Manuscripta mathematica manuscript No.
(will be inserted by the editor)

Monodromy of nonintegrable Hamiltonian system

Masafumi Yoshino

Received: date / Accepted: date

Abstract In this paper we are interested in the global behavior of solutions of certain analytically nonintegrable Hamiltonian systems. We study the monodromy function defined by W. Balser related to the so-called semi-formal solutions which corresponds to the fundamental solution in the case of linear ordinary differential equations. By using the convergent semi-formal solutions defined by multi-valued first integrals, we prove the formula of a monodromy function of a nonintegrable Hamiltonian system obtained by the nonlinear perturbation of confluent hypergeometric system.

Keywords monodromy function · nonintegrable Hamiltonian system · semi-formal solution

Mathematics Subject Classification (2000) 34M30 · 34M35 · 37F50

1 Introduction

In this paper we consider a Hamiltonian system with a Hamiltonian function H. We say that the Hamiltonian system is C^ω-Liouville integrable if there exist first integrals $\phi_j \in C^\omega$ $(j = 1, \ldots, n)$ which are functionally independent on an open dense set and Poisson commuting, i.e., $\{\phi_j, \phi_k\} = 0$, $\{H, \phi_k\} = 0$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket. If $\phi_j \in C^\infty$ $(j = 1, \ldots, n)$, then we say C^∞- Liouville integrable.

In [2] Bolsinov and Taimanov studied the Hamiltonian system related with geodesic flow on a Riemannian manifold which is C^∞-integrable and not C^ω-integrable. They also showed that non C^ω- integrability is closely related with

Partially supported by Grant-in-Aid for Scientific Research (No. 20540172), JSPS, Japan.

M. Yoshino
Department of Mathematics, Hiroshima University, Hiroshima 739-8526, Japan
Tel.: +81-82-424-7345
Fax: +81-82-424-0710
E-mail: yoshinom@hiroshima-u.ac.jp

the non Abelian property of a fundamental group of the manifold. Then Gorni and Zampieri (cf. [3]) showed similar results in the local setting, namely for a Hamiltonian system which has a certain kind of irregular singularity at the origin they showed the non C^ω- integrability. The latter result is also extended for a certain class of Hamiltonians in [5].

In this paper we are interested in the monodromy of non C^ω-integrable Hamiltonian system, namely, in the monodromy function. The monodromy function was defined in [1] as the formal power series of some parameter, which is a natural extension of the so-called monodromy of linear ordinary differential equations. We will prove formulas of the monodromy function. In proving the formulas we also show super integrability in the class of multi-valued first integrals. It naturally leads us to the existence and the expression of the so-called convergent semi-formal solutions in terms of a certain system of equations defined by functionally independent multi-valued first integrals. The explicit formula of the monodromy function is easily shown by the system. Although the super integrability in an analytic category is difficult to show, that in a class of multi-valued functions seems easier to verify for general class of Hamiltonians. Indeed, the Hamiltonian system studied in the last section is not integrable, while it is still super integrable in a class of multi-valued functions. (cf. [5]).

This paper is organized as follows. In Section 2 we prepare the notion of the convergent semi-formal solution and the monodromy function. In Section 3 we introduce the confluent hypergeometric system. In Section 4 we consider Hamiltonians derived from a linear confluent hypergeometric system. We first prove the super integrability in a class of multi-valued first integrals. Then we give the formula of the monodromy function. In Section 5 we calculate the monodromy function of a nonintegrable Hamiltonian which is a nonlinear perturbation of a system studied in Section 4.

2 Semi-formal solution via first integrals

Let $n \geq 2$ and $\sigma \geq 1$ be integers. Consider the Hamiltonian system

$$z^{2\sigma}\frac{dq}{dz} = \nabla_p \mathcal{H}(z,q,p), \quad z^{2\sigma}\frac{dp}{dz} = -\nabla_q \mathcal{H}(z,q,p), \tag{1}$$

where $q = {}^t(q_2, \dots, q_n)$, $p = {}^t(p_2, \dots, p_n)$, and $\mathcal{H}(z,q,p)$ is analytic with respect to $(z,q,p) \in \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$ in some neighborhood of the origin. By taking $q_1 = z$ as a new unknown function (1) is written in an equivalent form with Hamiltonian

$$H(q_1, q, p_1, p) := p_1 q_1^{2\sigma} + \mathcal{H}(q_1, q, p)$$

$$\begin{aligned} \dot{q_1} = H_{p_1} = q_1^{2\sigma}, \quad \dot{p_1} = -H_{q_1} = -2p_1 q_1^{2\sigma-1} - \partial_{q_1}\mathcal{H}(q_1,q,p), \\ \dot{q} = \nabla_p H = \nabla_p \mathcal{H}(q_1,q,p), \quad \dot{p} = -\nabla_q H = -\nabla_q \mathcal{H}(q_1,q,p). \end{aligned} \tag{2}$$

The solution of (1) is given in terms of that of (2) by taking $q_1 = z$ as an independent variable.

Semi-formal solution. We define the semi-formal solution of (1). (cf. [1]). Let $\mathcal{O}(\tilde{S}_0)$ be the set of holomorphic functions on $\tilde{S}_0$, where $\tilde{S}_0$ is the universal covering space of the punctured disk of radius r, $S_0 = \{|z| < r\} \setminus 0$ for some $r > 0$. The $(2n-2)$-vector $\check{x}(z,c)$ of formal power series of c

$$\check{x}(z,c) = \sum_{|\nu|\geq 0} \check{x}_\nu(z)c^\nu = \check{x}_0(z) + X(z)c + \sum_{|\nu|\geq 2} \check{x}_\nu(z)c^\nu \tag{3}$$

is said to be a *semi-formal solution* of (1) if $\check{x}_\nu \in (\mathcal{O}(\tilde{S}_0))^{2n-2}$ and $\check{x}(z,c) = (q(z,c), p(z,c))$ is a formal power series solution of (1). Here $X(z)$ is a $(2n-2)$-square matrix with component belonging to $\mathcal{O}(\tilde{S}_0)$. If $X(z)$ is invertible, then we say that $(q(z,c), p(z,c))$ is a *complete semi-formal solution.* We say that a semi-formal solution is a convergent semi-formal solution (at the origin) if the following condition holds. For every compact set K in $\tilde{S}_0$ there exists a neighborhood U such that the formal series converges for $z \in K$ and $c \in U$. The semi-formal solution at $z_0 \in \mathbb{C}$ is defined similarly.

Monodromy function. We will give the definition of the monodromy function of (1). Let z_0 be any point in $\mathbb{C}$ and let (q,p) be a semi-formal solutions of (1) about z_0. We define the monodromy function $v(c)$ around z_0 by

$$(q,p)((z-z_0)e^{2\pi i} + z_0, v(c)) = (q,p)(z,c), \tag{4}$$

where $v(c) = (v_j(c))_j$. The existence of $v(c)$ in the class of formal power series of c is proved in [1]. If we denote the linear part of $v(c)$ by Mc, then by considering the linear part of the monodromy relation we have $X((z-z_0)e^{2\pi i} + z_0)M = X(z)$. Hence M^{-1} is the so-called monodromy factor.

Construction of convergent semi-formal solution. In the following we will show that the convergent semi-formal solution of (1) is obtained by solving certain system of nonlinear equations given by first integrals. We consider (2). Given functionally independent first integrals $H(q_1, q, p_1, p)$ and $\psi_j \equiv \psi_j(q_1, q, p)$ $(j = 1, 2, \ldots, 2n-2)$ of (2), where ψ_j is holomorphic when $q_1 \in \tilde{S}_0$ and q and p in some neighborhood of the origin. The functional independentness means that there exists a neighborhood of the origin of $(q, p, p_1) \in V$ such that the matrix

$$^t(\nabla_{q,p,p_1} H, \nabla_{q,p,p_1} \psi_j)_{j\downarrow 1,2,\ldots,2n-2} \tag{5}$$

has full rank $2n-1$ on $(q_1, p_1, q, p) \in \tilde{S}_0 \times V$. We assume that every coefficient in the expansion of ψ_j in the powers of q and p is holomorphic with respect to q_1 on $\tilde{S}_0$.

Let the point $(q_{1,0}, p_{1,0}, q_0, p_0)$ and the values $c_{j,0}$ $(j = 1, 2, \ldots, 2n-2)$ satisfy that

$$H(q_{1,0}, p_{1,0}, q_0, p_0) = 0, \ \psi_j(q_{1,0}, q_0, p_0) = c_{j,0}, \ (j = 1, 2, \ldots, 2n-2). \tag{6}$$

For $c_j = \tilde{c}_j + c_{j,0}$, $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_{2n-2}) \in \mathbb{C}^{2n-2}$ we consider the system of equations of p_1, q and p

$$H(q_1, p_1, q, p) = 0,\ \psi_j(q_1, q, p) = c_j,\ (j = 1, 2, \ldots, 2n-2). \tag{7}$$

If (7) has a solution, then we denote it by $q \equiv q(q_1, c)$, $p \equiv p(q_1, c)$, $p_1 \equiv p_1(q_1, c)$. We see that q, p and p_1 are holomorphic functions of q_1 in $\tilde{S}_0$ and c in some neighborhood of $c = 0$ if we assume (5). The next theorem was proved in [6].

Theorem 1 *Suppose that $H(q_1, q, p_1, p)$ and $\psi_j \equiv \psi_j(q_1, q, p)$ $(j = 1, 2, \ldots, 2n-2)$ are functionally independent when $q_1 \in \tilde{S}_0$. Assume (6). Then the solutions of (7) gives the convergent complete semi-formal solution of (1), $(q(z, c), p(z, c))$ provided that q (resp. p) is not a constant function.*

Remark 1 Theorem 1 can be extended to the first order system of ordinary differential equations of n-unknown functions without Hamiltonian structure if one assumes the existence of functionally independent $(n-1)$-first integrals, which implies the super integrability in the Hamiltonian case. It should be noted that, because we take multi-valued first integrals into account, the so-called multi-valued super integrability holds in many examples. This enables us to calculate the monodromy even if the Hamiltonian is not integrable.

3 Confluent hypergeometric equation

We consider a class of hypergeometric system

$$(z - C)\frac{dv}{dz} = Av, \tag{8}$$

where C and A are diagonal and constant matrices of size m, respectively. The system has only regular singular points on the Riemann sphere.

The system contains a subclass written in a Hamiltonian form. Indeed, set $v = {}^t(q, p) \in \mathbb{C}^{2n-2}$ and assume that C and A are block diagonal matrices

$$C = \mathrm{diag}(\Lambda_1, \Lambda_1),\ A = \mathrm{diag}(A_1, -{}^tA_1) \tag{9}$$

where Λ_1 and A_1 are $(n-1)$-square diagonal and constant matrices, respectively. In order that (8) can be written in a Hamiltonian form we further assume that

$$\Lambda_1 A_1 = A_1 \Lambda_1. \tag{10}$$

Define

$$H := \langle (z - \Lambda_1)^{-1} p, A_1 q \rangle. \tag{11}$$

Then one can write (8) in the Hamiltonian form

$$\frac{dq}{dz} = H_p(z, q, p), \quad \frac{dp}{dz} = -H_q(z, q, p). \tag{12}$$

If we introduce the variable q_1 by $q_1 = z^{-1}$, then one can write (12) in the autonomous form for the Hamiltonian

$$H := p_1 q_1^2 - \langle (q_1^{-1} - \Lambda_1)^{-1} p, A_1 q \rangle. \tag{13}$$

We will introduce the irregular singularity at the origin $q_1 = 0$ by the confluence of singularities. Let λ_j $(j = 2, \ldots, n)$ be the diagonal elements of Λ_1. We assume $\lambda_j \neq 0$ for all j. Take nonempty sets J and J' such that $J \cup J' = \{2, 3, \ldots, n\}$. Without loss of generality one may assume $J = \{2, 3, \ldots, n_0\}$ for some $n_0 \geq 2$. We merge all regular singular points $q_1 = \lambda_\nu$ for $\nu \in J'$ to the origin. Let $\nu \in J'$. Substitute $q_1 \mapsto q_1 \varepsilon^{-1}$, $p_1 \mapsto p_1 \varepsilon$ in (13), and let $\varepsilon \to 0$. Note that the substitution extends to a symplectic transformation. One easily verifies that $(q_1^{-1}\varepsilon - \lambda_\nu)^{-1}$ tends to $-\lambda_\nu^{-1}$ because we assume $\lambda_\nu \neq 0$. We multiply the ν-th row of A_1 with ε^{-1}, similarly as in the case of the confluence of the hypergeometric equation.

On the other hand, if $\nu \in J$, then we require that the singular point λ_ν^{-1} does not move when $\varepsilon \to 0$ by replacing λ_ν with $\lambda_\nu \varepsilon$, to obtain $(q_1^{-1}\varepsilon - \lambda_\nu \varepsilon)^{-1} = \varepsilon^{-1}(q_1^{-1} - \lambda_\nu)^{-1}$. By taking the limit of εH as $\varepsilon \to 0$, we obtain the new Hamiltonian H

$$H(q_1, p_1, q, p) := p_1 q_1^2 - \langle \mathfrak{A}(q_1) A_1 q, p \rangle, \tag{14}$$

where

$$\mathfrak{A}_\nu(q_1) = \begin{cases} -\lambda_\nu^{-1} & \text{if } \nu \in J' \\ (q_1^{-1} - \lambda_\nu)^{-1} & \text{if } \nu \in J. \end{cases}$$

Note that, by the confluence precedure we obtain

$$-q_1^2 \frac{dq}{dq_1} = \mathfrak{A} A_1 q, \quad -q_1^2 \frac{dp}{dq_1} = -{}^t A_1 \mathfrak{A} p. \tag{15}$$

If λ_j are mutually distinct, then it follows from (10) that A_1 is a diagonal matrix. Denote the diagonal entries of A_1 by τ_j. Then we have

$$H(q_1, p_1, q, p) = p_1 q_1^2 + \sum_{j=2}^{n} \frac{\tau_j}{\lambda_j} q_j p_j + \sum_{j \in J} \frac{\tau_j}{\lambda_j^2} \frac{q_j p_j}{q_1 - \lambda_j^{-1}}. \tag{16}$$

4 First integrals and computation of monodromy

Let H be given by (14). Then the Hamiltonian vector field χ_H is given by

$$\begin{aligned} \chi_H := q_1^2 \frac{\partial}{\partial q_1} - 2 q_1 p_1 \frac{\partial}{\partial p_1} + \langle \partial_{q_1} \mathfrak{A}(q_1) A_1 q, p \rangle \frac{\partial}{\partial p_1} \\ - \sum_{\nu=2}^{n} (\mathfrak{A}(q_1) A_1 q)_\nu \frac{\partial}{\partial q_\nu} + \sum_{\nu=2}^{n} ({}^t A_1 \mathfrak{A}(q_1) p)_\nu \frac{\partial}{\partial p_\nu}, \end{aligned} \tag{17}$$

where $(\mathfrak{A}(q_1)A_1q)_\nu$ denotes the ν-th component of $\mathfrak{A}(q_1)A_1q$ and so on. We assume (10). First we look for first integrals of the form $\psi = \sum_{j=2}^n \psi_j(q_1)q_j$. Let $a_{\nu,j}$ be the (ν, j)-component of A_1 and write the ν-th component of $\mathfrak{A}$ by $\mathfrak{A}_\nu$. We consider $\sum_{\nu=2}^n (\mathfrak{A}A_1q)_\nu \frac{\partial}{\partial q_\nu}\psi$. Since $\frac{\partial}{\partial q_\nu}\psi = \psi_\nu(q_1)$, we have

$$\sum_{\nu=2}^{n}(\mathfrak{A}A_1q)_\nu \frac{\partial}{\partial q_\nu}\psi = \sum_\nu \left(\sum_j \mathfrak{A}_\nu a_{\nu,j} q_j \psi_\nu\right) = \sum_j \left(\sum_\nu \mathfrak{A}_\nu a_{\nu,j}\psi_\nu\right) q_j. \quad (18)$$

Hence $\chi_H\psi = 0$ is equivalent to

$$q_1^2 \frac{d\psi_j}{dq_1} + \sum_\nu \mathfrak{A}_\nu a_{\nu,j}\psi_\nu = 0, \quad j = 2, \dots, n, \quad (19)$$

or equivalently, with $\Psi := (\psi_\nu(q_1))_{\nu\downarrow 2,\dots,n}$

$$\mathfrak{A}^{-1} q_1^2 \frac{d\Psi}{dq_1} + {}^tA_1\Psi = 0. \quad (20)$$

By (10) we have $a_{i,j}\mathfrak{A}_i = a_{i,j}\mathfrak{A}_j$ for every i and j. Hence, if $\mathfrak{A}_i \neq \mathfrak{A}_j$, then we have $a_{i,j} = 0$. Indeed, the condition $\mathfrak{A}_i \neq \mathfrak{A}_j$ holds, if $i \in J$ and $j \in J'$ or $i \in J'$ and $j \in J$, or more generally if $\lambda_i \neq \lambda_j$. Hence, by suitable permutation of λ_j one may assume that A_1 is a block diagonal matrix each of which blocks are assigned by some k and those j's such that $\lambda_j = \lambda_k$. Moreover, we may assume that there exist positive integers, ν, μ, $n_1, n_2, \dots, n_\nu, n_{\nu+1}, \dots, n_\mu$ such that

$$n_1 + \cdots + n_\nu = \#J', \quad n_{\nu+1} + \cdots + n_\mu = \#J, \quad \#J + \#J' = n - 1$$

and that, there exist $k_1, k_2, \dots, k_\nu \in J'$ and $k_{\nu+1}, \dots, k_\mu \in J$ so that $\mathfrak{A}_i$'s are given by

$$\begin{aligned}
&-\lambda_{k_1}^{-1}\ (1 \le i \le n_1),\ -\lambda_{k_2}^{-1}\ (n_1 + 1 \le i \le n_1 + n_2),\ \cdots, \qquad (21)\\
&-\lambda_{k_\nu}^{-1}\ (n_1 + \cdots + n_{\nu-1} + 1 \le i \le n_1 + \cdots + n_\nu),\\
&(q_1^{-1} - \lambda_{k_{\nu+1}})^{-1} \quad (n_1 + \cdots + n_\nu + 1 \le i \le n_1 + \cdots + n_{\nu+1}), \cdots,\\
&(q_1^{-1} - \lambda_{k_\mu})^{-1} \quad (n_1 + \cdots + n_{\mu-1} + 1 \le i \le n_1 + \cdots + n_\mu).
\end{aligned}$$

We take a non singular constant matrix P such that $P^tA_1P^{-1} =: B_1$ is a Jordan canonical form. Set $\Phi = P\Psi$. Because $\mathfrak{A}$ and A_1 commute, (20) can be written in

$$\mathfrak{A}^{-1} q_1^2 \frac{d\Phi}{dq_1} + B_1\Phi = 0. \quad (22)$$

First we consider the rows of (22) corresponding to some $-\lambda_k^{-1}$ in $\mathfrak{A}$, $k \in J'$, where $k = k_j$ $(1 \le j \le \nu)$. The block of B_1 corresponding to $-\lambda_k^{-1}$ in $\mathfrak{A}$ can be decomposed into the sum of Jordan blocks with size $m(k, s)$ and diagonal elements $-\tau(k, s)$ $(s = 1, 2, \dots, j_k)$ for some $m(k, s)$ and $-\tau(k, s)$, where j_k is the number of Jordan blocks in B_1 corresponding to $-\lambda_k^{-1}$.

For simplicity, assume that the block of B_1 corresponding to $-\lambda_k^{-1}$ in $\mathfrak{A}$ has one Jordan block of size ℓ with diagonal component $-\tau_k$ and the lower off-diagonal element 1 for some ℓ. Set $\Phi = {}^t(\Phi_1, \dots, \Phi_\ell)$. Then (22) gives the system of equations for Φ_j

$$-\lambda_k q_1^2 \frac{d\Phi_j}{dq_1} - \tau_k \Phi_j + \Phi_{j-1} = 0, \quad j = 1, 2, \dots, \ell. \tag{23}$$

Let m, $1 \le m \le \ell$ be given. We will solve (23) by defining $\Phi_j = 0$ for $j < m$. Indeed, for $j = m$ (23) becomes the equation of Φ_m, $-\lambda_k q_1^2 (d\Phi_m/dq_1) - \tau_k \Phi_m = 0$. Hence the solution is given by $\Phi_m = \exp(\tau_k/(\lambda_k q_1))$. Then one can inductively determine Φ_j for $j > m$ and one obtains a first integral for each m, $1 \le m \le \ell$. They are functionally independent solutions of (23). More precisely, we obtain Φ_j for $j > m$ as follows. Φ_{m+1} is given by $\Phi_{m+1} = -(\lambda_k q_1)^{-1} \exp(\tau_k/(\lambda_k q_1))$. Then, one can easily see, by induction, that Φ_{m+i} is given by

$$\Phi_{m+i} = \tilde{E}_i(q_1) \exp\left(\frac{\tau_k}{\lambda_k q_1}\right), \quad i = 0, 1, \dots, \ell - m, \quad \tilde{E}_i(q_1) := \frac{(-1)^i}{\lambda_k^i i! q_1^i}. \tag{24}$$

Next we consider the case where the block of A_1 is assigned by some $k \in J$. We make a similar argument as in the case $k \in J'$. Namely, instead of (23) we have

$$(q_1^{-1} - \lambda_k) q_1^2 \frac{d\Phi_j}{dq_1} - \tau_k \Phi_j + \Phi_{j-1} = 0, \quad j = 1, 2, \dots, \ell. \tag{25}$$

Let $1 \le m \le \ell$ and define $\Phi_j = 0$ for $j < m$. By (25) with $j = m$ we easily see that Φ_m is given by

$$w_k(q_1) := \left(\frac{q_1}{q_1 - \lambda_k^{-1}}\right)^{\tau_k}. \tag{26}$$

We determine Φ_j for $j > m$ inductively. Consider (25) with $j = m+1$. Recalling that Φ_m is the solution of the inhomogeneous equation of (25) with $j = m+1$ and that

$$-\frac{1}{q_1^2(q_1^{-1} - \lambda_k)} = -\frac{1}{q_1} + \frac{1}{q_1 - \lambda_k^{-1}},$$

we have

$$\Phi_{m+1} = \left(\frac{q_1}{q_1 - \lambda_k^{-1}}\right)^{\tau_k} \log\left(\frac{q_1 - \lambda_k^{-1}}{q_1}\right). \tag{27}$$

When we determine Φ_{m+2} by (25) with $j = m+2$, we use the relation

$$\int^{q_1} \left(\frac{1}{t - \lambda_k^{-1}} - \frac{1}{t}\right) \log\left(\frac{t - \lambda_k^{-1}}{t}\right) dt = \frac{1}{2}\left(\log\left(\frac{q_1 - \lambda_k^{-1}}{q_1}\right)\right)^2 + C,$$

where C is a constant. Then we take

$$\Phi_{m+2} = \frac{1}{2!}\left(\frac{q_1}{q_1-\lambda_k^{-1}}\right)^{\tau_k}\left(\log\left(\frac{q_1-\lambda_k^{-1}}{q_1}\right)\right)^2. \tag{28}$$

In the same way, one can show that

$$\Phi_{m+j} \;=\; E_j(q_1)\left(\frac{q_1}{q_1-\lambda_k^{-1}}\right)^{\tau_k}, \tag{29}$$

$$E_j(q_1) := \frac{1}{j!}\left(\log\left(\frac{q_1-\lambda_k^{-1}}{q_1}\right)\right)^j, j = 0, 1, \dots, \ell - m.$$

Next we will construct the first integrals of the form $\sum_{j=2}^n \tilde{\psi}_j(q_1)p_j$. Because the argument is almost identical to the case of the first integral $\sum_{j=2}^n \psi_j(q_1)q_j$ we will give the sketch of the proof. For the sake of simplicity we write $\sum_{j=2}^n \psi_j(q_1)p_j$ instead of $\sum_{j=2}^n \tilde{\psi}_j(q_1)p_j$. The condition $\chi_H\psi = 0$ is equivalent to (20) with ${}^tA_1$ replaced by $-A_1$. Take B_1 and P as in (22). Then we have

$$\mathfrak{A}^{-1}q_1^2\frac{d\Phi}{dq_1} - {}^tB_1\Phi = 0. \tag{30}$$

Consider the block of A_1 which is assigned by some $k \in J'$. Then, by (30) we have

$$-\lambda_k q_1^2\frac{d\Phi_j}{dq_1} + \tau_k\Phi_j - \Phi_{j+1} = 0, \quad j = 1, 2, \dots, \ell \tag{31}$$

where ℓ is the size of B_1. We can solve (31) by the same method as in (23). Namely, let an integer m, $1 \le m \le \ell$ be given. Define $\Phi_j = 0$ for $j > m$ and determine $\Phi_m, \Phi_{m-1}, \dots, \Phi_1$ recurrently via (31). Then we have

$$\Phi_{m-s} = (-1)^s\tilde{E}_s(q_1)\exp\left(-\frac{\tau_k}{\lambda_k q_1}\right), \quad s = 0, 1, \dots, m-1. \tag{32}$$

Next, we consider the block of A_1 assigned by some $k \in J$. We see that Φ_j's satisy the equation similar to (25)

$$(q_1^{-1} - \lambda_k)q_1^2\frac{d\Phi_j}{dq_1} + \tau_k\Phi_j = \Phi_{j+1}, \quad j = 1, 2, \dots, \ell, \tag{33}$$

where $\Phi_{\ell+1} = 0$. Let m, $1 \le m \le \ell$ be an integer. Define $\Phi_j = 0$ for $j > m$. Then one can easily see that

$$\Phi_{m-s} = (-1)^s E_s(q_1)\left(\frac{q_1-\lambda_k^{-1}}{q_1}\right)^{\tau_k}, \quad s = 0, 1, \dots, m-1. \tag{34}$$

Hence we have the first integral as desired. Moreover, by choosing $m = 1, \dots, \ell$ we obtain ℓ functionally independent first integrals.

We will define the first integrals $\psi_j(q_1, q, p)$ $(j = 1, 2, \dots, 2n-2)$. Choose $k = k_j$ in (21) and a Jordan block with diagonal element $-\tau_k$. Corresponding to the transformation $\Phi = P\Psi$ we define the variable $\tilde{q}$ by $\tilde{q} = {}^tP^{-1}q$. If $k \in J'$, then, by (24) with $m = \ell, \ell-1, \ldots, 1$ the set of first integrals corresponding to the Jordan block are given by

$$\begin{aligned} &\exp\left(\frac{\tau_k}{\lambda_k q_1}\right)\tilde{q}_\kappa,\ \exp\left(\frac{\tau_k}{\lambda_k q_1}\right)\left(\tilde{q}_{\kappa-1} + \tilde{E}_1\tilde{q}_\kappa\right), \ldots, \\ &\exp\left(\frac{\tau_k}{\lambda_k q_1}\right)\left(\tilde{q}_{\kappa-\ell+1} + \tilde{E}_1\tilde{q}_{\kappa-\ell+1} + \cdots + \tilde{E}_{\ell-1}\tilde{q}_\kappa,\right), \end{aligned} \tag{35}$$

where κ is some integer. If $k \in J$, then, by (29) we obtain first integrals

$$\begin{aligned} &\left(\frac{q_1}{q_1 - \lambda_k^{-1}}\right)^{\tau_k}\tilde{q}_\kappa,\ \left(\frac{q_1}{q_1 - \lambda_k^{-1}}\right)^{\tau_k}\left(\tilde{q}_{\kappa-1} + E_1\tilde{q}_\kappa\right), \ldots, \\ &\left(\frac{q_1}{q_1 - \lambda_k^{-1}}\right)^{\tau_k}\left(\tilde{q}_{\kappa-\ell+1} + E_1\tilde{q}_{\kappa-\ell+2} + \cdots + E_{\ell-1}\tilde{q}_\kappa,\right). \end{aligned} \tag{36}$$

In view of (21) we can construct functionally independent $(n-1)$-first integrals $\psi_1, \ldots, \psi_{n-1}$.

Next we construct first integrals $\psi_n, \ldots, \psi_{2n-2}$ depending on p. If $k \in J'$, then we use (32) to obtain

$$\begin{aligned} &\exp\left(-\frac{\tau_k}{\lambda_k q_1}\right)\tilde{p}_{\kappa-\ell+1},\ \exp\left(-\frac{\tau_k}{\lambda_k q_1}\right)\left(\tilde{p}_{\kappa-\ell+2} - \tilde{E}_1\tilde{p}_{\kappa-\ell+1}\right), \\ &\exp\left(-\frac{\tau_k}{\lambda_k q_1}\right)\left(\tilde{p}_{\kappa-\ell+3} - \tilde{E}_1\tilde{p}_{\kappa-\ell+2} + \tilde{E}_2\tilde{p}_{\kappa-\ell+1}\right), \ldots, \\ &\exp\left(-\frac{\tau_k}{\lambda_k q_1}\right)\left(\tilde{p}_\kappa - \tilde{E}_1\tilde{p}_{\kappa-1} + \cdots + (-1)^{\ell-1}\tilde{E}_{\ell-1}\tilde{p}_{\kappa-\ell+1}\right) \end{aligned} \tag{37}$$

where κ is an integer. On the other hand, if $k \in J$, then we use (34), to obtain

$$\begin{aligned} &\left(\frac{q_1 - \lambda_k^{-1}}{q_1}\right)^{\tau_k}\tilde{p}_{\kappa-\ell+1},\ \left(\frac{q_1 - \lambda_k^{-1}}{q_1}\right)^{\tau_k}\left(\tilde{p}_{\kappa-\ell+2} - E_1\tilde{p}_{\kappa-\ell+1}\right), \ldots, \\ &\left(\frac{q_1 - \lambda_k^{-1}}{q_1}\right)^{\tau_k}\left(\tilde{p}_\kappa - E_1\tilde{p}_{\kappa-1} + \cdots + (-1)^{\ell-1}E_{\ell-1}\tilde{p}_{\kappa-\ell+1}\right). \end{aligned} \tag{38}$$

Monodromy. Let ψ_j be the first integrals given by (35), (36), (37) and (38). We look for the monodromy function. For this purpose, consider the analytic continuation of ψ_j with respect to q_1 around the small circle at $q_1 = 0$. We want to expand the analytic continuation of every ψ_j in terms of ψ_ν's. Clearly, if these first integrals are given in terms of (35) or (37), then the first integrals are invariant under the analytic continuation around the origin. Therefore we will consider first integrals (36) and (38). Because the argument is similar we consider (36). For the sake of simplicity we denote the first integrals (36) by $\psi_1, \psi_2, \ldots, \psi_\ell$ in this order.

For the sake of clearity we first consider the case $\ell = 1$. (36) reduces to $\psi_1 \equiv q_1^{\tau_k}(q_1 - \lambda_k^{-1})^{-\tau_k}\tilde{q}_\kappa$. Clearly we have $\psi_1(q_1 e^{2\pi i}) = e^{2\pi i\tau_k}\psi_1(q_1)$. Next we consider the case $\ell = 2$. We have first integrals ψ_1 and $\psi_2(q_1) := q_1^{\tau_k}(q_1 - \lambda_k^{-1})^{-\tau_k}(\tilde{q}_{\kappa-1} + E_1(q_1)\tilde{q}_\kappa)$. Noting that $E_1(q_1 e^{2\pi i}) = E_1(q_1) - 2\pi i$ we have

$$\begin{aligned}\psi_2(q_1 e^{2\pi i}) &= e^{2\pi i\tau_k} q_1^{\tau_k}(q_1 - \lambda_k^{-1})^{-\tau_k}\left(\tilde{q}_{\kappa-1} + E_1(q_1)\tilde{q}_\kappa - 2\pi i\tilde{q}_\kappa\right) \\ &= e^{2\pi i\tau_k}\psi_2(q_1) - 2\pi i e^{2\pi i\tau_k}\psi_1(q_1).\end{aligned} \tag{39}$$

We will consider the general case. We note

$$E_s(q_1 e^{2\pi i}) = \frac{1}{s!}(E_1(q_1) - 2\pi i)^s = \sum_{j=0}^{s}\frac{E_1^j(-2\pi i)^{s-j}}{j!(s-j)!} = \sum_{j=0}^{s} E_j\frac{(-2\pi i)^{s-j}}{(s-j)!}. \tag{40}$$

Hence we have the following relation for first integrals given by (36)

$$\begin{aligned}&\psi_\ell(q_1 e^{2\pi i}) \\ &= \left(\frac{q_1 e^{2\pi i}}{q_1 - \lambda_k^{-1}}\right)^{\tau_k}\left(\tilde{q}_{\kappa-\ell+1} + E_1(q_1 e^{2\pi i})\tilde{q}_{\kappa-\ell+2} + \cdots + E_{\ell-1}(q_1 e^{2\pi i})\tilde{q}_\kappa\right) \\ &= e^{2\pi i\tau_k}\left(\frac{q_1}{q_1 - \lambda_k^{-1}}\right)^{\tau_k}\sum_{r=0}^{\ell-1}\frac{(-2\pi i)^r}{r!}\left(\tilde{q}_{\kappa-\ell+1+r} + \cdots + E_{\ell-1-r}(q_1)\tilde{q}_\kappa\right) \\ &= \sum_{r=0}^{\ell-1} e^{2\pi i\tau_k}\frac{(-2\pi i)^r}{r!}\psi_{\ell-r}(q_1),\end{aligned} \tag{41}$$

where $E_0 = 1$ and $E_s = 0$ for $s < 0$. In the same way, for the first integrals given by (38) we have

$$\begin{aligned}&\psi_\ell(q_1 e^{2\pi i}) \\ &= \left(\frac{q_1 - \lambda_k^{-1}}{q_1 e^{2\pi i}}\right)^{\tau_k}\left(\tilde{p}_\kappa - E_1(q_1 e^{2\pi i})\tilde{p}_{\kappa-1} + \cdots + (-1)^{\ell-1}E_{\ell-1}(q_1 e^{2\pi i})\tilde{p}_{\kappa-\ell+1}\right) \\ &= \sum_{r=0}^{\ell-1} e^{-2\pi i\tau_k}\frac{(2\pi i)^r}{r!}\psi_{\ell-r}(q_1).\end{aligned} \tag{42}$$

Let $v(c) = (v_{k,j}(c))_{k,j}$ and $w(c) = (w_{k,j}(c))_{k,j}$ be the monodromy function, where k and j mean that $v_{k,j}$ is the j-th component in the block corresponding to $k = k_\mu$ in (21). We also write $c = (c_{k,j})_{k,j}$ with the same convention. v and w are monodromy functions corresponding to q and p, respectively. Define

$$v_{k,j}(c) = \sum_{r=0}^{j-1} e^{2\pi i\tau_k}\frac{(-2\pi i)^r}{r!}c_{k,j-r} \quad w_{k,j}(c) = \sum_{r=0}^{j-1} e^{-2\pi i\tau_k}\frac{(2\pi i)^r}{r!}c_{k,j-r}. \tag{43}$$

We have

Theorem 2 *Assume (10). Then the functions $(v(c), w(c))$ in (43) is the monodromy function around $q_1 = 0$ of the semi-formal solution of (1) defined by (7) with Hamiltonian (14).*

Remark 2 We can also show, by a similar argument as in Theorem 2 that the monodromy function around $q_1 = \lambda_k^{-1}$ is given by $(\tilde{v}(c), \tilde{w}(c))$, where the (k,j) component of $\tilde{v}(c)$ is given by $w_{k,j}(c)$ and (μ, j) component for $\mu \neq k$ is given by $c_{\mu,j}$. The factor $\tilde{w}(c)$ is similarly defined as $\tilde{v}(c)$ with $w_{k,j}(c)$ replaced by $v_{k,j}(c)$. Indeed, one may consider the analytic continuation around λ_k^{-1} instead of the origin. The form of the first integrals yields the assertion.

Proof of Theorem 2. By Theorem 1 $(q(z,c), p(z,c))$ is the unique solution of (7). On the other hand, by (41), (42) and (43) we see that $(q(z,c), p(z,c))$ satisfies the relations $\psi_\nu(ze^{2\pi i}, q(z,c), p(z,c)) = v_\nu(c)$, where $v_\nu(c)$ is the ν-th component of $v(c)$. It follows from Theorem 1 that $q(ze^{2\pi i}, v(c))$ coincides with $q(z,c)$. We have the same relation for $p(z,c)$. Hence we have (4), and the assertion follows. This ends the proof.

Example. We will consider the Hamiltonian (16) assuming that λ_j's are mutually distinct. First we will determine the convergent semi-formal solution of (1). For $k = 2, \ldots, n$, the first integrals of the form $q_k w_k(q_1)$ are given by

$$w_k(q_1) = \begin{cases} \left(\dfrac{q_1}{q_1 - \lambda_k^{-1}}\right)^{\tau_k} & \text{if } k \in J \\ \exp\left(\dfrac{\tau_k}{\lambda_k q_1}\right) & \text{if } k \notin J. \end{cases} \tag{44}$$

Similarly, the first integrals of the form $p_k u_k(q_1)$ are given by

$$u_k(q_1) = w_k(q_1)^{-1}, \quad k = 2, \ldots, n. \tag{45}$$

By (44) and (45) we have the first integrals ψ_j $(j = 1, 2, \ldots, 2n-2)$

$$\psi_j = \begin{cases} q_{j+1} w_{j+1}(q_1) & (j = 1, 2, \ldots, n-1) \\ p_{j-n+2} w_{j-n+2}(q_1)^{-1} & (j = n, n+1, \ldots, 2n-2). \end{cases} \tag{46}$$

We define the convergent non constant semi-formal solution $q(z,c)$ and $p(z,c)$ of (1) by (7) with $q_1 = z$. Let $v(c)$ be the monodromy function defined by (4). We will study the monodromy around the origin $z_0 = 0$ or around $z_0 = \lambda_k^{-1}$ for some $k \in J$. Note that λ_k^{-1} is a regular singular point of our equation which remains unchanged under the confluence procedure.

We consider the case $z_0 = 0$. In order to determine $v(c)$, we first note $H(q_1 e^{2\pi i}, p_1, q, p) = H(q_1, p_1, q, p)$. On the other hand, for $1 \leq j \leq n-1$ we have

$$\begin{aligned} &\psi_j(q_1 e^{2\pi i}, q, p) = q_{j+1} w_{j+1}(q_1 e^{2\pi i}) = \\ &= \begin{cases} e^{2\pi i \tau_{j+1}} q_{j+1} w_{j+1}(q_1) = c_j e^{2\pi i \tau_{j+1}}, & \text{if } j+1 \in J \\ q_{j+1} w_{j+1}(q_1) = c_j, & \text{if } j+1 \notin J. \end{cases} \end{aligned} \tag{47}$$

If $n \leq j \leq 2n-2$, then we have

$$\begin{aligned} &\psi_j(q_1 e^{2\pi i}, q, p) = q_{j-n+2} w_{j-n+2}(q_1 e^{2\pi i})^{-1} = \\ &= \begin{cases} e^{-2\pi i \tau_{j-n+2}} p_{j-n+2} w_{j-n+2}(q_1)^{-1} = c_j e^{-2\pi i \tau_{j-n+2}}, & \text{if } j-n+2 \in J \\ p_{j-n+2} w_{j-n+2}(q_1)^{-1} = c_j, & \text{if } j-n+2 \notin J. \end{cases} \end{aligned} \tag{48}$$

We define $v(c) = (v_j(c))_j$ by

$$v_j(c) = \begin{cases} c_j e^{2\pi i \tau_{j+1}}, & \text{if } 1 \le j \le n-1,\ j+1 \in J \\ c_j, & \text{if } 1 \le j \le n-1,\ j+1 \notin J \\ c_j e^{-2\pi i \tau_{j-n+2}} & \text{if } n \le j \le 2n-2,\ j-n+2 \in J \\ c_j, & \text{if } n \le j \le 2n-2,\ j-n+2 \notin J. \end{cases} \tag{49}$$

Similarly, we define $\tilde{v}(c) = (\tilde{v}_j(c))_j$ by the right-hand side of (49) with τ_{j+1} and τ_{j-n+2} in (49) replaced by $-\tau_{j+1}\delta_{k,j+1}$ and $-\tau_{j-n+2}\delta_{k,j-n+2}$, respectively. Here $\delta_{k,j+1}$ and $\delta_{k,j-n+2}$ are Kronecker's delta. Then, by Theorem 2 and the remark which follows we have

Corollary 1 *Assume that $\lambda_j \ne 0$ for all j and that λ_j's are mutually distinct. Then the monodromy functions for the Hamiltonian (16) around the origin and λ_k^{-1} $(k \in J)$ are given by (49) and $\tilde{v}(c)$, respectively.*

5 Monodromy for Hamiltonians with nonlinear perturbations

Consider the Hamiltonian $H + H_1$, where H and H_1 are given, respectively, by (16) and

$$H_1 = \sum_{j=2}^{n} q_j^2 B_j(q_1, q), \tag{50}$$

where $B_j(q_1, q)$'s are holomorphic at the origin with respect to $(q_1, q) \in \mathbb{C} \times \mathbb{C}^{n-1}$. One can see that H is integrable, while $H + H_1$ is not integrable for generic $H_1 \ne 0$. (cf. [5]).

In order to give the formula of the monodromy we will construct first integrals of the Hamiltonian vector field $\chi_H + \chi_{H_1}$ in the forms $q_k w_k(q_1)$ $(k = 2, \ldots, n)$ and $p_k u_k(q_1) + W_k(q_1, q)$ $(k = 2, \ldots, n)$. Note that χ_{H_1} is given by

$$\chi_{H_1} = \sum_{j=2}^{n} \left(-2q_j B_j \frac{\partial}{\partial p_j} - q_j^2 \sum_{\nu=2}^{n} \partial_{q_\nu} B_j \frac{\partial}{\partial p_\nu} - q_j^2 (\partial_{q_1} B_j) \frac{\partial}{\partial p_1} \right). \tag{51}$$

As for the first integrals $q_k w_k(q_1)$ we have $\chi_{H_1}(q_k w_k(q_1)) = 0$ because the first integrals do not contain p and p_1. Hence $q_k w_k(q_1)$'s are first integrals of $\chi_H + \chi_{H_1}$, where w_k is given by (44).

We will construct the first integrals $p_k u_k(q_1) + W_k(q_1, q)$ by solving $(\chi_H + \chi_{H_1})(p_k u_k + W_k) = 0$, where $k = 2, \ldots, n$. We compare the coefficients of p_k in the equation. Because no term containing p_k appears from $\chi_{H_1}(p_k u_k + W_k)$, we may consider $\chi_H(p_k u_k) = 0$. We easily see that u_k is given by $u_k = w_k^{-1}(q_1)$, where $w_k(q_1)$ is given by (44). Next we construct W_k by comparing the coefficients of the powers of $p_k^0 = 1$ in the equation $(\chi_H + \chi_{H_1})(p_k u_k +$

$W_k) = 0$. Because $\chi_{H_1} W_k = 0$ by definition, it follows that W_k is determined by the equation

$$\chi_H W_k = -\chi_{H_1}(p_k u_k) = u_k \left(2q_k B_k + \sum_{j=2}^{n} q_j^2 \partial_{q_k} B_j \right).$$

By expanding $B_j(q_1, q) = \sum_\ell B_j^{(\ell)}(q_1) q^\ell$ and $W_k(q_1, q) = \sum_\ell W_k^{(\ell)}(q_1) q^\ell$ and setting

$$\mathcal{R}^{(\ell)}(q_1) = \left(2B_k^{(\ell - e_k)}(q_1) + \sum_{j=2}^{n} (\ell + e_k - 2e_j) B_j^{(\ell + e_k - 2e_j)}(q_1) \right),$$

where e_k is the k-th unit vector, we see that $W_k^{(\ell)}(q_1)$ satisfies

$$\left(q_1^2 \frac{d}{dq_1} + \sum_{j=2}^{n} \frac{\tau_j}{\lambda_j} \ell_j + \sum_{j \in J} \frac{\tau_j}{\lambda_j^2} \frac{\ell_j}{q_1 - \lambda_j^{-1}} \right) W_k^{(\ell)} = w_k(q_1)^{-1} \mathcal{R}^{(\ell)}(q_1). \tag{52}$$

The solution of the inhomogeneous equation is given by $\prod_{j=2}^{n} w_j(q_1)^{\ell_j}$. Hence $W_k^{(\ell)}$ is given by

$$W_k^{(\ell)} = \left(\prod_{\nu=2}^{n} w_\nu(q_1)^{\ell_\nu} \right) \int_a^{q_1} t^{-2} w_k(t)^{-1} \mathcal{R}^{(\ell)}(t) \prod_{\nu=2}^{n} w_\nu(t)^{-\ell_\nu} dt, \tag{53}$$

where $a \in \mathbb{C} \setminus 0$ is some fixed point. Note that $W_k^{(\ell)}$ is analytic on the universal covering space of $\mathbb{C} \setminus \{0, \lambda_j^{-1}(j \in J)\}$. The series $\sum_\ell W_k^{(\ell)}(q_1) q^\ell$ converges if q_1 is on some compact set in the universal covering space of $\mathbb{C} \setminus \{0, \lambda_j^{-1}(j \in J)\}$ and q is in some neighborhood of the origin. Note that $\sum_\ell W_k^{(\ell)}(q_1) q^\ell$ is the convergent semi-formal series. Summing up the above we have

Theorem 3 *The Hamiltonian system with the Hamiltonian function $H + H_1$ has $(2n-1)$ functionally independent first integrals of the form, $H + H_1$, $q_k w_k(q_1)$, $p_k w_k(q_1)^{-1} + W_k(q_1, q)$ $(k = 2, \ldots, n)$.*

Monodromy function We will detemine the monodromy function. Define the first integrals ψ_j by (46) with $p_{j-n+2} w_{j-n+2}(q_1)^{-1}$ replaced by $p_{j-n+2} \times w_{j-n+2}(q_1)^{-1} + W_{j-n+2}(q_1, q)$. We first consider the monodromy around the origin. Suppose that $q = q(q_1, c)$ and $p = p(q_1, c)$ satisfy (7). We shall show that there exists $v_j(c)$ such that q and p satisfy

$$\psi_j(q_1 e^{2\pi i}, q, p) = v_j(c) \quad \text{for} \quad 1 \leq j \leq 2n-2. \tag{54}$$

If one can show the relation, then, by the uniqueness of semi-formal solution we have $q(q_1 e^{2\pi i}, v(c)) = q(q_1, c)$ and $p(q_1 e^{2\pi i}, v(c)) = p(q_1, c)$. Hence $v(c)$ is the desired monodromy function.

The relation (54) is clear if $1 \le j \le n-1$ by definition. Indeed, $v_j(c)$'s $(1 \le j \le n-1)$ are given by (49). Next we consider

$$\psi_j(q_1 e^{2\pi i}, q, p) = p_j w_j(q_1 e^{2\pi i})^{-1} + W_j(q_1 e^{2\pi i}, q), \quad \text{for } n \le j \le 2n-2.$$

By (53) we have

$$\begin{aligned} W_j(q_1 e^{2\pi i}, q) &= \sum_\ell W_j^{(\ell)}(q_1 e^{2\pi i}) q^\ell \qquad (55) \\ &= \sum_\ell I_{j,\ell}(q_1 e^{2\pi i}, a) \prod_{\nu=2}^n \left(q_\nu w_\nu(q_1 e^{2\pi i})\right)^{\ell_\nu} \\ &= \sum_\ell I_{j,\ell}(q_1 e^{2\pi i}, a e^{2\pi i}) \prod_{\nu=2}^n \left(q_\nu w_\nu(q_1 e^{2\pi i})\right)^{\ell_\nu} \\ &\quad + \sum_\ell I_{j,\ell}(a e^{2\pi i}, a) \prod_{\nu=2}^n \left(q_\nu w_\nu(q_1 e^{2\pi i})\right)^{\ell_\nu}, \end{aligned}$$

where a is sufficiently close to the origin and

$$I_{j,\ell}(q_1, a) = \int_a^{q_1} t^{-2} w_j(t)^{-1} \mathcal{R}^{(\ell)}(t) \prod_{\nu=2}^n w_\nu(t)^{-\ell_\nu} dt. \qquad (56)$$

The integral $I_{j,\ell}(a e^{2\pi i}, a)$ is taken along the circle with center at the origin and radius $|a|$. By definition there exists a complex number m_j such that $w_j(q_1 e^{2\pi i}) = m_j w_j(q_1)$. On the other hand, by (7) we have $q_\nu w_\nu(q_1) = c_\nu$. Hence we have

$$\begin{aligned} &\sum_\ell I_{j,\ell}(a e^{2\pi i}, a) \prod_{\nu=2}^n \left(q_\nu w_\nu(q_1 e^{2\pi i})\right)^{\ell_\nu} \qquad (57) \\ &= \sum_\ell I_{j,\ell}(a e^{2\pi i}, a) \prod_{\nu=2}^n \left(q_\nu w_\nu(q_1) m_\nu\right)^{\ell_\nu} = \sum_\ell I_{j,\ell}(a e^{2\pi i}, a) \prod_{\nu=2}^n \left(c_\nu m_\nu\right)^{\ell_\nu}. \end{aligned}$$

Note that the sum in the right-hand side converges for sufficiently small c.

Next we consider the first term in the right-hand side of (55). By the change of variables like $t = s e^{2\pi i}$ in the integral we have

$$\begin{aligned} &\sum_\ell I_{j,\ell}(q_1 e^{2\pi i}, a e^{2\pi i}) \prod_{\nu=2}^n \left(q_\nu w_\nu(q_1 e^{2\pi i})\right)^{\ell_\nu} \qquad (58) \\ &= \sum_\ell m_j^{-1} I_{j,\ell}(q_1, a) \left(\prod_{\nu=2}^n m_\nu^{-\ell_\nu}\right) \prod_{\nu=2}^n \left(q_\nu w_\nu(q_1) m_\nu\right)^{\ell_\nu} \\ &= \sum_\ell m_j^{-1} I_{j,\ell}(q_1, a) \prod_{\nu=2}^n \left(q_\nu w_\nu(q_1)\right)^{\ell_\nu}. \end{aligned}$$

Note that the right-hand side term is equal to $m_j^{-1}W_j(q_1, q)$. Therefore, by (57) and (58) we have

$$W_j(q_1 e^{2\pi i}, q) = m_j^{-1} W_j(q_1, q) + \sum_{\ell} I_{j,\ell}(ae^{2\pi i}, a) \prod_{\nu=2}^{n} (c_\nu m_\nu)^{\ell_\nu} . \tag{59}$$

By the definition of ψ_j we have

$$\begin{aligned} \psi_j(q_1 e^{2\pi i}, q, p) &= m_j^{-1} \psi_j(q_1, q, p) + \sum_{\ell} I_{j,\ell}(ae^{2\pi i}, a) \prod_{\nu=2}^{n} (c_\nu m_\nu)^{\ell_\nu} \qquad (60) \\ &= m_j^{-1} c_j + \sum_{\ell} I_{j,\ell}(ae^{2\pi i}, a) \prod_{\nu=2}^{n} (c_\nu m_\nu)^{\ell_\nu} =: v_j(c). \end{aligned}$$

Therefore we have (54). We have

Theorem 4 *Let $v_j(c)$ be defined by (49) for $1 \le j \le n-1$ and by (60) for $n \le j \le 2n-2$. Then $v(c) = (v_j(c))_j$ is the monodromy function of the Hamiltonian $H + H_1$.*

References

1. Balser, W., Semi-formal theory and Stokes' phenomenon of nonlinear meromorphic systems of ordinary differential equations, Formal and analytic solutions of differential and difference equations, *Banach Center Publications*, **97** (2012), 11-28.
2. Bolsinov, A.V. and Taimanov, I.A.: Integrable geodesic flows with positive topological entropy. Invent Math. **140** (3) (2000), 639-650.
3. Gorni, G. and Zampieri, G.: Analytic-non-integrability of an integrable analytic Hamiltonian system. Differ. Geom. Appl. **22** (2005), 287-296.
4. Yoshino, M., Smooth-integrable and analytic- nonintegrable resonant Hamiltonians, RIMS Kokyuroku Bessatsu, **B**40, (2013), 177-189.
5. Yoshino, M., Analytic- nonintegrable resonant Hamiltonians which are integrable in a sector, K. Matsuzaki and T. Sugawa (eds.), Proceedings of the 19th ICFIDCAA Hiroshima 2011, Tohoku University Press (2012), 85-96.
6. Yoshino, M., Semi-formal solution and monodromy of some confluent hypergeometric equations, *Accepted for publication in RIMS Kôkyûroku Bessatsu*.