特異的領域変形と楕円型作用素の固有値の挙動

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研究テーマの背景

物理現象において, 生じる波,振動の特性は媒質,空間の形状に大いに依存する

$\Longrightarrow$ 偏微分方程式の大域解析のテーマ: 固有値と特異的な形状との関係が興味深い課題

光, 色彩 (視覚) – 物体の形状, 散乱など

音 (聴覚) — 楽器による音の発生(音程, 音色)

物体の振動特性 – 物体の形状や構造

方程式の主部にある楕円型作用素の固有値(スペクトル)が重要 —— ラプラス作用素, ラメの作用素の固有値の領域依存性を考える

(Equation) $$\rho \frac{\partial^2 u}{\partial t^2} - L[u] = 0$$

(Special solution) $$u(t,x) = e^{i\omega t}\Phi(x)$$

$\Longrightarrow$

$$\rho(i\omega)^2 e^{i\omega t}\Phi - e^{i\omega t}L[\Phi] = 0 \Longleftrightarrow L[\Phi] + \rho\omega^2\Phi = 0$$

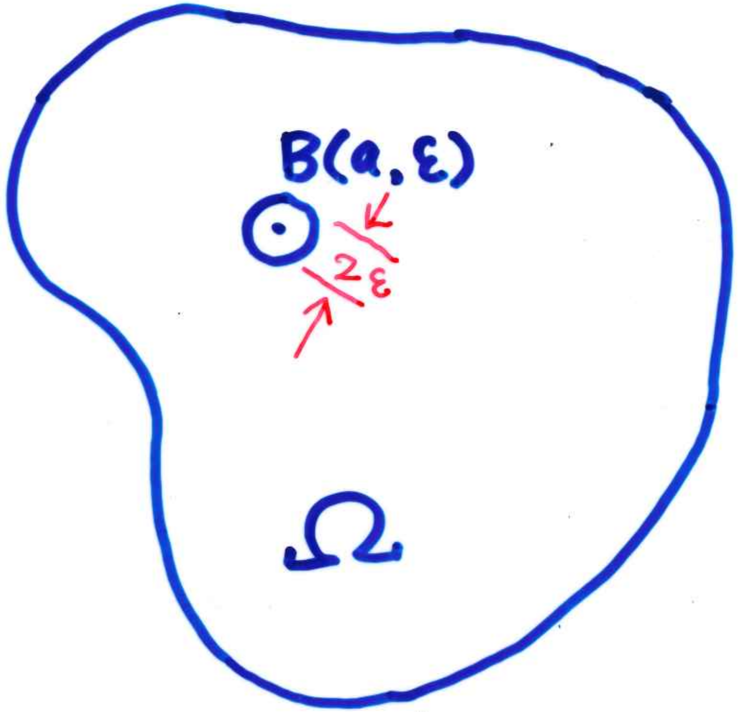
今回考える課題

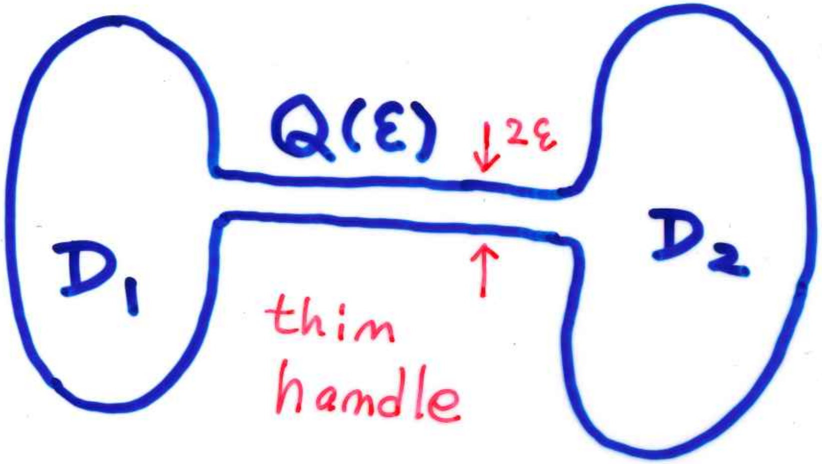
(I-A) 穴や欠陥をもつ領域上のラプラシアンの固有値の摂動

(I-B) 部分的に細い (あるいは薄い) 領域のラプラシアンの固有値の漸近挙動

(II) 細い (あるいは薄い) ３次元弾性体の曲げ振動特性

Two typical cases (2 dimension)





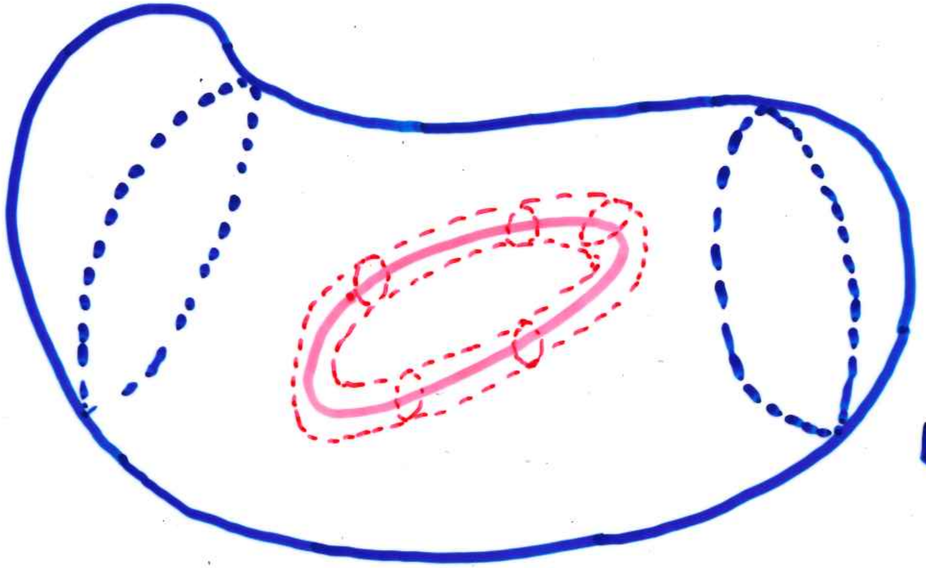
$\Omega(\varepsilon) = \Omega \setminus B(a, \varepsilon)$

Domain with a small hole

$\Omega(\varepsilon) = D_1 \cup D_2 \cup Q(\varepsilon)$

Domain with a thin handle

$\Omega \subset \mathbb{R}^3$



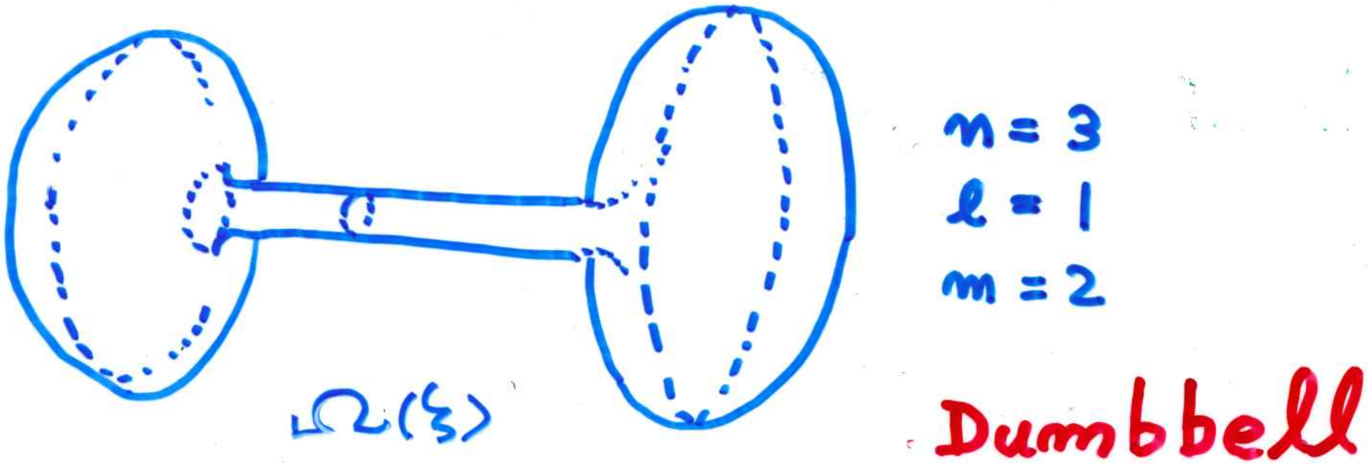
$M =$ 

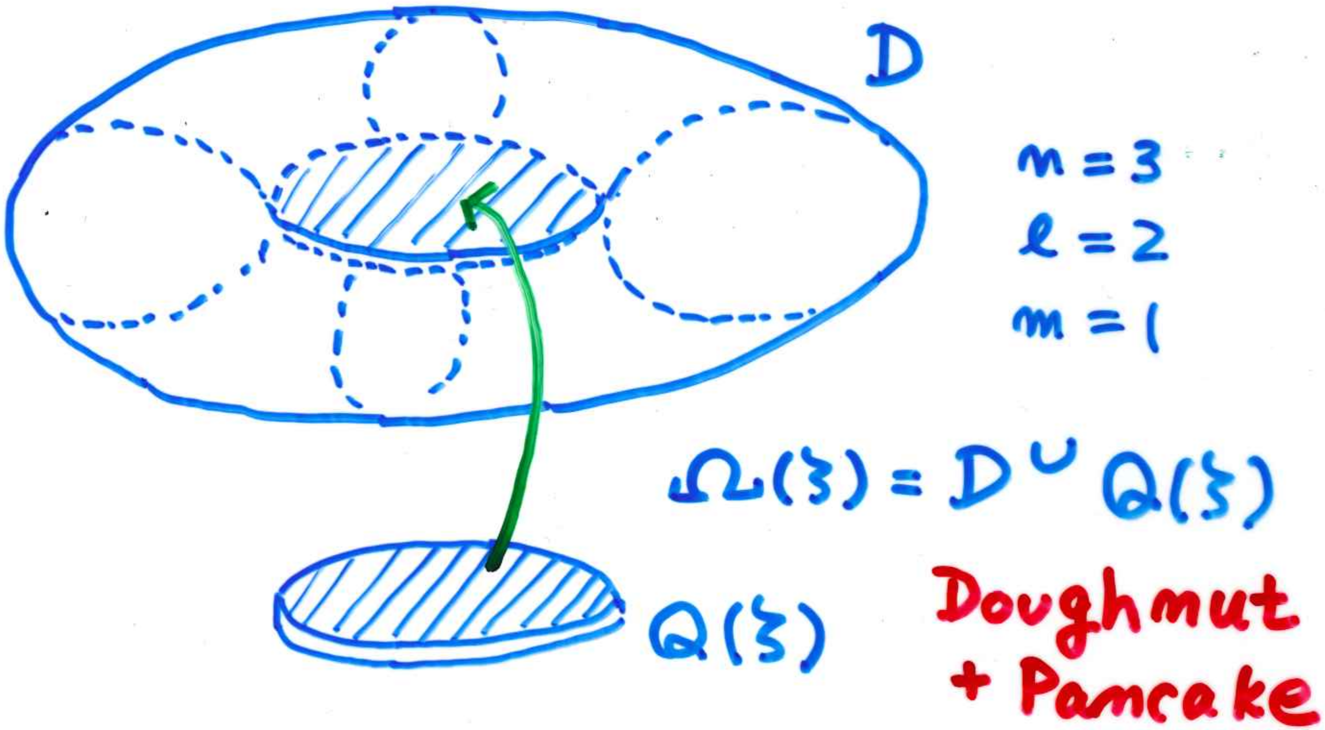
circle in Ω

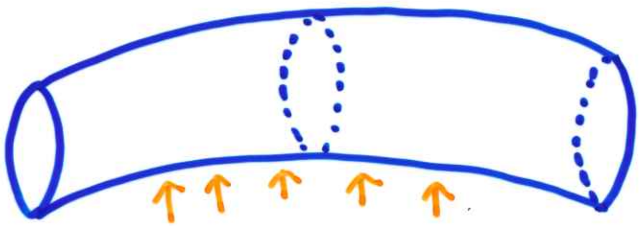
$B(M, \varepsilon)$: ε-neighborhood of M

$\Omega(\varepsilon) = \Omega \setminus B(M, \varepsilon)$

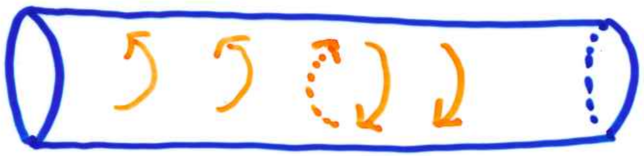
Example


$\Omega(\xi)$
$n = 3$
$\ell = 1$
$m = 2$
Dumbbell

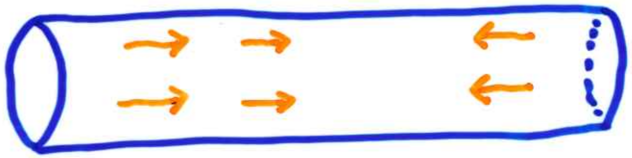

D
$n = 3$
$\ell = 2$
$m = 1$
$\Omega(\xi) = D \cup Q(\xi)$
$Q(\xi)$
Doughnut
+ Pancake



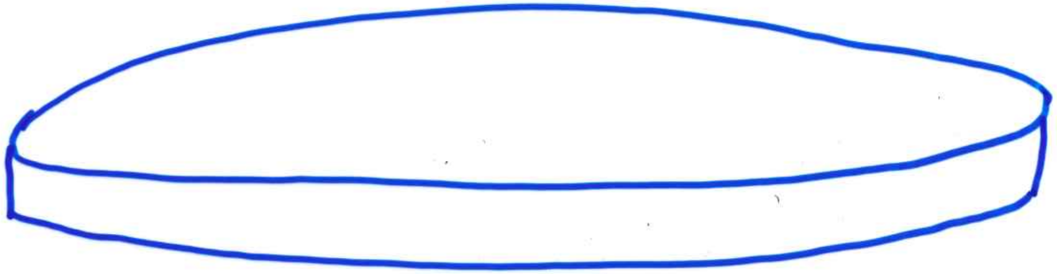
曲げ(Bending)モード



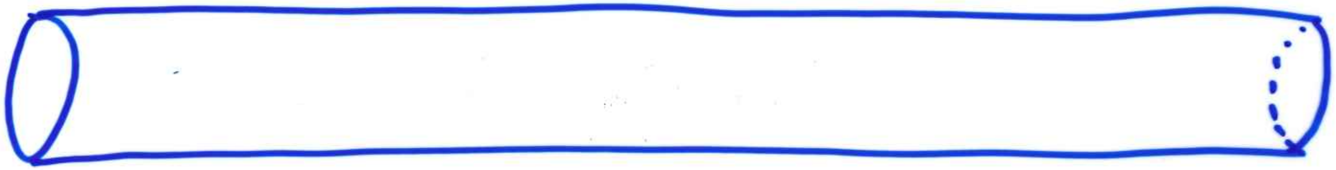
ねじれ(Torsion)モード



伸縮(stretching)モード



薄い板
Thin Slab



細い棒
Thin Rod

Part I : Eigenvalues of the Laplacian in a singularly perturbed domain

Eigenvalue problem

Ω : a bounded domain in $\mathbb{R}^n$ $(n \geqq 2)$ with a smooth boundary $\partial\Omega$

$$(1) \qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega, \\ \text{Dirichlet or Neumann or Robin B.C.} \quad \text{on} \quad \partial\Omega \end{cases}$$

Eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ which are arranged in increasing order with counting multiplicities. Denote a corresponding complete system of orthonormal eigenfunctions by $\{\Phi_k\}_{k=1}^{\infty} \subset L^2(\Omega)$.
(cf. Books of Courant-Hilbert, L.C.Evans, Edmunds-Evans)

Basic Problem–Singular deformation of domains

A singularly perturbed bounded domain $\Omega(\epsilon) \subset \mathbb{R}^n$ ($\epsilon > 0$) small parameter

(A) $\Omega(\epsilon)$ has a small hole or a thin defect (tunnel)
(with some B.C. for emerging boundary).
$\Omega(\epsilon)$ increases as $\epsilon \to 0$.

(B) Some portion of $\Omega(\epsilon)$ shrinks to a low dimensional set.
$\Omega(\epsilon)$ decreases as $\epsilon \to 0$.

How does each eigenvalue $\lambda_k(\epsilon)$ of the Laplacian behaves when $\epsilon \to 0$?

See (A) : Swanson ('63'77), Rauch-Taylor('75), Ozawa ('81, '83),..., (B) : Beale ('75), Chavel-Feldman('81), Ram ('85), ... for early works. See Jimbo ('15) and Jimbo-Kosugi ('09) for the details of I-(A) and I-(B) of the lecture (cf. Nazarov-Mazya-Plamenvsky ('90) other topics)

(A1) Domain with a small hole

Let $\boldsymbol{a} \in \Omega$ be a point and

$$\Omega(\epsilon) = \Omega \setminus \overline{B(\boldsymbol{a}, \epsilon)}, \quad \Gamma(\epsilon) = \partial B(\boldsymbol{a}, \epsilon), \quad \Gamma = \partial\Omega.$$

Dirichlet B.C. on $\Gamma(\epsilon)$

$$(2-D) \qquad \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \qquad \Phi = 0 \text{ on } \Gamma(\epsilon) \cup \Gamma$$

Neumann B.C. on $\Gamma(\epsilon)$

$$(2-N) \qquad \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \qquad \Phi = 0 \text{ on } \Gamma, \quad \partial\Phi/\partial\nu = 0 \text{ on } \Gamma(\epsilon)$$

The $k-$th eigenvalue of (2-D) and (2-N) are denoted by $\lambda_k^D(\epsilon)$ and $\lambda_k^N(\epsilon)$, respectively.

Theorem ($n = 2$ or $n = 3$). Assume λ_k is simple in (1)

$$\lambda_k^D(\epsilon) = \lambda_k + \begin{cases} 4\pi\Phi_k(\boldsymbol{a})^2\epsilon + \text{H.O.T.} & (n = 3) \\ (2\pi/\log(1/\epsilon))\Phi_k(\boldsymbol{a})^2 + \text{H.O.T.} & (n = 2) \end{cases}$$

Theorem ($n = 2$ or $n = 3$). Assume λ_k is simple in (1)

$$\lambda_k^N(\epsilon) = \lambda_k + \begin{cases} \pi(-2|\nabla\Phi_k(\boldsymbol{a})|^2 + (4\lambda_k/3)\Phi_k(\boldsymbol{a})^2)\epsilon^3 + \text{H.O.T.} & (n = 3) \\ \pi\,(-2|\nabla\Phi_k(\boldsymbol{a})|^2 + \lambda_k\Phi_k(\boldsymbol{a})^2)\epsilon^2 + \text{H.O.T.} & (n = 2) \end{cases}$$

cf. S.Ozawa ('81,'83) for the above results.

Remark. Swanson ('63) gave "some perturbation formula" for $\lambda_{k,\epsilon}^{D}$, previously.
These results are proved by the method of "Approximate Green function". There are also results for Robin condition on $\Gamma(\epsilon)$ (cf. Ozawa ('83,'92), Roppongi ('93), Ozawa-Roppongi ('92)). 小澤真の方法は独創的である. 一方, 空間次元が高いとか, あるいは変数係数の場合等に一般化するにはあまり適さない. Swanson の方法は汎用性があり"第一近似"を求めるには非常に役に立つ. しかし, 高精度の摂動論を目指すには限界あり.

There are many related works in different situations (generalization or elaboration). See Maz'ya-Nazarov-Plamenevsky('85, '00), Flucher('95), Ammari-Kang-Lim-Zribi ('10), Lanza de Christoforis ('12), ...

Proof of Perturbation formula of the eigenvalue $n = 2$, Neumann B.C. (Skip)

A bounded domain $\Omega \subset \mathbb{R}^2$, a fixed point $\boldsymbol{a} \in \Omega$. The eigenvalue problem

$$(3) \qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \quad \Phi = 0 \quad \text{on} \quad \partial\Omega \\ \partial\Phi/\partial\nu = 0 \quad \text{on} \quad \Gamma(\epsilon) = \partial B(\boldsymbol{a}, \epsilon) \end{cases}$$

$\{\lambda_k(\epsilon)\}_{k=1}^{\infty}$: the set of eigenvalues.
$\{\Phi_{k,\epsilon}\}_{k=1}^{\infty}$: system of corresponding eigenfunctions with $(\Phi_{p,\epsilon}, \Phi_{q,\epsilon})_{L^2(\Omega(\epsilon))} = \delta(p, q)$.

$$(4) \qquad \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega, \quad \Phi = 0 \quad \text{on} \quad \partial\Omega$$

$\{\lambda_k\}_{k=1}^{\infty}$: the set of eigenvalues
$\{\Phi_k\}_{k=1}^{\infty}$: system of corresponding eigenfunctions with $(\Phi_p, \Phi_q)_{L^2(\Omega)} = \delta(p, q)$.

We want to closely look at $\lambda_k(\epsilon) - \lambda_k$. There are two parts in the process of proof.

(i) Characterization of the behavior of the true eigenfunction $\Phi_{k,\epsilon}$

(ii) Construction a good approximate eigenfunction $\widetilde{\Phi}_{k,\epsilon}$

(i) Characterization of $\Phi_{k,\epsilon}$

Proposition. For any sequence of positive numbers $\{\epsilon(p)\}_{p=1}^{\infty}$ with $\lim_{p\to\infty}\epsilon(p)=0$, there exist a subsequence $\{\epsilon(p(q))\}_{q=1}^{\infty}$ and a sequence $\{\lambda'_k\}_{k=1}^{\infty}$ with an complete orthonormal system $\{\Phi'_k\}_{k=1}^{\infty}\subset L^2(\Omega)$ such that

$$\Delta\Phi'_k+\lambda'_k\Phi'_k=0 \text{ in } \Omega, \quad \Phi'_k=0 \text{ on } \partial\Omega,$$

and

$$\lim_{q\to\infty}\lambda_k(\epsilon(p(q)))=\lambda'_k, \quad \lim_{q\to\infty}\sup_{x\in\Omega(\epsilon(p(q)))}|\Phi_{k,\epsilon(p(q))}(x)-\Phi'_k(x)|=0 \quad (\forall k\in\mathbb{N}).$$

We omit the details of the proof.

Estimation of solutions of elliptic equations away from the small hole.

By the aid of "Barrier functions", we prove a uniform bound in $\Omega(\epsilon)$.

Proof of uniform convergence of $\Phi_{k,\epsilon}$ of $\Omega(\epsilon)$.

Proposition. $\lambda_k=\lambda'_k$ $(k\geqq 1)$ and $\lim_{\epsilon\to 0}\lambda_k(\epsilon)=\lambda_k$.

(ii) Construction of approximate eigen function $\widetilde{\Phi}_{k,\epsilon}$. Modify the eigenfunction Φ_k of Ω around the hole $B(\boldsymbol{a},\epsilon)$. Prepare the function

$$\eta_k(x) = \frac{\langle \nabla \Phi_k(\boldsymbol{a}), x - \boldsymbol{a} \rangle}{|x - \boldsymbol{a}|^2} \quad \text{(harmonic in} \quad \mathbb{R}^2 \setminus \{\boldsymbol{a}\}).$$

It is easy to calculate

$$\nabla \eta_k(x) = \frac{\nabla \Phi_k(\boldsymbol{a})}{|x - \boldsymbol{a}|^2} + \langle \nabla \Phi_k(\boldsymbol{a}), x - \boldsymbol{a} \rangle \frac{(x - \boldsymbol{a})}{|x - \boldsymbol{a}|} \frac{(-2)}{|x - \boldsymbol{a}|^3}$$

$$= \frac{\nabla \Phi_k(\boldsymbol{a})}{|x - \boldsymbol{a}|^2} - 2\langle \nabla \Phi_k(\boldsymbol{a}), x - \boldsymbol{a} \rangle \frac{(x - \boldsymbol{a})}{|x - \boldsymbol{a}|^4}$$

$$\Delta \eta_k(x) = \frac{-2\langle \nabla \Phi_k(\boldsymbol{a}), x - \boldsymbol{a} \rangle}{|x - \boldsymbol{a}|^4} - 2\langle \nabla \Phi_k(\boldsymbol{a}), \frac{x - \boldsymbol{a}}{|x - \boldsymbol{a}|^4} \rangle$$

$$-2\langle \nabla \Phi_k(\boldsymbol{a}), x - \boldsymbol{a} \rangle \left(\frac{2}{|x - \boldsymbol{a}|^4} - \frac{4|x - \boldsymbol{a}|^2}{|x - \boldsymbol{a}|^6} \right) = 0 \qquad (x \neq \boldsymbol{a})$$

Put a function $\widetilde{\Phi}_{k,\epsilon}$ as follows

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) + \epsilon^2 \eta_k(x) & (x \in B(\boldsymbol{a}, r_0) \setminus B(\boldsymbol{a}, \epsilon)) \\ \Phi_k(x) + \epsilon^2 \widehat{\eta}_k(x) & (x \in \Omega \setminus B(\boldsymbol{a}, r_0)) \end{cases}$$

where $\widehat{\eta}_k$ is the unique solution $\widehat{\eta}$ of

$$\Delta \widehat{\eta} = 0 \text{ in } \Omega \setminus B(\boldsymbol{a}, r_0), \qquad \widehat{\eta}(x) = 0 \text{ on } \partial\Omega, \qquad \widehat{\eta}(x) = \eta_k(x) \text{ on } \partial B(\boldsymbol{a}, r_0).$$

Other choice of approximate eigenfunction

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) + \eta_{k,\epsilon}(x) & (x \in B(\boldsymbol{a}, r_0) \setminus B(\boldsymbol{a}, \epsilon)) \\ \Phi_k(x) & (x \in \Omega \setminus B(\boldsymbol{a}, r_0)) \end{cases}$$

where $\eta = \eta_{k,\epsilon} \in C^2(B(\boldsymbol{a}, r_0) \setminus B(\boldsymbol{a}, \epsilon))$ is the unique solution of

$$\Delta \eta = 0 \text{ in } B(\boldsymbol{a}, r_0) \setminus B(\boldsymbol{a}, \epsilon), \quad \eta = 0 \text{ on } \partial B(\boldsymbol{a}, r_0), \ \frac{\partial \eta}{\partial \nu_1} = -\frac{\partial \Phi_k}{\partial \nu_1} \quad \text{on} \quad \partial B(\boldsymbol{a}, \epsilon).$$

Here ν_1 is the inward unit normal vector on $\partial B(\boldsymbol{a}, \epsilon)$.

The equation is written as

$$\int_{\Omega(\epsilon)} (\langle \nabla \Phi_{k,\epsilon}, \nabla \varphi \rangle - \lambda_k(\epsilon)\Phi_{k,\epsilon}\varphi)dx = 0 \qquad (\forall \varphi \in H^1(\Omega(\epsilon)) \text{ with } \varphi = 0 \text{ on } \partial\Omega)$$

Substitute $\varphi = \widetilde{\Phi}_{k,\epsilon}$, we have

$$\int_{\Omega(\epsilon)} (\langle \nabla \Phi_{k,\epsilon}, \nabla \widetilde{\Phi}_{k,\epsilon} \rangle - \lambda_k(\epsilon)\Phi_{k,\epsilon}\widetilde{\Phi}_{k,\epsilon})dx = 0$$

Swanson trick : One method to deduce the perturbation of the eigenvalue.

Looking into this integral equality leads us to see the details of $\lambda_k(\epsilon) - \lambda_k$.

$$\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \langle \nabla\Phi_{k,\epsilon}, \nabla(\Phi_k+\epsilon^2\eta_k)\rangle dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \langle \nabla\Phi_{k,\epsilon}, \nabla(\Phi_k+\epsilon^2\widehat{\eta}_k)\rangle dx$$
$$-\lambda_k(\epsilon)\left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon}\,(\Phi_k+\epsilon^2\eta_k)dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\,(\Phi_k+\epsilon^2\widehat{\eta}_k)dx\right) = 0$$

Gauss-Green formula gives

$$\int_{\partial(B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon))} \Phi_{k,\epsilon}\frac{\partial}{\partial\nu}(\Phi_k+\epsilon^2\eta_k)dS - \int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon}\Delta\Phi_k dx$$
$$+\int_{\partial(\Omega\setminus B(\boldsymbol{a},r_0))} \Phi_{k,\epsilon}\frac{\partial}{\partial\nu}(\Phi_k+\epsilon^2\widehat{\eta}_k)dS - \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\Delta\Phi_k dx$$
$$-\lambda_k(\epsilon)\left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon}\,(\Phi_k+\epsilon^2\eta_k)dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\,(\Phi_k+\epsilon^2\widehat{\eta}_k)dx\right) = 0$$

Using $\Delta\Phi_k = -\lambda_k\Phi_k$ we get

$$(\lambda_k(\epsilon) - \lambda_k)\int_{\Omega(\epsilon)} \Phi_{k,\epsilon}\Phi_k dx = \int_{\partial B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon}\frac{\partial}{\partial\nu_1}(\Phi_k + \epsilon^2\eta_k)dS + \int_{\partial B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\frac{\partial}{\partial\nu_2}(\epsilon^2\eta_k)dS$$

$$+\int_{\partial B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\frac{\partial}{\partial\nu_3}(\epsilon^2\eta_k)dS - \lambda_k(\epsilon)\left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon}\,(\epsilon^2\eta_k)dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\,(\epsilon^2\widehat{\eta}_k)dx\right)$$

$$= I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon) + I_4(\epsilon)$$

ν_1 is the unit outward normal vector of $\partial B(\boldsymbol{a},\epsilon)$ at $|x-\boldsymbol{a}| = \epsilon$
ν_2 is the unit outward normal vector of $\partial B(\boldsymbol{a},r_0)$ at $|x-\boldsymbol{a}| = r_0$
ν_3 is the unit outward normal vector of $\partial(\Omega\setminus B(\boldsymbol{a},r_0))$ at $|x-\boldsymbol{a}| = r_0$

Similarly, we get

$$(\lambda_\ell(\epsilon) - \lambda_k)\int_{\Omega(\epsilon)} \Phi_{\ell,\epsilon}\Phi_k dx = \int_{\partial B(\boldsymbol{a},\epsilon)} \Phi_{\ell,\epsilon}\frac{\partial}{\partial\nu_1}(\Phi_k + \epsilon^2\eta_k)dS + \int_{\partial B(\boldsymbol{a},r_0)} \Phi_{\ell,\epsilon}\frac{\partial}{\partial\nu_2}(\epsilon^2\eta_k)dS$$

$$+\int_{\partial B(\boldsymbol{a},r_0)} \Phi_{\ell,\epsilon}\frac{\partial}{\partial\nu_3}(\epsilon^2\eta_k)dS - \lambda_\ell(\epsilon)\left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{\ell,\epsilon}\,(\epsilon^2\eta_k)dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{\ell,\epsilon}\,(\epsilon^2\widehat{\eta}_k)dx\right)$$

for any $k,\ell \geqq 1$.

Estimate the right hand side, we can prove

$$(\lambda_\ell(\epsilon) - \lambda_k)(\Phi_{\ell,\epsilon}, \Phi_k)_{L^2(\Omega(\epsilon))} = O(\epsilon^2)$$

(with the aid of calucation)and accordingly , we can also see

$$\lim_{\epsilon \to 0} \lambda_k(\epsilon) = \lambda_k$$

for any $k \geqq 1$.

Evalueate and estimate the terms $I_1(\epsilon), I_2(\epsilon), I_3(\epsilon), I_4(\epsilon)$ of the right hand side.

On $\partial B(\boldsymbol{a},\epsilon)$ (i.e. $|x-\boldsymbol{a}|=\epsilon$), we have

$$\frac{\partial}{\partial \nu_1}(\Phi_k+\epsilon^2\eta_k)=\langle \nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle = O(\epsilon)$$

and we get

$$\frac{1}{\epsilon^2}I_1(\epsilon)=\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}\Phi_{k,\epsilon}\frac{\partial}{\partial \nu_1}(\Phi_k+\epsilon^2\eta_k)dS=\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}\Phi_{k,\epsilon}\langle \nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle dS$$

$$=\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}\Phi_k'\langle \nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle dS+\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}(\Phi_{k,\epsilon}-\Phi_k')\langle \nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle dS$$

$$=\widetilde{I}_{1,1}(\epsilon)+\widetilde{I}_{1,2}(\epsilon)=\widetilde{I}_{1,1}(\epsilon)+o(1)$$

For $\epsilon = \epsilon(p(q))$, we have

$$I_4(\epsilon) = -\lambda_k \epsilon^2 \int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi'_k \eta_k dx - \lambda_k \epsilon^2 \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi'_k \widehat{\eta}_k dx + o(\epsilon^2)$$

$$= \epsilon^2 \int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Delta\Phi'_k \eta_k dx - \lambda_k \epsilon^2 \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi'_k \widehat{\eta}_k dx + o(\epsilon^2)$$

$$= \epsilon^2 \int_{\partial B(\boldsymbol{a},\epsilon)} \frac{\partial \Phi'_k}{\partial \nu_1} \eta_k dS + \epsilon^2 \int_{\partial B(\boldsymbol{a},r_0)} \frac{\partial \Phi'_k}{\partial \nu_2} \eta_k dS$$

$$-\epsilon^2 \int_{\partial B(\boldsymbol{a},\epsilon)} \Phi'_k \frac{\partial \eta_k}{\partial \nu_1} dS - \epsilon^2 \int_{\partial B(\boldsymbol{a},r_0)} \Phi'_k \frac{\partial \eta_k}{\partial \nu_2} dS - \lambda_k \epsilon^2 \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi'_k \widehat{\eta}_k dx + o(\epsilon^2)$$

Comparing the terms of the right hand side with $I_2(\epsilon)$, $I_3(\epsilon)$, we get

$$\frac{1}{\epsilon^2}(I_2(\epsilon)+I_3(\epsilon)+I_4(\epsilon)) = \int_{\partial B(\boldsymbol{a},\epsilon)} \frac{\partial \Phi'_k}{\partial \nu_1} \eta_k dS - \int_{\partial B(\boldsymbol{a},\epsilon)} \Phi'_k \frac{\partial \eta_k}{\partial \nu_1} dS + o(1) = \widetilde{I}_{2,1}(\epsilon) + \widetilde{I}_{2,2}(\epsilon) + o(1)$$

Lemma.

$$\widetilde{I}_{1,1}(\epsilon) = \lambda_k \pi \Phi_k'(\boldsymbol{a})\Phi_k(\boldsymbol{a}) + o(1) \quad (\epsilon = \epsilon(p(q)) \to 0),$$

$$\widetilde{I}_{2,1}(\epsilon) = -\pi\langle \nabla\Phi_k'(\boldsymbol{a}), \nabla\Phi_k(\boldsymbol{a})\rangle + o(1) \quad (\epsilon = \epsilon(p(q)) \to 0),$$

$$\widetilde{I}_{2,2}(\epsilon) = -\pi\langle \nabla\Phi_k'(\boldsymbol{a}), \nabla\Phi_k(\boldsymbol{a})\rangle + o(1) \quad (\epsilon = \epsilon(p(q)) \to 0).$$

(Sketch of the roof) Evaluate each quantity by Taylor expansiion.

$$\begin{aligned}\widetilde{I}_{1,1} &= \frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)} \Phi_k'\langle \nabla\Phi_k(x) - \nabla\Phi_k(\boldsymbol{a}), \nu_1\rangle dS \\ &= \frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)} \Phi_k'(\boldsymbol{a})\langle \nabla\Phi_k(x) - \nabla\Phi_k(\boldsymbol{a}), \nu_1\rangle dS \\ &+ \frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)} (\Phi_k'(x) - \Phi_k'(\boldsymbol{a}))\langle \nabla\Phi_k(x) - \nabla\Phi_k(\boldsymbol{a}), \nu_1\rangle dS\end{aligned}$$

$$= \frac{1}{\epsilon^2} \int_{\partial B(\boldsymbol{a},\epsilon)} \Phi_k'(\boldsymbol{a}) \sum_{\ell=1}^{2} (x_\ell - a_\ell) \langle \frac{\partial}{\partial x_\ell} (\nabla \Phi_k)(\boldsymbol{a}) + O(\epsilon), (-1) \frac{(x-\boldsymbol{a})}{|x-\boldsymbol{a}|} \rangle dS + o(1)$$

$$= -\frac{1}{\epsilon^2} \int_{\partial B(\boldsymbol{a},\epsilon)} \Phi_k'(\boldsymbol{a}) \sum_{\ell_1,\ell_2=1}^{2} \frac{\partial^2 \Phi_k'}{\partial x_{\ell_1} \partial x_{\ell_2}} (\boldsymbol{a})(x_{\ell_1} - a_{\ell_1})(x_{\ell_2} - a_{\ell_2}) dS + o(1)$$

$$= -\pi \Phi_k'(\boldsymbol{a}) \Delta \Phi_k(\boldsymbol{a}) + o(1) = \pi \lambda_k \Phi_k'(\boldsymbol{a}) \Phi_k(\boldsymbol{a}) + o(1)$$

The remaining two terms $\widetilde{I}_{2,1}(\epsilon)$, $\widetilde{I}_{2,2}(\epsilon)$ are evaluated similarly with the aid of Taylor expansion. $\square$

Eventually we have

$$\lim_{q\to\infty}\frac{\lambda_k(\epsilon(p(q)))-\lambda_k}{\epsilon(p(q))^2}(\Phi_{k,\epsilon(p(q))},\Phi_k)_{L^2(\Omega(\epsilon(p(q))))}$$
$$=\pi(-2\langle\nabla\Phi'_k(\boldsymbol{a}),\nabla\Phi_k(\boldsymbol{a})\rangle+\lambda_k\pi\Phi'_k(\boldsymbol{a})\Phi_k(\boldsymbol{a}))$$

Since λ_k is simple, we accordingly have $\Phi'_k=\Phi_k$ or $\Phi'_k=-\Phi_k$ and the $\{\epsilon(p)\}_{p=1}^{\infty}$ is arbitrary and conclude

$$\lim_{\epsilon\to 0}\frac{\lambda_k(\epsilon)-\lambda_k}{\epsilon^2}=\pi(-2|\nabla\Phi_k(\boldsymbol{a})|^2+\lambda_k\pi\Phi_k(\boldsymbol{a})^2)$$

(A2) Domain with a thin tubular hole

Let M be a $m-$dimensional smooth compact orientable manifold such that $M \subset \Omega$ and $0 \leqq m \leqq n-2$ and put

$$B(M,\epsilon) = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x,M) < \epsilon\}, \qquad \Gamma = \partial\Omega, \quad \Gamma(M,\epsilon) = \partial B(M,\epsilon).$$

Note $|B(M,\epsilon)| = O(\epsilon^{n-m})$.

Let $\Omega(\epsilon) = \Omega \setminus \overline{B(M,\epsilon)}$ and $\lambda_k^D(\epsilon)$ be the $k-$th eigenvalue of the Laplacian in $\Omega(\epsilon)$ with the Dirichlet B.C. on $\partial\Omega(M,\epsilon)$.

Due to **G.Besson ('85), I.Chavel-D.Feldman ('88), C.Courtois ('95),** the following results have been established.

Theorem. Assume λ_k is simple in (1)

$$\lambda_k^D(\epsilon) - \lambda_k = \begin{cases} \left((n-m-2)|S^{n-m-1}| \int_M \Phi_k(\xi)^2 ds\right) \epsilon^{n-m-2} + \text{H.O.T.} & \text{for } n-m \geqq 3 \\ \left(2\pi \int_M \Phi_k(\xi)^2 ds\right) / \log(1/\epsilon) + \text{H.O.T.} & \text{for } n-m = 2 \end{cases}$$

Here S^{n-m-1} is the unit sphere in $\mathbb{R}^{n-m}$ and "H.O.T." implies "a higher order term".

The case of Neumann B.C., Robin B.C. on $\Gamma(M,\epsilon)$

Perturbed eigenvalue problems

$$(5)\qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \qquad \Phi = 0 \quad \text{on} \quad \Gamma, \\ \dfrac{\partial\Phi}{\partial\nu} + \sigma\epsilon^{\tau}\Phi = 0 \quad \text{on} \quad \Gamma(M,\epsilon). \qquad (<= \text{Robin B.C.}) \end{cases}$$

$$(6)\qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \quad \Phi = 0 \quad \text{on} \quad \Gamma, \\ \quad \dfrac{\partial\Phi}{\partial\nu} = 0 \quad \text{on} \quad \Gamma(M,\epsilon). \qquad (<= \ \text{Neumann B.C.}) \end{cases}$$

Here ν is the unit outward normal vector on $\partial\Omega(\epsilon)$ and $\sigma > 0, \tau \in \mathbb{R}$ are parameters.

Eigenvalues and Eigenfunctions in $\Omega(\epsilon)$

Definition. We denote the eigenvalues of (3) by $\{\lambda_k^R(\epsilon)\}_{k=1}^\infty$ and the corresponding complete orthonormal system by $\{\Phi_{k,\epsilon}^R\}_{k=1}^\infty \subset L^2(\Omega(\epsilon))$, respectively.

$$(\Phi_{k,\epsilon}^R, \Phi_{\ell,\epsilon}^R)_{L^2(\Omega(\epsilon))} = \delta(k,\ell) \quad (k,\ell \geqq 1).$$

Definition. We denote the eigenvalues of (4) by $\{\lambda_k^N(\epsilon)\}_{k=1}^\infty$ and the corresponding complete orthonormal system $\{\Phi_{k,\epsilon}^N\}_{k=1}^\infty \subset L^2(\Omega(\epsilon))$, respectively.

$$(\Phi_{k,\epsilon}^N, \Phi_{\ell,\epsilon}^N)_{L^2(\Omega(\epsilon))} = \delta(k,\ell) \quad (k,\ell \geqq 1).$$

Proposition. For $k \in \mathbb{N}$, $\lambda_k^N(\epsilon) \leqq \lambda_k^R(\epsilon) \leqq \lambda_k^D(\epsilon) \leqq \lambda_k + o(1)$ for $\epsilon \to 0$.

(Sketch of the proof) This is proved by a (rough) test functions

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) & \text{for} \quad x \in \Omega \setminus B(M, r_0) \\ \Phi_k(x)\frac{\log(\epsilon/r)}{\log(\epsilon/r_0)} & \text{for} \quad x \in B(M, r_0) \setminus B(M, \epsilon), r = \mathrm{dist}(x, M) \end{cases}.$$

with the max-min principle through the Rayleigh quotient

$$\mathcal{R}_\epsilon(\Phi) = \|\nabla\Phi\|^2_{L^2(\Omega(\epsilon))}/\|\Phi\|^2_{L^2(\Omega(\epsilon))}$$

$$\lambda_k^D(\epsilon) = \sup_{\dim E \leqq k-1} \inf\{\mathcal{R}_\epsilon(\Phi) \mid \Phi \in H_0^1(\Omega(\epsilon)),\ \Phi \perp E \text{ in } L^2(\Omega(\epsilon))\}$$

Here E is a subspace of $L^2(\Omega(\epsilon))$.

$$\|\widetilde{\Phi}_{k,\epsilon}-\Phi_k\|^2_{L^2(\Omega(\epsilon))}=O(1/|\log\epsilon|^2),\quad \|\nabla(\widetilde{\Phi}_{k,\epsilon}-\Phi_k)\|^2_{L^2(\Omega(\epsilon))}=\begin{cases}O(1/|\log\epsilon|^2) & \text{if} \quad q\geqq 3\\ O(1/|\log\epsilon|) & \text{if} \quad q=2\end{cases}$$

$$(\widetilde{\Phi}_{k,\epsilon},\widetilde{\Phi}_{k',\epsilon})_{L^2(\Omega(\epsilon))}=\delta(k,k')+O(\frac{1}{|\log\epsilon|}),$$

$$(\nabla\widetilde{\Phi}_{k,\epsilon},\nabla\widetilde{\Phi}_{k',\epsilon})_{L^2(\Omega(\epsilon))}=\lambda_\ell\,\delta(k,k')+\begin{cases}O(\frac{1}{|\log\epsilon|^{1/2}}) & \text{for } q=2\\ O(\frac{1}{|\log\epsilon|}) & \text{for } q\geqq 3\end{cases}$$

Put $F=L.H.[\widetilde{\Phi}_{1,\epsilon},\widetilde{\Phi}_{2,\epsilon},\cdots,\widetilde{\Phi}_{k,\epsilon}]$ and see $\dim(F)=k$.
Take any subspace $E\subset L^2(\Omega(\epsilon))$ with $\dim(E)\leqq k-1$, then there exists

$$\Psi=\sum_{\ell=1}^{k}c_k\widetilde{\Phi}_{\ell,\epsilon}\in F,\quad \Psi\perp E \text{ in } L^2(\Omega(\epsilon)),\quad \sum_{\ell=1}^{k}c_\ell^2=1.$$

Then we have

$$\inf_{\Phi\in H_0^1(\Omega(\epsilon)),\Phi\perp E\,\text{in}\,L^2(\Omega(\epsilon))}\mathcal{R}_\epsilon(\Phi)\leqq\mathcal{R}_\epsilon(\Psi)=\frac{\|\nabla(\sum_{\ell=1}^{k}c_k\widetilde{\Phi}_{\ell,\epsilon})\|^2_{L^2(\Omega(\epsilon))}}{\|\sum_{\ell=1}^{k}c_k\widetilde{\Phi}_{\ell,\epsilon}\|^2_{L^2(\Omega(\epsilon))}}$$

$$= \frac{\sum_{1\leqq \ell,\ell'\leqq k}(\nabla\widetilde{\Phi}_{\ell,\epsilon},\nabla\widetilde{\Phi}_{\ell',\epsilon})_{L^2(\Omega(\epsilon))}c_\ell c_{\ell'}}{\sum_{1\leqq \ell,\ell'\leqq k}(\widetilde{\Phi}_{\ell,\epsilon},\widetilde{\Phi}_{\ell',\epsilon})_{L^2(\Omega(\epsilon))}c_\ell c_{\ell'}} = \frac{\sum_{\ell=1}^k \lambda_\ell(1+o(1))c_\ell^2 + \sum_{1\leqq \ell\neq\ell'\leqq k} o(1)c_\ell c_{\ell'}}{\sum_{\ell=1}^k (1+o(1))c_\ell^2 + \sum_{1\leqq \ell\neq\ell'\leqq k} o(1)c_\ell c_{\ell'}}$$

$$\leqq \frac{\lambda_k + o(1)}{1-k^2 o(1)} \leqq \lambda_k + o(1)$$

Note that the right hand side is independent of choice of E. Taking sup for all choices of $E\subset L^2(\Omega(\epsilon)), \mathrm{dim} E\leqq k-1$ with the max min principle

$$\lambda_k^D(\epsilon)\leqq \lambda_k + o(1)$$

Since $\lambda_k\leqq \lambda_k^D(\epsilon)$, $\lim_{\epsilon\to 0}\lambda_k^D(\epsilon)=\lambda_k$ follows.

Proposition (Convergence). For $k \in \mathbb{N}$, we have

$$\lim_{\epsilon \to 0} \lambda_k^R(\epsilon) = \lambda_k \quad (k \in \mathbb{N}), \quad \lim_{\epsilon \to 0} \lambda_k^N(\epsilon) = \lambda_k \quad (k \in \mathbb{N}).$$

Proposition (Uniform bound). For each $k \in \mathbb{N}$, there exist $\epsilon_0 > 0$ and $c(k) > 0$ such that

$$|\Phi_{k,\epsilon}^R(x)| \leqq c(k), \quad |\Phi_{k,\epsilon}^N(x)| \leqq c(k) \quad (x \in \Omega(\epsilon),\, 0 < \epsilon \leqq \epsilon_0).$$

Notation

∇ : the gradient in $\mathbb{R}^n$
∇_M : the tangential gradient on M
∇_N : the normal gradient at a point of the manifold M

$$\nabla\phi = \nabla_M\phi + \nabla_N\phi \quad \text{on} \quad M$$

Notation

Denote the **mean curvature vector** field on M by H. H is a normal vector field on M. As an operator, for a function ϕ defined in a neighborhood of M, H acts on ϕ as a differential in H direction as follows

$$H[\phi](\xi) = \lim_{t\to 0}(\phi(\xi + tH(\xi)) - \phi(\xi))/t \quad \text{at each} \quad \xi \in M.$$

Theorem. Assume that $n - m = q \geqq 3$ and λ_k is simple in (1).

(0) We have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^N(\epsilon)-\lambda_k}{\epsilon^q}=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+\lambda_k\Phi_k^2-\Phi_kH[\Phi_k]\right\}ds(\xi).$$

(i) Assume $\tau > 1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^q}=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+\lambda_k\Phi_k^2-\Phi_kH[\Phi_k]\right\}ds(\xi).$$

(ii) Assume $\tau = 1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^q}=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+(\lambda_k+q\sigma)\Phi_k^2-\Phi_kH[\Phi_k]\right\}ds(\xi).$$

(iii) Assume $-1 < \tau < 1$, then we have

$$\lim_{\epsilon\to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{q+\tau-1}} = \sigma |S^{q-1}| \int_M \Phi_k(\xi)^2 ds(\xi).$$

(iv) Assume $\tau = -1$, then we have

$$\lim_{\epsilon\to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{q-2}} = \frac{\sigma(q-2)}{q-2+\sigma} |S^{q-1}| \int_M \Phi_k(\xi)^2 ds(\xi).$$

(v) Assume $\tau < -1$, then we have

$$\lim_{\epsilon\to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{q-2}} = (q-2)|S^{q-1}| \int_M \Phi_k(\xi)^2 ds(\xi).$$

Here $|S^{q-1}| = 2\pi^{q/2}/\mathbf{\Gamma}(q/2)$, which is is the measure of S^{q-1} and $\mathbf{\Gamma}(s) = \int_0^\infty t^{s-1}e^{-t}\,dt$ is the Gamma function.

Theorem. Assume that $n-m=q=2$ and λ_k is simple in (1).
(0) We have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^N(\epsilon)-\lambda_k}{\epsilon^2}=\pi\int_M\big(-2|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+\lambda_k\Phi_k^2-\Phi_k H[\Phi_k]\big)\,ds(\xi).$$

(i) Assume $\tau>1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^2}=\pi\int_M\big(-2|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+\lambda_k\Phi_k^2-\Phi_k H[\Phi_k]\big)\,ds(\xi).$$

(ii) Assume $\tau=1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^2}=\pi\int_M\big(-2|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+(\lambda_k+2\sigma)\Phi_k^2-\Phi_k H[\Phi_k]\big)\,ds(\xi).$$

(iii) Assume $-1 < \tau < 1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{1+\tau}} = 2\pi\sigma \int_M \Phi_k(\xi)^2 ds(\xi).$$

(iv) Assume $\tau \leqq -1$, then we have

$$\lim_{\epsilon \to 0} (\lambda_k^R(\epsilon) - \lambda_k) \log(1/\epsilon) = 2\pi \int_M \Phi_k(\xi)^2 ds(\xi).$$

The above theorems are in S.Jimbo, Eigenvalues of the Laplacian in a domain with a thin tubular hole, J. Elliptic, Parabolic, Equations **1** (2015).

Remark. It should be noted that in the case $\tau < -1$ in Theorem 3 and Theorem 4, the formula takes the same form as $\lambda_k^D(\epsilon)$ (the case of the Dirichlet B.C. on $\Gamma(M, \epsilon)$). In this case the Robin B.C. is close to the Dirichlet B.C. On the other hand, the formulas for $\lambda_k^R(\epsilon)$ for $\tau > 1$ (in (i)) takes the same form as $\lambda_k^N(\epsilon)$ (in (0)).

Remark. S. Ozawa dealt with $n = 3$, $\dim M = 1$ and proved (iii) in Theorem 4 with other method in his preprint: S. Ozawa, Spectra of the Laplacian and singular variation of domain - removing an $\epsilon-$ neighborhood of a curve, unpublished note (1998).

Sketch of the proof for the case of thin tubular hole (skip)

[I] Characterization of the eigenfunction $\Phi^R_{k,\epsilon}(x)$, $\Phi^N_{k,\epsilon}(x)$
Estimates for uniform bound and convergence

[II] Construction of the approximate eigenfunction $\widetilde{\Phi}^R_{k,\epsilon}(x)$, $\widetilde{\Phi}^N_{k,\epsilon}(x)$
Explicit expression of the approximation

[**Coordinate system in** $B(M, r_0)$]

M : a compact $m-$dimensional smooth manifold (M has a finite covering)

$$\mathbb{R}^n = T_\xi M \oplus N_\xi M \quad (\xi \in M) \qquad \text{(orthogonal decomposition)}$$

Here $\dim(T_\xi M) = m, \ \dim(N_\xi M) = q$.

Let $(e_1(\xi), e_2(\xi), \cdots, e_q(\xi))$ be an orthonormal frame in $N_\xi M$ (smooth in ξ) in a chart of the covering. $\exists r_0 > 0$ such that

$$B(M, r_0) \ni x = \xi + \sum_{\ell=1}^{q} \eta_\ell \, e_\ell(\xi).$$

Denote the second term by $\eta \cdot e(\xi)$.

[Mean curvature operator (vector) on M]

The second fundamental form $h_\xi(X,Y)$ of M is defined by the following formula

$$\nabla_Y X = \nabla_Y^M X + h_\xi(X,Y) \in T_\xi M \oplus N_\xi M \quad \text{(orthogonal decomposition)}$$

for any C^1 vector fields X, Y which are defined in a neighborhood of M and tangent to M. The mean curvature vector H of M is defined by

$$H_\xi = \sum_{i=1}^{m} h_\xi(E_i, E_i)$$

for each $\xi \in M$. Here $\{E_1, E_2, \cdots, E_m\}$ is an orthonormal frame of $T_\xi M$ (cf. Kobayashi-Nomizu ('63)).

Lemma. In this coordinate system $(\xi, \eta) = (\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_q)$, the mean curvature operator of M is expressed as follows.

$$H_\xi = -\sum_{\ell=1}^{q} \frac{1}{\sqrt{g(\xi,0)}} \left(\frac{\partial \sqrt{g(\xi,\eta)}}{\partial \eta_\ell} \right)_{|M} \frac{\partial}{\partial \eta_\ell} = -\sum_{\ell=1}^{q} \sum_{1 \leqq i,j \leqq m} \frac{g^{ij}(\xi,0)}{2} \frac{\partial g_{ij}}{\partial \eta_\ell}(\xi,0) \frac{\partial}{\partial \eta_\ell}$$

It is also expressed as a normal vector field

$$H_\xi = -\sum_{\ell=1}^{q} \frac{1}{\sqrt{g(\xi,\mathbf{0})}} \left(\frac{\partial \sqrt{g(\xi,\eta)}}{\partial \eta_\ell} \right)_{\eta=\mathbf{0}} e_\ell(\xi).$$

Proposition. For a C^2 function u which is defined in $B(M, r_0)$, we have

$$(\Delta u)_{|M} = \Delta_M(u_{|M}) - H[u] + \sum_{\ell=1}^{q} \left(\frac{\partial^2 u}{\partial \eta_\ell^2} \right)_{|\eta=\mathbf{0}} \quad \text{on} \quad M.$$

[I] Uniform bound for the eigenfunction $\Phi^R_{k,\epsilon}$, $\Phi^N_{k,\epsilon}$

Lemma (Barrier function). There exsits a function $\psi_\epsilon(x)$ (defined from K_1) satisfies the following properties. For any $m_2 > 0$, there exist $\epsilon_1 > 0$, $r_1 \in (0, r_0]$ and $\epsilon_1 > 0$ such that

$$\Delta\psi_\epsilon + m_2\psi_\epsilon \leqq 0 \quad \text{in} \quad B(M, r_1) \setminus B(M, \epsilon),$$

$$\frac{\partial \psi_\epsilon}{\partial \nu} \geqq 0 \quad \text{on} \quad \Gamma(M, \epsilon), \quad 1 \leqq \psi_\epsilon(x) \leqq 3 \quad \text{in} \quad B(M, r_1) \setminus B(M, \epsilon),$$

for any $\epsilon \in (0, \epsilon_1)$.

Estimates for $\Phi^R_{k,\epsilon}$, $\Phi^N_{k,\epsilon}$

For any $r_1 > 0$, there exists $c > 0$ such that $|\Phi^R_{k,\epsilon}(x)| \leqq c$ in $\Omega \setminus B(M, r_1)$ and $0 < \epsilon \leqq r_1/2$ (Elliptic estimates).

By the comparison argument, we have

$$-c\,\psi_\epsilon(x) \leqq \Phi^R_{k,\epsilon}(x) \leqq c\,\psi_\epsilon(x) \quad \text{in} \quad B(M, r_1) \setminus B(M, \epsilon).$$

for $\epsilon > 0$.

Same argument applies to $\Phi^N_{k,\epsilon}$.

[II] Approximate eigenfunction $\widetilde{\Phi}^R_{k,\epsilon}$

We first construct an approximate eigenfunction $\widetilde{\Phi}_{k,\epsilon}$, by modifying Φ_k around M according to the Robin B.C. on $\Gamma(M,\epsilon)$. We consider $\phi(\eta) = \phi(\eta_1, \cdots, \eta_q)$ satisfying

$$\begin{cases} \Delta_\eta \phi = 0 \quad \text{for} \quad \epsilon < |\eta| < r_0, \quad \phi = 0 \quad \text{for} \quad |\eta| = r_0, \\ \left(\dfrac{\partial \phi}{\partial \nu_\eta} + \sigma \epsilon^\tau \phi\right)_{|\eta|=\epsilon} = \left(\dfrac{\partial}{\partial \nu_\eta} \Phi_k(\xi + \eta \cdot \boldsymbol{e}(\xi)) + \sigma \epsilon^\tau \Phi_k(\xi + \eta \cdot \boldsymbol{e}(\xi))\right)_{|\eta|=\epsilon} \end{cases}$$

for each $\xi \in M$. Here $\Delta_\eta = \partial^2/\partial\eta_1^2 + \cdots + \partial^2/\partial\eta_q^2$. Basic harmonic functions in η space solutions are given by

$$r^\ell \varphi_{\ell,p}(\omega), \quad r^{-\ell-q+2}\varphi_{\ell,p}(\omega) \quad (\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)) \quad \text{harmonic functions in} \quad \mathbb{R}^q \setminus \{\mathbf{0}\}.$$

where $\{\varphi_{\ell,p}(\omega)\}_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)}$ are eigenfunctions of the Laplace-Beltrami operator in S^{q-1}. The eigenvalues $\gamma(\ell)$ and its multiplicity $\iota(\ell)$ are given as follows

$$\gamma(\ell) = \ell(\ell + q - 2), \ \iota(\ell) = \frac{(2\ell + q - 2)(q + \ell - 3)!}{(q-2)!\ell!} \quad (\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)$$

The solution of the Laplace equation

$$\phi(\eta) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} (a_{l,p} r^{\ell} + b_{\ell,p} r^{-\ell-q+2}) \varphi_{\ell,p}(\omega) \quad (\epsilon < r < r_0, \omega \in S^{q-1}).$$

The coefficients $a_{\ell,p}$, $b_{\ell,p}$ can be calculated by the infinite series of relations determined by the boundary condition. From the boundary condition on $|\eta| = r_0$, we have

$$\sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} (a_{\ell,p} r_0^{\ell} + b_{\ell,p} r_0^{-\ell-q+2}) \varphi_{\ell,p}(\omega) = 0 \quad (\omega \in S^{q-1})$$

which gives

$$a_{\ell,p} r_0^{\ell} + b_{\ell,p} r_0^{-\ell-q+2} = 0 \quad \text{for} \quad \ell \geqq 0, 1 \leqq p \leqq \iota(\ell).$$

ϕ is written by

$$\phi(\eta) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p} (r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell}) \varphi_{\ell,p}(\omega) \quad (\epsilon < r < r_0, \omega \in S^{q-1}).$$

We calculate the Robin condition on $|\eta| = \epsilon$. Noting

$$\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial r} = -\sum_{i=1}^{q} \frac{\eta_i}{|\eta|}\frac{\partial}{\partial \eta_i} \quad \text{on} \quad \Gamma(M,\epsilon) = \{x = \xi + \eta \cdot e(\xi) \mid \xi \in M, |\eta| = \epsilon\}$$

we have the equations for the coefficients $a_{\ell,p}, b_{\ell,p}$ as follows.

$$- \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p} \left((-\ell - q + 2) r^{-\ell-q+1} - \ell r_0^{-2\ell-q+2} r^{\ell-1} \right)_{r=\epsilon} \varphi_{\ell,p}(\omega)$$

$$+ \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p} \sigma \epsilon^{\tau} \left(r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell} \right)_{r=\epsilon} \varphi_{\ell,p}(\omega)$$

$$= -\sum_{i=1}^{q} \langle \nabla \Phi_k(\xi + (\epsilon\omega) \cdot e(\xi)), e_i(\xi) \rangle \eta_i / |\eta| + \sigma \epsilon^{\tau} \Phi_k(\xi + (\epsilon\omega) \cdot e(\xi))$$

for $\omega \in S^{q-1}$. Multiply both sides by $\varphi_{p,\ell}$ and integrate on S^{q-1} and we get

$$b_{\ell,p} \left\{ (\ell + q - 2) \epsilon^{-\ell-q+1} + \ell r_0^{-2\ell-q+2} \epsilon^{\ell-1} + \sigma (\epsilon^{-\ell-q+2+\tau} - r_0^{-2\ell-q+2} \epsilon^{\ell+\tau}) \right\}$$

$$= \int_{S^{q-1}} \left\{ -\sum_{i=1}^{q} \{ (\nabla \Phi_k(\xi + (\epsilon\omega) \cdot e(\xi)), e_i(\xi)) \omega_i \} + \sigma \epsilon^{\tau} \Phi_k(\xi + (\epsilon\omega) \cdot e(\xi)) \right\} \varphi_{\ell,p}(\omega) d\omega$$

From these equations we get $a_{\ell,p}, b_{\ell,p}$ as follows

$$a_{\ell,p} = -r_0^{-2\ell-q+2} b_{\ell,p}$$

$$b_{\ell,p} = \frac{1}{(\ell+q-2)\epsilon^{-\ell-q+1} + \ell r_0^{-2\ell-q+2}\epsilon^{\ell-1} + \sigma(\epsilon^{-\ell-q+2+\tau} - r_0^{-2\ell-q+2}\epsilon^{\ell+\tau})}$$

$$\times \int_{S^{q-1}} \left\{ -\sum_{i=1}^{q} \{(\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi))\omega_i\} + \sigma\epsilon^{\tau}\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)) \right\} \varphi_{\ell,p}(\omega) d\omega$$

We remark that these ($\epsilon-$dependent) coefficients $a_{\ell,p}$, $b_{\ell,p}$ are smoothly dependent on $\xi \in M$ since Φ_k is smooth. So we denote this function $\phi(x)$ in $B(M, r_0) \setminus B(M, \epsilon)$ by $G_{k,\epsilon}(x)$. That is

$$G_{k,\epsilon}(x) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p}(r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell}) \varphi_{\ell,p}(\omega) \quad (x = \xi + (r\omega)\cdot e(\xi)).$$

Definition. The approximate eigenfunction is defined by

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) & \text{for} \quad x \in \Omega \setminus B(M, r_0) \\ \Phi_k(x) - G_{k,\epsilon}(x) & \text{for} \quad x = \xi + \eta \cdot e(\xi) \in B(M, r_0) \setminus B(M, \epsilon) \end{cases}$$

Lemma. (i) $\ell = 0$

$$b_{0,1} = |S^{q-1}|^{1/2} \begin{cases} \frac{-\epsilon^q}{q(q-2)}\{\sum_{i=1}^q \langle e_i(\xi), \nabla^2\Phi_k(\xi)e_i(\xi)\rangle + O(\epsilon)\} & (\tau > 1) \\ \frac{\epsilon^q}{q-2}\{(-1/q)\sum_{i=1}^q \langle e_i(\xi), \nabla^2\Phi_k(\xi)e_i(\xi)\rangle + \sigma\Phi_k(\xi) + O(\epsilon)\} & (\tau = 1) \\ \frac{\sigma\epsilon^{q-1+\tau}}{q-2}(\Phi_k(\xi) + O(\epsilon)) & (-1 < \tau < 1) \\ \frac{\sigma\epsilon^{q-2}}{q-2+\sigma}(\Phi_k(\xi) + O(\epsilon)) & (\tau = -1) \\ \epsilon^{q-2}(\Phi_k(\xi) + O(\epsilon)) & (\tau < -1) \end{cases}$$

(ii) $\ell = 1$

$$b_{1,p} = \frac{|S^{q-1}|^{1/2}}{q^{1/2}} \langle \nabla\Phi_k(\xi), e_p(\xi)\rangle \epsilon^q (1 + O(\epsilon)) \times \begin{cases} -1/(q-1) & (\tau > -1) \\ (\sigma - 1)/(q - 1 + \sigma) & (\tau = -1) \\ 1 & (\tau < -1) \end{cases}$$

Lemma. For any $N \in \mathbb{N}$, there exists $d_N > 0$ (independent of $\xi \in M$) such that

$$|b_{\ell,p}| \leqq \frac{d_N}{\gamma(\ell)^N} \begin{cases} \epsilon^{\ell+q} & (\tau \geqq 0) \\ \epsilon^{\ell+q+\tau} & (-1 < \tau < 0) \\ \epsilon^{\ell+q-1} & (\tau \leqq -1) \end{cases}, \quad (1 \leqq p \leqq \iota(\ell), \ell \geqq 2).$$

Proof for the theorems

For any sequence of positive values $\{\epsilon_p\}_{p=1}^{\infty}$ with $\lim_{p\to\infty}\epsilon_p = 0$, there exists a subsequence $\{\zeta_p\}_{p=1}^{\infty}$ and orthonormal systems of eigenfunctions $\{\Phi'_k\}_{k=1}^{\infty}$ and $\{\Phi''_k\}_{k=1}^{\infty}$ of (1) corresponding to $\{\lambda_k\}_{k=1}^{\infty}$, respectively such that

$$(\Phi'_k, \Phi'_\ell)_{L^2(\Omega)} = \delta(k,\ell), \quad (\Phi''_k, \Phi''_\ell)_{L^2(\Omega)} = \delta(k,\ell) \quad (k,\ell \in \mathbb{N}),$$

$$\lim_{p\to\infty} \|\Phi^R_{k,\zeta_p} - \Phi'_k\|_{L^2(\Omega(\zeta_p))} = 0, \quad \lim_{p\to\infty} \|\Phi^N_{k,\zeta_p} - \Phi''_k\|_{L^2(\Omega(\zeta_p))} = 0.$$

Calculation of the limit behavior of $\lambda_k^R(\epsilon) - \lambda_k$.

$$\int_{\Omega(\epsilon)} (\Delta \Phi_{k,\epsilon}^R + \lambda_k^R(\epsilon)\Phi_{k,\epsilon}^R)\widetilde{\Phi}_{k,\epsilon} dx = 0 \tag{7}$$

Assume the situation $\Phi_{k,\epsilon}^R \longrightarrow \Phi_k'$ for $\epsilon = \zeta_p \to 0$ as in Proposition 2.

Calculation on the above integral relation gives

$$(\lambda_k(\epsilon) - \lambda_k) \int_{\Omega(\epsilon)} \Phi_k(x)\Phi_{k,\epsilon}(x) dx = I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon) + I_4(\epsilon). \tag{8}$$

where

$$I_1(\epsilon) = -\int_{\Gamma(M,r_0)} \frac{\partial G_{k,\epsilon}}{\partial \nu_1}(\Phi_{k,\epsilon}(x) - \Phi_k'(x)) dS$$

$$I_2(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} G_{k,\epsilon}(x)(\Delta \Phi_k'(x) + \lambda_k(\epsilon)\Phi_{k,\epsilon}(x)) dx$$

$$I_3(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} (\Delta G_{k,\epsilon}(x))(\Phi_{k,\epsilon}(x) - \Phi_k'(x)) dx$$

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} \left(\frac{\partial G_{k,\epsilon}}{\partial \nu_1} \Phi'_k - G_{k,\epsilon} \frac{\partial \Phi'_k}{\partial \nu_1} \right) dS$$

$I_4(\epsilon)$ is also written

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} \left((\frac{\partial \Phi_k}{\partial \nu_1} + \sigma \epsilon^\tau \Phi_k) \Phi'_k - G_{k,\epsilon} (\frac{\partial \Phi'_k}{\partial \nu_1} + \sigma \epsilon^\tau \Phi'_k) \right) dS.$$

Careful evaluation on I_1, I_2, I_3, I_4 gives the perturbation formula in Theorem.

(B) Domain with partial degeneration

$D \subset \mathbb{R}^n$: a bounded domain (or a finite union of bounded domains) with a smooth boundary. The perturbed domain

$$\Omega(\zeta) = D \cup Q(\zeta) \subset \mathbb{R}^n$$

Here $Q(\zeta)$ is a thin set which approaches a lower dimensional set L as $\zeta \to 0$.

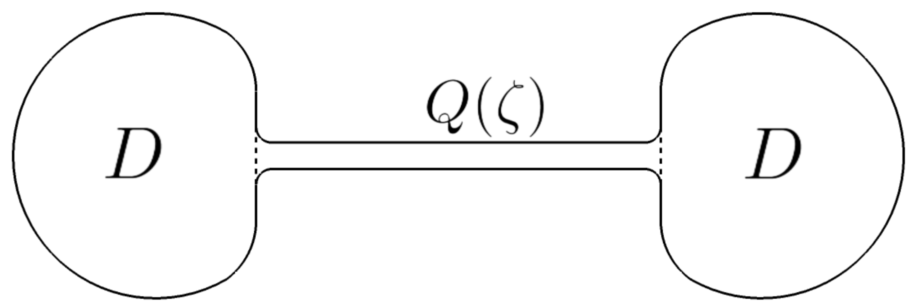


FIGURE 1. Sample of $\Omega(\varepsilon)$

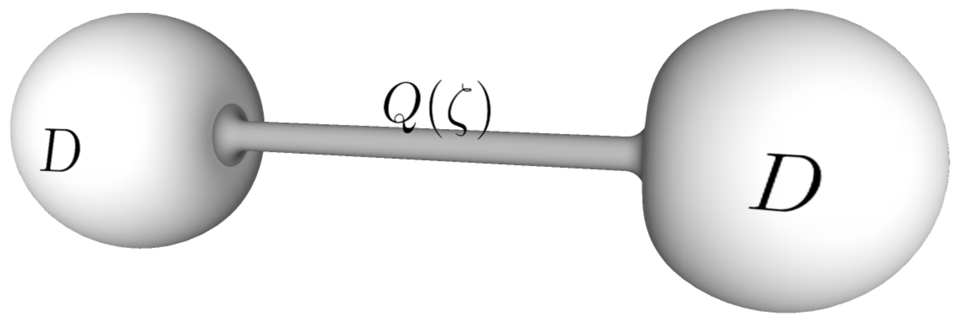


FIGURE 2. Sample of $\Omega(\varepsilon)$

Eigenvalue problem

$$\Delta\Phi + \mu\Phi = 0 \quad \text{in} \quad \Omega(\zeta), \qquad \partial\Phi/\partial\nu = 0 \quad \text{on} \quad \partial\Omega(\zeta) \tag{9}$$

Let $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ be the eigenvalues with the corresponding eigenfunctions $\Phi_{k,\zeta}$ $(k \geqq 1)$ such that

$$(\Phi_{k,\zeta}, \Phi_{k',\zeta})_{L^2(\Omega(\zeta))} = \delta(k, k') \quad (k, k' \geqq 1) \quad \text{(Kronecker's delta)}$$

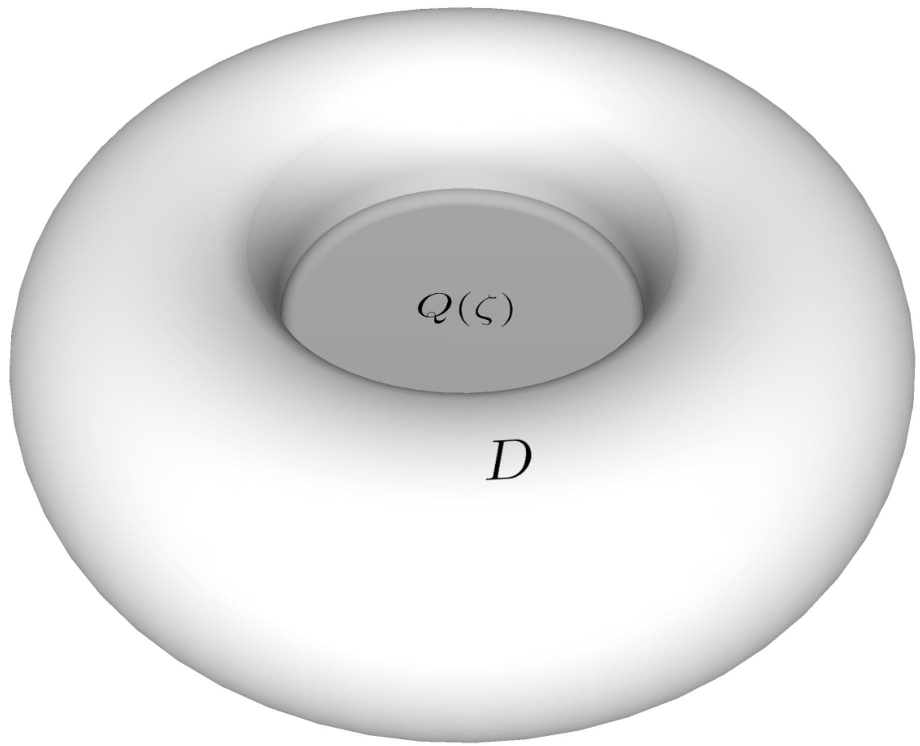


FIGURE 3. Sample of $\Omega(\varepsilon)$

Basic question : What is the limitting behavior of $\mu_k(\zeta)$ for $\zeta \to 0$?

For the Dumbbell domain, there are results. Beale('73), Fang('93), Jimbo('93), Gadylshin('93), Arrieta('95), Jimbo-Morita('95), Anné('87),...

Convergence of the eigenvalues. Perturbation formula (first order approximation) is studied.

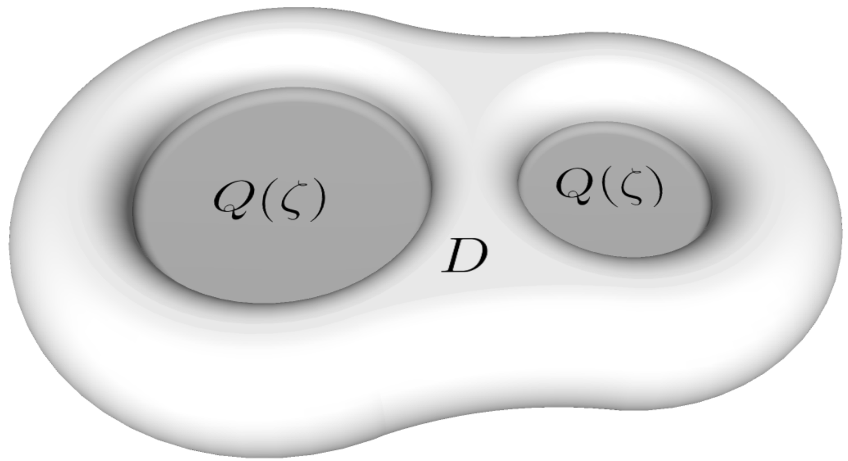


FIGURE 4. Sample of $\Omega(\varepsilon)$

In this lecture I deal with more general cases. Hereafter I mainly talk about the results in Jimbo-Kosugi('09).

The construction of $\Omega(\zeta) = D \cup Q(\zeta)$

Let $n, \ell, m \in \mathbb{N}$ with $n = \ell + m$. $x = (x', x'') \in \mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^m$
$D \subset \mathbb{R}^n$, $L \subset \mathbb{R}^\ell$ bounded domains (finite disjoint union of bounded domains) with smooth boundaries.

Some symbols:

$$B^{(m)}(s) := \{x'' \in \mathbb{R}^m \mid |x''| < s\}, \quad L(s) := \{x' \in L \mid \mathrm{dist}(x', \partial L) > s\}$$

Assumption: There exists $\zeta_0 > 0$ such that

$$(\overline{L} \times B^{(m)}(3\zeta_0)) \cap D = \partial L \times B^{(m)}(3\zeta_0) \subset \partial D$$

There exists a function $\rho = \rho(t) \in C^3((-\infty, 0)) \cap C^0((-\infty, 0])$ such that

$$\rho(t) = 1 \ (t \leqq -1), \ \rho'(t) > 0 \ (-1 < t < 0), \ \rho(0) = 2, \ \lim_{s \uparrow 2} d^k \rho^{-1}(s)/ds^k = 0 \quad (1 \leqq k \leqq 3)$$

Put $Q(\zeta) = Q_1(\zeta) \cup Q_2(\zeta)$ where $Q_1(\zeta) = L(2\zeta) \times B^{(m)}(\zeta)$ and

$$Q_2(\zeta) = \{(\xi + s\nu'(\xi), \eta) \mid \mathbb{R}^\ell \times \mathbb{R}^m \mid -2\zeta \leqq s \leqq 0, \xi \in \partial L, |\eta| < \zeta\rho(s/\zeta)\}$$

To express the limit of $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ we prepare the notation.

Definition. $\{\omega_k\}_{k=1}^{\infty}$ is the system of eigenvalues of

$$\Delta\phi + \omega\phi = 0 \text{ in } D, \quad \partial\phi/\partial\nu = 0 \text{ on } \partial D \tag{10}$$

Definition. $\{\lambda_k\}_{k=1}^{\infty}$ is the system of eigenvalues of

$$\Delta'\psi + \lambda\psi = 0 \text{ in } L, \quad \psi = 0 \text{ on } \partial L \tag{11}$$

where

$$\Delta' = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_\ell^2}$$

The limit of $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ is given by the following result.

Proposition. $\lim_{\zeta\to 0}\mu_k(\zeta) = \mu_k$ for any $k \geqq 1$ where $\{\mu_k\}_{k=1}^{\infty}$ is given by rearranging $\{\omega_k\}_{k=1}^{\infty} \cup \{\lambda_k\}_{k=1}^{\infty}$ in increasing order with counting multiplicity.

Remark. μ_k is written as

$$\mu_k = \max_{1\leqq j\leqq k}\left(\min(\omega_{k+1-j}, \lambda_j)\right).$$

Classification of eigenvalues

Definition

$$E_I = \{\omega_k\}_{k=1}^{\infty} \setminus \{\lambda_k\}_{k=1}^{\infty}, \qquad E_{II} = \{\lambda_k\}_{k=1}^{\infty} \setminus \{\omega_k\}_{k=1}^{\infty}, \qquad E_{III} = \{\omega_k\}_{k=1}^{\infty} \cap \{\lambda_k\}_{k=1}^{\infty}$$

Relation to the eigenfunctions

Let $\{\Phi_{k,\zeta}\}_{k=1}^{\infty} \subset L^2(\Omega(\zeta))$ be the (complete) orthonormal system corresponding to $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ of (9).

Proposition.

$$\lim_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{L^2(Q(\zeta))} = 0 \Longleftrightarrow \mu_k \in E_I$$

$$\lim_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{L^2(D)} = 0 \Longleftrightarrow \mu_k \in E_{II}$$

$$\liminf_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{L^2(Q(\zeta))} > 0, \ \liminf_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{L^2(D)} > 0 \Longleftrightarrow \mu_k \in E_{III}$$

Proposition (Convergence rate)

$$\mu_k \in E_I \Longrightarrow \mu_k(\zeta) - \mu_k = O(\zeta^m)$$

$$\mu_k \in E_{II} \Longrightarrow \mu_k(\zeta) - \mu_k = \begin{cases} O(\zeta) & (m \geqq 2) \\ O(\zeta \log(1/\zeta)) & (m = 1) \end{cases}$$

For $\mu_k \in E_{III}$, a mixed situation occurs (as seen later).

Some preparation(uniform converegence)

Consider the following semilinear elliptic equation in $\Omega(\zeta)$.

$$\Delta u + f_\zeta(u) = 0 \quad \text{in} \quad \Omega(\zeta), \quad \partial u/\partial\nu = 0 \quad \text{on}$$

Here $\zeta > 0$ is a parameter and the nonlinear term $f_\zeta(u)$ is assumed to be a C^1 function in $\mathbb{R}$ such that $(\partial f_\zeta/\partial u)(u)$ is uniformly bounded in $\mathbb{R}$ and $f_\zeta(u)$ converges locally uniformly to a C^1 function $f_0(u)$ for $\zeta \to 0$.

Theorem. Let $\{\zeta_p\}_{p=1}^\infty$ be a positive sequence which converges to 0 as $p \to \infty$ and let $u_{\zeta_p} \in C^2(\overline{\Omega(\zeta_p)})$ be a solution of the above equation for $\zeta = \zeta_p$ such that

$$\sup_{p \geqq 1} \sup_{x \in \Omega(\zeta_p)} |u_{\zeta_p}(x)| < \infty.$$

Then there exists a subsequence $\{\sigma_p\}_{p=1}^\infty \subset \{\zeta_p\}_{p=1}^\infty$ and functions $w \in C^2(\overline{D})$ and $V \in C^2(\overline{L})$ such that

$$
\begin{gathered}
\Delta w + f_0(w) = 0 \quad \text{in} \quad D, \ \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial D \quad \text{(Neumann B.C.)}, \\
\Delta' V + f_0(V) = 0 \quad \text{in} \quad L, \ V(x') = w(x', o'') \quad \text{for} \quad x' \in \partial L, \\
\lim_{p\to\infty} \sup_{x\in D} |u_{\sigma_p}(x) - w(x)| = 0, \\
\lim_{p\to\infty} \sup_{(x',x'')\in Q(\sigma_p)} |u_{\sigma_p}(x', x'') - V(x')| = 0,
\end{gathered}
$$

where $\Delta' = \sum_{k=1}^{\ell} \partial^2/\partial x_k^2$. Note that $\partial L \times \{o''\} \subset \partial D$.

Perturbation formula [Type (I)]

Let $\{\phi_k\}_{k=1}^{\infty}$ be the system of eigenfunctions of (10) (eigenvalue problem in D) orthonormalized in $L^2(D)$.

Assume $\mu_k \in E_I$ and there exists $k' \in \mathbb{N}$ such that $\mu_k = \omega_{k'}$. Assume also that $\omega_{k'}$ is a simple eigenvalue of (10).

Theorem.

$$\mu_k(\zeta) - \mu_k = S(m)\alpha(k)\zeta^m + o(\zeta^m)$$

where

$$\alpha(k) = \int_{\partial L} \frac{\partial V_{k'}}{\partial \nu'}(\xi)\phi_{k'}(\xi, o'')dS'$$

$V_{k'}(x')$ is the unique solution $V \in C^2(\overline{L})$ of

$$\Delta' V + \omega_{k'} V = 0 \text{ in } L, \quad V(\xi) = \phi_{k'}(\xi, o'') \text{ for } \xi \in \partial L.$$

$S(m)$ is the m-dimensional volume of the unit ball in $\mathbb{R}^m$.

Perturbation formula [Type (II)]

Let $\{\psi_k\}_{k=1}^{\infty}$ be the system of eigenfunctions of (11) (eigenvalue problem in L) orthonormalized in $L^2(L)$.

Assume $\mu_k \in E_{II}$ and there exists $k'' \in \mathbb{N}$ such that $\mu_k = \lambda_{k''}$. Assume also that $\lambda_{k''}$ is a simple eigenvalue of (11).

Theorem.

$$\mu_k(\zeta) - \mu_k = -\frac{2}{\pi}\beta(k'')\zeta \log(1/\zeta) + o(\zeta \log(1/\zeta)) \quad (m = 1),$$

$$\mu_k(\zeta) - \mu_k = -T(\rho, m)\beta(k'')\zeta + o(\zeta) \quad (m \geqq 2).$$

where

$$\beta(k'') = \int_{\partial L} \left(\frac{\partial \psi_{k''}}{\partial \nu'}(\xi)\right)^2 dS'$$

and $T(\rho, m)$ is the number which depends on $\Omega(\zeta)$ (to be explained later).

Remark. For the case of Dumbbell, Gadylshin('93) obtained this result $m = 2$ and Arrieta ('95) obtained this result for $m = 1$.

Quantity $T(\rho, m)$ $(m \geqq 2)$

Harmonic function G in the set $H = H_1 \cup H_2 \subset \mathbb{R} \times \mathbb{R}^m$ where H_1, H_2 are given

$$H_1 = (0, \infty) \times \mathbb{R}^m, \qquad H_2 = \{(s, \eta) \in \mathbb{R} \times \mathbb{R}^m \mid |\eta| < \rho(s),\ s \leqq 0\}.$$

Proposition. There exists a solution G to

$$\frac{\partial^2 G}{\partial s^2} + \sum_{j=1}^{m} \frac{\partial^2 G}{\partial \eta_j^2} = 0 \quad ((s, \eta) \in H) \qquad \frac{\partial G}{\partial \boldsymbol{n}} = 0 \quad ((s, \eta) \in \partial H)$$

such that

$$G(z) = G(s, \eta) \longrightarrow 0 \quad \text{for} \quad (z \in H_1, |z| \to \infty)$$
$$G(s, \eta) - (-\kappa_1 s + \kappa_2) \longrightarrow 0 \quad \text{for} \quad (z \in H_2, |z| \to \infty)$$

where $\kappa_1 > 0, \kappa_2$ are real constants. κ_2/κ_1 is uniquely determined by H.

Definition. $T(\rho, m) = \kappa_2/\kappa_1$.

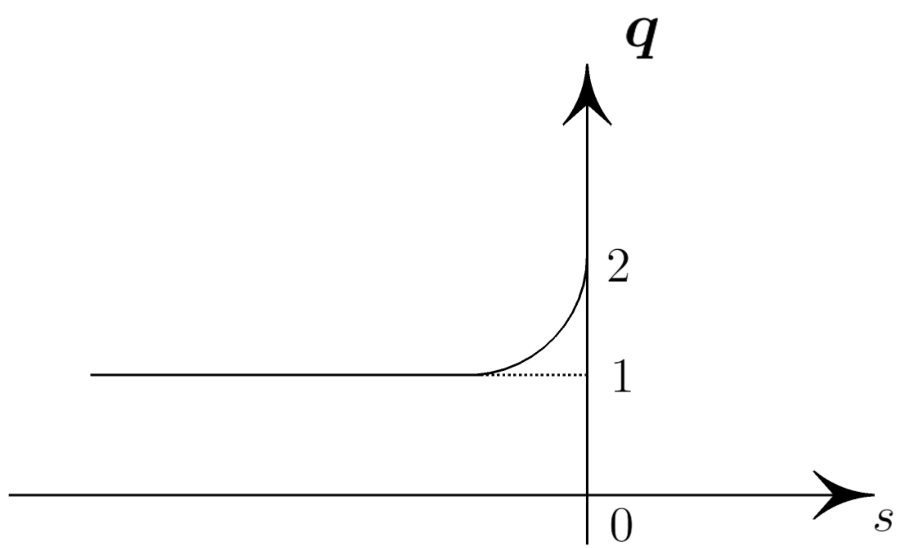


FIGURE 5. Function $q = q(s)$

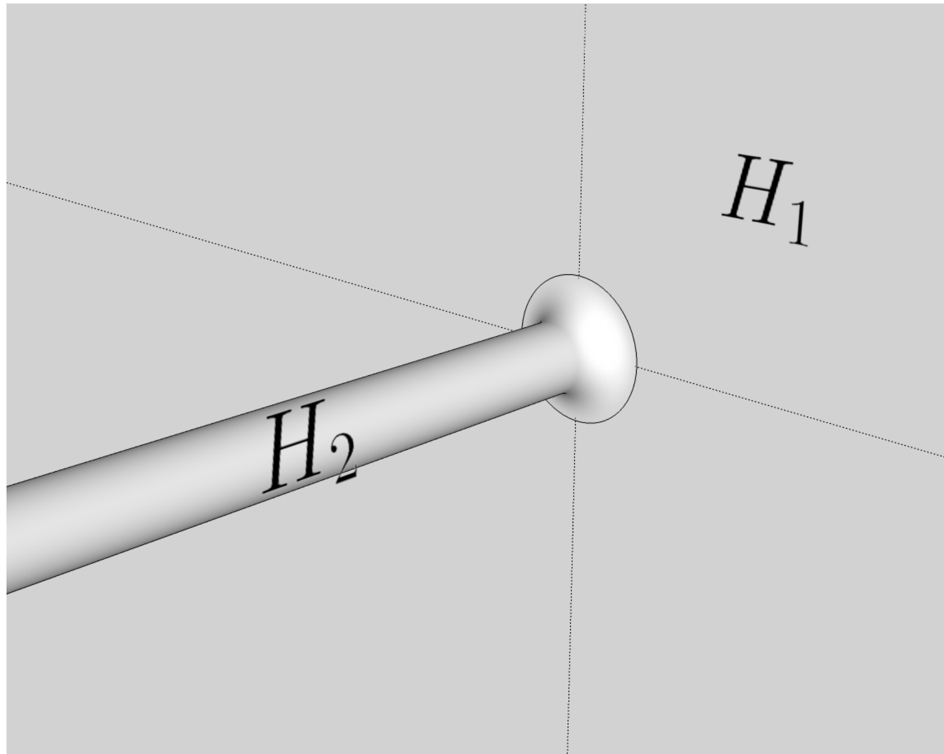


FIGURE 6. Picture of H

Perturbation formula [Type (III)]

Assume $\mu_k \in E_{III}$ and there exists $k', k'' \in \mathbb{N}$ such that $\mu_k = \omega_{k'} = \lambda_{k''}$. Assume also that $\omega_{k'}$ is simple eigenvalue of (10) and $\lambda_{k''}$ is a simple eigenvalue of (11).

We have the situation

$$\mu_{k-1} < \mu_k = \mu_{k+1} < \mu_{k+2}.$$

Theorem. For $m = 1$, we have

$$\mu_k(\zeta) - \mu_k = \gamma_1^-(k', k'')\zeta^{1/2} + o(\zeta^{1/2})$$
$$\mu_{k+1}(\zeta) - \mu_{k+1} = \gamma_1^+(k', k'')\zeta^{1/2} + o(\zeta^{1/2})$$

where $\gamma_1^\pm(k', k'')$ are eigenvalues of

$$\begin{pmatrix} 0 & \sqrt{2}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' \\ \sqrt{2}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' & 0 \end{pmatrix}$$

Theorem. For $m = 2$, we have

$$\mu_k(\zeta) - \mu_k = \gamma_2^-(k', k'')\zeta + o(\zeta)$$
$$\mu_{k+1}(\zeta) - \mu_{k+1} = \gamma_2^+(k', k'')\zeta + o(\zeta)$$

where $\gamma_2^\pm(k', k'')$ are eigenvalues of

$$\begin{pmatrix} 0 & \sqrt{\pi}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' \\ \sqrt{\pi}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' & -T(\rho, 2)\int_{\partial L}((\partial\psi_{k''}/\partial\nu')(\xi))^2 dS' \end{pmatrix}$$

Remark. For the case of Dumbbell ($m = 2, n = 3$), Gadylshin ('05) got this result. See Jimbo-Kosugi('09) for more genral cases.

Theorem. Assume $T(\rho, m) > 0$. For $m \geqq 3$, we have

$$\mu_k(\zeta) - \mu_k = \gamma_m^-(k', k'')\zeta + o(\zeta)$$
$$\mu_{k+1}(\zeta) - \mu_{k+1} = \gamma_m^+(k', k'')\zeta^{m-1} + o(\zeta)$$

where

$$\gamma_m^-(k', k'') = -T(\rho, m) \int_{\partial L} \left(\frac{\partial \psi_{k''}}{\partial \nu'}(\xi) \right)^2 dS'$$

$$\gamma_m^+(k', k'') = S(m) T(\rho, m)^{-1} \left(\int_{\partial L} (\frac{\partial \psi_{k''}}{\partial \nu'}(\xi))^2 dS' \right)^{-1} \left(\int_{\partial L} \frac{\partial \psi_{k''}}{\partial \nu'}(\xi) \phi_{k'}(\xi, o'') dS' \right)^2$$

In the case $T(\rho, m) < 0$, the right hand sides are exchanged.

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