. Define $\phi(x) := \prod_{j=1}^{n} \phi_0(x_j)$, and write

$$\delta_x^{\gamma}\partial_z^{\theta}\nabla a(x,z)\delta^{\xi+\alpha} w_k = \left(\phi(x)\delta_x^{\gamma}\partial_z^{\theta}\nabla a(x,z)\right)\left(\phi(x)^{-1}\delta^{\xi+\alpha} w_k\right).$$

By Lemma 3.3 we have

$$\|\delta_x^{\gamma}\partial_z^{\theta}\nabla a(x,z)\delta^{\xi+\alpha} w_k\|_0 \le \|\phi(x)\delta_x^{\gamma}\partial_z^{\theta}\nabla a(x,z)\|_0 \|\phi(x)^{-1}\delta^{\xi+\alpha} w_k\|_0.$$

We will estimate $\|\phi(x)\delta_x^{\gamma}\partial_z^{\theta}\nabla a(x,z)\|_0$. By integrating (3.4) with

$$s = n+1, \ u = \phi(x)\delta_x^{\gamma}\partial_z^{\theta}\nabla a(x,z), \ z = (\delta^{\alpha} w_k),$$

and by recalling the definition of the norm we have, for $\Gamma'' \supset\supset \Gamma_0$ and some constants $C > 0$ and $C' > 0$,

$$\begin{aligned} \|u\|_{0,\Gamma_0} &= \|\hat{u}\|_{0,\Gamma_0} \le C \sup_{x\in\mathbb{R}_+^n, |\alpha|\le n+1, \eta\in\overline{\Gamma''}} |x^{\eta}\delta^{\alpha} u(x)| \int \langle \eta + i\xi \rangle^{-n-1} d\xi \\ &\le C' \sup_{x\in\mathbb{R}_+^n, |\alpha|\le n+1, \eta\in\overline{\Gamma''}} |x^{\eta}\delta^{\alpha} u(x)|. \end{aligned}$$

We note that, if $|\eta_j|$ is sufficiently small, the quantity

$$x^{\eta}\delta^{\alpha} u(x) = x^{\eta}\delta^{\alpha}\left(\phi(x)\delta_x^{\gamma}\partial_z^{\theta}\nabla a(x,z)\right)$$

tends to zero when $x_j \to \infty$ by (A.2) and the decay property of $\phi(x)$. It follows that, if we take Γ_0 and Γ'' sufficiently small we have

$$\sup_{x\in\mathbb{R}_+^n, |\alpha|\le n+1, \eta\in\overline{\Gamma''}} \left|x^{\eta}\delta^{\alpha}\left(\phi(x)\delta_x^{\gamma}\partial_z^{\theta}\nabla a(x,z)\right)\right| \le C'',$$

for some $C'' > 0$. It follows that

$$\|\phi(x)\delta_x^\gamma \partial_z^\theta \nabla a(x,z)\|_0 \le C'C''.$$

On the other hand, we can easily show that $\|\phi(x)^{-1}\delta^{\xi+\alpha}w_k\|_0$ is bounded by a constant multiplied by $\|w_k\|_{s+n+1+|\xi|}$. Therefore the term $\|\delta_x^\gamma \partial_z^\theta \nabla a(x,z)\delta^{\xi+\alpha}w_k\|_0$ is estimated by $\|w_k\|_{s+n+1+|\xi|}$. By the interpolation and the similar calculations as in the proof of Lemma 4.2 the product of the terms which appear by differentiating $\delta^\alpha w_k$ in $\nabla a(x,z)$ is estimated by $\|w_k\|_{s+n+1+q-\mu}$. Hence, for every $\varepsilon' > 0$ there exists a constant $C > 0$ such that

$$\|\delta^\beta(Bg_k)\|_0 \le C\sum_{\mu=0}^{q}(\varepsilon' + \|w_k\|_{s+n+1+q-\mu})\|g_k\|_\mu.$$

Especially, we have $\|Bg_k\|_0 \le C\varepsilon'\|g_k\|_\mu$ for some constant $C > 0$, because $\|w_k\|_{s+n+1} < \varepsilon$.

In order to estimate $\delta^\beta(B^j g_k)$ we note that

$$\delta^\beta(B^j g_k) = \sum_{|\beta_\ell|=\mu, 0\le\mu\le q}\ \sum_{\beta=\beta_1+\cdots+\beta_\ell, 1\le\ell\le q}(\delta^{\beta_1}B)B\cdots B(\delta^{\beta_2}B)B\cdots B\cdots(\delta^{\beta_{\ell-1}}B)(\delta^{\beta_\ell}g_k).$$

We can estimate the right-hand side terms by Lemma 3.3, the interpolation lemma and the above estimate of $\|\delta^\beta(Bv)\|_0$. Because the number of combinations in $\delta^\beta(B^j g_k)$ can be estimated by a constant multiplied by $j^{q-\mu}$ there exists $C_1 > 0$ such that

$$\|\delta^\beta(B^j g_k)\|_0 \le C_1\sum_{\mu=0}^{q} j^{q-\mu}(C\varepsilon')^{j-q}C^q(\varepsilon'^q + \|w_k\|_{s+n+1+q-\mu})\|g_k\|_\mu,$$

if $\|w_k\|_{s+n+1} < \varepsilon$ and $w_k \in (\mathcal{H}_{q+s+n+1})^N$, which follows from (4.15) if Λ_ν is sufficiently small.

Let $s+n+1+q-\mu \ge \kappa + s$. Then it follows from (4.15), (4.18) and $\mu_k = \mu_{k-1}^\tau$ that

$$\begin{aligned} &\|g_k\|_\mu\|w_k\|_{s+n+1+q-\mu} \le C^2\Lambda_\nu^2\mu_{k-1}^{n+2+q-\mu-\kappa}(1+\mu_k^{(\mu+m+n+2-s-\kappa)/\tau}) \\ \le\ & C^2\Lambda_\nu^2\mu_k^{(q-\mu+n+2-\kappa)/\tau}(1+\mu_k^{(\mu+m-s+n+2-\kappa)/\tau}) \le C_2\Lambda_\nu^2(1+\mu_k^{(q+m-s+n+2-\kappa)/\tau}) \end{aligned} \tag{4.26}$$

for some $C_2 > 0$, where we have used $n+2-\kappa \le 0$ by (4.13), and

$$q-\mu+n+2-\kappa \le q+m-s+n+2-\kappa.$$

We can similarly argue in case $s+n+1+q-\mu < \kappa+s$. It follows that there exists $C_3 > 0$ such that

$$\|\delta^\beta(B^j g_k)\|_{0,\Gamma} \le C_3 j^q r_1^j \Lambda_\nu^2(1+\mu_k^{(q+m+n+2-s-\kappa)/\tau}).$$

Because

$$\|\delta^\beta u_k\|_{0,\Gamma} \le \sum_{j=0}^{\infty}\|\delta^\beta(B^j g_k)\|_{0,\Gamma}, \quad \|\rho_k\|_\nu \le C\|u_k\|_\nu,\ q \le \nu,$$

we obtain (4.21).

Step 5. We will estimate $\|g_{k+1}\|_0$. By (4.12) we have

$$\begin{aligned} -g_{k+1} &= G(w_{k+1}) = G(w_k + S_k\rho_k) = G(w_k) + L_{w_k}S_k\rho_k + Q(w_k, S_k\rho_k) \\ &= G(w_k) + L_{w_k}\rho_k + L_{w_k}(S_k - I)\rho_k + Q(w_k, S_k\rho_k), \end{aligned} \tag{4.27}$$

where $Q(w_k, S_k\rho_k)$ is the quadratic term of $S_k\rho_k$. Hence we have

$$\|g_{k+1}\|_0 \le \|L_{w_k}(S_k - I)\rho_k\|_0 + \|Q(w_k, S_k\rho_k)\|_0 \equiv I_1 + I_2. \tag{4.28}$$

By (4.21) and Proposition 3.2 we have

$$I_1 \le C\|(S_k - I)\rho_k\|_m \le CC'\mu_k^{m-\nu}\|\rho_k\|_\nu \le C^2C'\Lambda_\nu\mu_k^{m-\nu}(1+\mu_k^a) \tag{4.29}$$

for some constants $C > 0$ and $C' > 0$, where

$$a = (\nu + m + n + 2 - s - \kappa)/\tau > 0.$$

Because $1 \le \mu_k^a$ the right-hand side is bounded by $C''\Lambda_\nu\mu_k^{m-\nu+a}$ for some $C'' > 0$. By (4.13) we have $m - \nu + a < -\kappa\tau$. Hence it follows from $\mu_{k+1} = \mu_k^\tau$ that

$$\mu_k^{m-\nu+a} = \mu_k^{m-\nu+a+\kappa\tau}\mu_k^{-\kappa\tau} = \mu_k^{m-\nu+a+\kappa\tau}\mu_{k+1}^{-\kappa}. \tag{4.30}$$

If we take $d \ge d_0$ sufficiently large, we can absorb constants independent of d by the term $\mu_k^{m-\nu+a+\kappa\tau} \le d^{-1}$. Therefore we have $I_1 \le C\Lambda_\nu\mu_{k+1}^{-\kappa}/2$.

As to I_2 we have, by Propositions 3.1, 3.2, 4.1 and the inductive assumption of g_k

$$I_2 \le C_1\|S_k\rho_k\|_{s+n+1}^2 \le C_2\mu_k^{2n+2}\|\rho_k\|_s^2 \le C_3\mu_k^{2n+2}\|g_k\|_0^2 \le C_3\mu_k^{2n+2-2\kappa}(C\Lambda_\nu)^2. \tag{4.31}$$

If we take $\|g_0\|_{\nu+1,\Gamma}$ so small that

$$2C_3C\Lambda_\nu = 2C_3Cd^\kappa\|g_0\|_{\nu+1,\Gamma} \le 1,$$

then we have $I_2 \le C\Lambda_\nu\mu_{k+1}^{-\kappa}/2$. Therefore it follows from (4.28) that $\|g_{k+1}\|_0 \le C\Lambda_\nu\mu_{k+1}^{-\kappa}$. This proves (4.14).

It follows from (4.12), (4.14), (1) of Proposition 3.2 and Proposition 4.1 that, for every $\ell < \kappa + s$

$$\|w_{k+1} - w_k\|_\ell \le \|S_k\rho_k\|_\ell \le C\mu_k^{\ell-s}\|\rho_k\|_s \le C'\mu_k^{\ell-s}\|g_k\|_0 \le C''\mu_k^{\ell-s-\kappa}\Lambda_\nu.$$

Clearly $w = \lim_k w_k$ exists in $(\mathcal{H}_\ell)^N$, and w satisfies $G(w) = \lim G(w_k) = -\lim g_k = 0$. Since $\kappa > \sigma - s$ we have $\sigma < \kappa + s$ and $w \in (\mathcal{H}_\sigma)^N$. This proves Theorem 2.1. □

Proof of Theorem 2.2. Let $\phi(x/\lambda)$ $(\lambda > 0)$ be a cutoff function given in Lemma 3.4. Instead of (2.1) we consider the equation

$$\tilde{G}_j(u) := p_j(\delta)u_j + \phi(x/\lambda)a_j(x, \delta^\alpha u; |\alpha| \le s) = 0, \quad j = 1, \dots, N. \tag{4.32}$$

We can easily see that $\phi(x/\lambda)a(x,z)$ satisfies (A.2). Let ν be a positive integer. Let $\psi \in C^\infty(\mathbb{R}^n)$ be a function with compact support which is identically equal to 1 in some neighborhood of the origin such that

$$\psi a \in (\mathcal{H}_{\nu+n+1,\Gamma})^N, \qquad \psi(x)\nabla_z a(x,0) \in (\mathcal{H}_{\nu+n+1,\Gamma})^{kN}.$$

We take $\lambda > 0$ so small that supp $\phi(x/\lambda) \subset \{x; \psi \equiv 1\}$. Then we have $\phi(x/\lambda)\psi(x) \equiv \phi(x/\lambda)$. It follows from Lemma 3.4 that there exists $\lambda > 0$ such that

$$\|\phi a(\cdot,0)\|_{\nu,\Gamma'} < \varepsilon, \quad \|\phi\nabla_z a_j(\cdot,0)\|_{\nu,\Gamma'} < \varepsilon \quad (\Gamma' \subset\subset \Gamma).$$

Therefore, by Theorem 2.1 Eq. (4.32) has a solution $u \in (\mathcal{H}_{\sigma,\Gamma'})^N$. This yields a solution of (2.1) because $\phi \equiv 1$ in some neighborhood of the origin. □

Proof of Theorem 2.10. By definition the dual cone of $\mathcal{C}$ is not empty. Hence there exist $c_\mu \in \mathbb{R}(\mu = 1, \dots, d)$ such that $\sum_{\mu=1}^d c_\mu\lambda_j^\mu > 0$ for all $j = 1, \dots, n$. It follows that $\chi^0 = \sum_{\mu=1}^d c_\mu\chi^\mu$ commutes with χ^μ and χ^0 is a Poincaré vector fields since all eigenvalues of the linear part of χ^0 are positive. In the following we assume that χ^1 is a Poincaré vector field in χ. By Theorem 2.7, there exists a change of variables, $y \mapsto x = y + v(y)$ which linearizes χ^1, namely $R^1 = 0$. For simplicity we assume that χ^1 is linearized. In view of the commutativity and the definition of the homology equation, we have

$$\mathcal{L}_1R^\mu(x) - \mathcal{L}_\mu R^1(x) = \partial R^1(x)R^\mu(x) - \partial R^\mu(x)R^1(x) = 0, \quad \mu = 1, \dots, d.$$

It follows that we have $\mathcal{L}_1 R^\mu = 0$ for $\mu = 1, \dots, d$.

We define $w(\sigma)$ and $X(\sigma)$, respectively by

$$w(\sigma) := R^\mu(x_1\sigma^{\lambda_1^1}, \dots, x_n\sigma^{\lambda_n^1}), \quad X(\sigma) := \mathrm{diag}(\sigma^{-\lambda_1^1}, \dots, \sigma^{-\lambda_n^1}).$$

It is easy to verify that $dX/d\sigma = -\Lambda^1 X\sigma^{-1}$. Hence we have

$$\frac{d}{d\sigma}(wX) = \frac{dw}{d\sigma}X + w\frac{dX}{d\sigma} = \frac{dw}{d\sigma}X - w\Lambda^1 X\sigma^{-1}. \tag{4.33}$$

By the relation $\mathcal{L}_1 R^\mu = 0$ with x_j replaced by $x_j\sigma^{\lambda_j^1}$ $(0 < \sigma \le 1)$ the right-hand side of (4.33) is zero. By assumption, $R^\mu \in \mathcal{H}_{\nu,\Gamma}$ and R^μ vanishes at the origin $x = 0$. Therefore, by integrating (4.33) with respect to σ from 0 to 1 we have $R^\mu = 0$. □

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