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包括的レーザープラズマシミュレーションに向けた数値計算法の再構築

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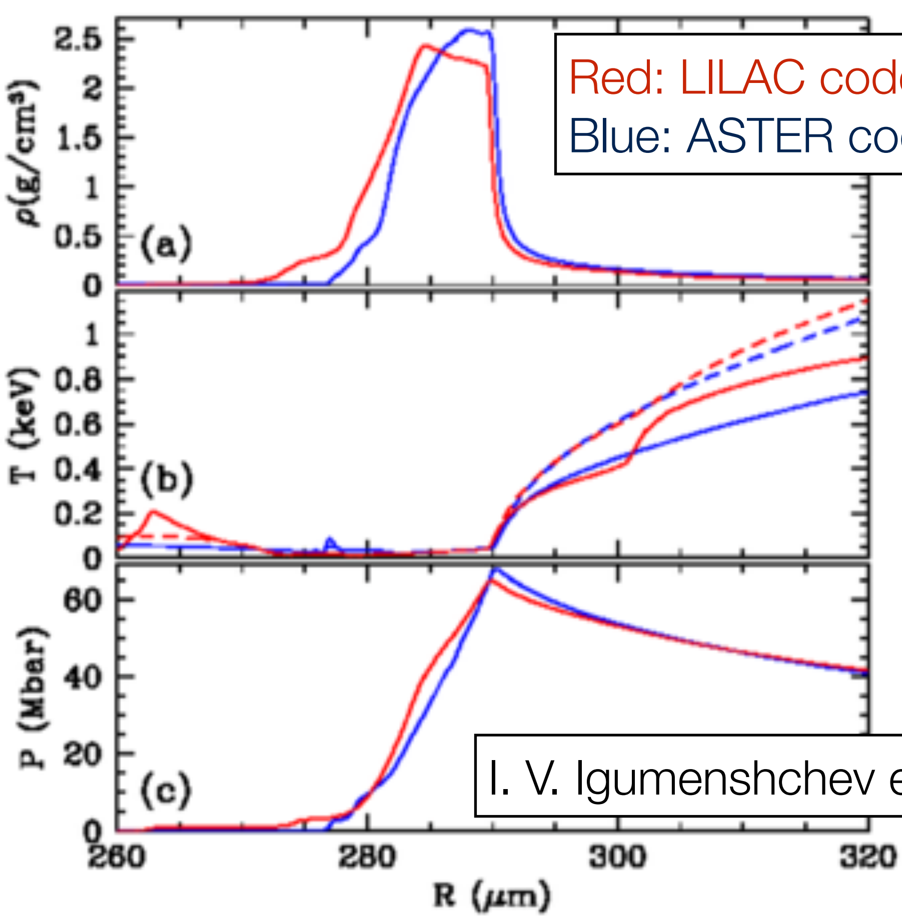
Summary

- ✓ Violation of energy conservation on the two-temperature hydrodynamic model was under the control of machine epsilon
- ✓ Structure-preserving scheme also follows the jump condition of Rankine–Hugoniot relationship
- ✓ Conservative PIC scheme detests discrimination between the electric field and magnetic field

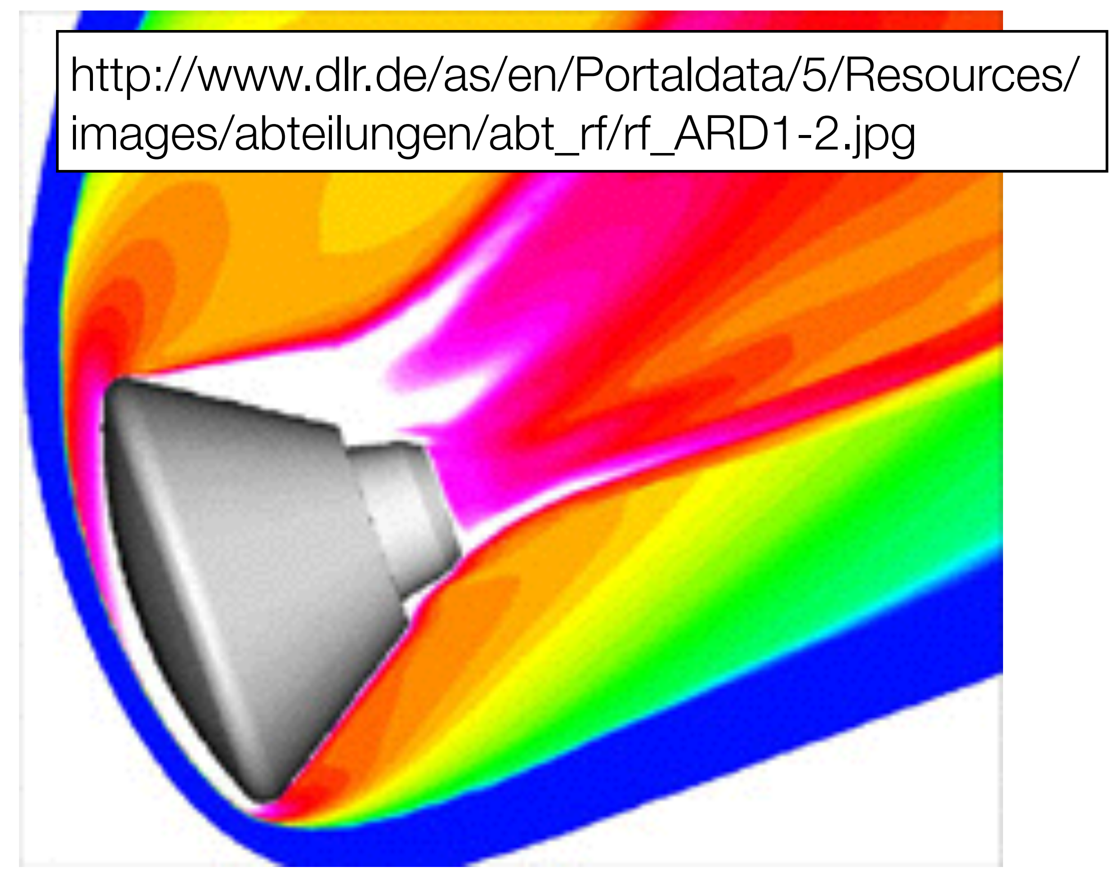
Structure-preserving scheme for non-equilibrium hydrodynamics

Rankine–Hugoniot relationship is violated in one-fluid two-temperature (1F2T) model

- ✓ Global conservation hasn't been satisfied in ICF
- ✓ RH jump condition is still violated in ASTER code
- ✓ Multi-temperature is also used in hypersonic CFD



I. V. Igumenshchev et al., PoP (2016)



http://www.dlr.de/as/en/Portaldata/5/Resources/images/abteilungen/abt_rf/rf_ARD1-2.jpg

Discrete calculus is required to construct a structure-preserving scheme

e.g. 1D continuity equation in the differential form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{(\rho u)_{j+1}^n - (\rho u)_{j-1}^n}{2\Delta x} = 0$$

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + u_j^n \frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\Delta x} + \rho_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{u_{j+1}^n + u_{j-1}^n}{2}\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\Delta x} + \frac{\rho_{j+1}^n + \rho_{j-1}^n}{2}\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

Euler equation including artificial dissipation in the discrete form

$$\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}+\frac{\langle\rho u\rangle_{j^+}^n-\langle\rho u\rangle_{j^-}^n}{\Delta x}=0$$

$$j^+=j+\frac{1}{2}, j^-=j-\frac{1}{2}$$

$$\frac{(\rho u)_j^{n+1}-(\rho u)_j^n}{\Delta t}+\frac{\langle\rho u^2+p\rangle_{j^+}^n-\langle\rho u^2+p\rangle_{j^-}^n}{\Delta x}=\frac{\langle A\rangle_{j^+}^n-\langle A\rangle_{j^-}^n}{\Delta x}$$

$$\frac{(\rho e+\frac{1}{2}\rho u^2)_j^{n+1}-(\rho e+\frac{1}{2}\rho u^2)_j^n}{\Delta t}+$$
$$\frac{\langle\rho eu+\frac{1}{2}\rho u^3+pu\rangle_{j^+}^n-\langle\rho eu+\frac{1}{2}\rho u^3+pu\rangle_{j^-}^n}{\Delta x}=\frac{\langle B\rangle_{j^+}^n-\langle B\rangle_{j^-}^n}{\Delta x}$$

ρ : Density, u : Velocity, e : Internal energy, $p=p(\rho,e)$: Pressure,

A, B : Artificial dissipation, $\langle Q\rangle$: Midpoint-interpolated value of Q

Condition of constraint for energy conservation in the discrete form

$$\frac{(\rho e)_j^{n+1}-(\rho e)_j^n}{\Delta t}+\frac{\langle\rho e u\rangle_{j+}^n-\langle\rho e u\rangle_{j-}^n}{\Delta x}+\frac{\langle p u\rangle_{j+}^n-\langle p u\rangle_{j-}^n}{\Delta x}$$

Advection terms

$$-\frac{\rho_j^{n+1}+\rho_j^n}{4\rho_j^{n+1}\rho_j^n}\left\{(\rho u)_j^{n+1}+(\rho u)_j^n\right\}\frac{\langle p\rangle_{j+}^n-\langle p\rangle_{j-}^n}{\Delta x}$$

$$=-\frac{\rho_j^{n+1}+\rho_j^n}{4\rho_j^{n+1}\rho_j^n}\left\{(\rho u)_j^{n+1}+(\rho u)_j^n\right\}\frac{\langle A\rangle_{j+}^n-\langle A\rangle_{j-}^n}{\Delta x}+\frac{\langle B\rangle_{j+}^n-\langle B\rangle_{j-}^n}{\Delta x}+$$

Artificial dissipation terms

$$\frac{\rho_j^{n+1}+\rho_j^n}{4\rho_j^{n+1}\rho_j^n}\left\{(\rho u)_j^{n+1}+(\rho u)_j^n\right\}\frac{\langle\rho u^2\rangle_{j+}^n-\langle\rho u^2\rangle_{j-}^n}{\Delta x}$$

$$-\frac{1}{2}\frac{\langle\rho u^3\rangle_{j+}^n-\langle\rho u^3\rangle_{j-}^n}{\Delta x}-\frac{\left\{(\rho u)^2\right\}_j^{n+1}+\left\{(\rho u)^2\right\}_j^n}{4\rho_j^{n+1}\rho_j^n}\frac{\langle\rho u\rangle_{j+}^n-\langle\rho u\rangle_{j-}^n}{\Delta x}$$

Error terms: key of structure-preservation

Proposed structure-preserving scheme

T. Shiroto, S. Kawai and N. Ohnishi, J. Comput. Phys.: Regular Article (in prep.)

$$
\begin{aligned}
&\frac{(\rho e_\mathrm{s})_j^{n+1} - (\rho e_\mathrm{s})_j^{n}}{\Delta t} + \frac{\langle \rho e_\mathrm{s} u\rangle_{j+}^n - \langle \rho e_\mathrm{s} u\rangle_{j-}^n}{\Delta x} + \frac{\langle p_\mathrm{s} u\rangle_{j+}^n - \langle p_\mathrm{s} u\rangle_{j-}^n}{\Delta x} \\
&\quad - \frac{\rho_j^{n+1}+\rho_j^n}{4\rho_j^{n+1}\rho_j^n}\left\{(\rho u)_j^{n+1} + (\rho u)_j^n\right\}\frac{\langle p_\mathrm{s}\rangle_{j+}^n - \langle p_\mathrm{s}\rangle_{j-}^n}{\Delta x} \\
&= -\frac{\rho_j^{n+1}+\rho_j^n}{8\rho_j^{n+1}\rho_j^n}\left\{(\rho u)_j^{n+1} + (\rho u)_j^n\right\}\frac{\langle A\rangle_{j+}^n - \langle A\rangle_{j-}^n}{\Delta x} + \frac{\langle B_\mathrm{s}\rangle_{j+}^n - \langle B_\mathrm{s}\rangle_{j-}^n}{\Delta x} \\
&\quad + \frac{\rho_j^{n+1}+\rho_j^n}{8\rho_j^{n+1}\rho_j^n}\left\{(\rho u)_j^{n+1} + (\rho u)_j^n\right\}\frac{\langle \rho u^2\rangle_{j+}^n - \langle \rho u^2\rangle_{j-}^n}{\Delta x} \\
&\quad - \frac{1}{4}\frac{\langle \rho u^3\rangle_{j+}^n - \langle \rho u^3\rangle_{j-}^n}{\Delta x} - \frac{\left\{(\rho u)^2\right\}_j^{n+1} + \left\{(\rho u)^2\right\}_j^{n}}{8\rho_j^{n+1}\rho_j^n}\frac{\langle \rho u\rangle_{j+}^n - \langle \rho u\rangle_{j-}^n}{\Delta x}
\end{aligned}
$$

$$
A = \beta^*\frac{\partial u}{\partial x} \equiv C_\beta \overline{\rho f_\mathrm{sw}\left|\frac{\partial^r}{\partial x^r}\left(\frac{\partial u}{\partial x}\right)\right|\Delta x^{r+2}}\frac{\partial u}{\partial x}
$$

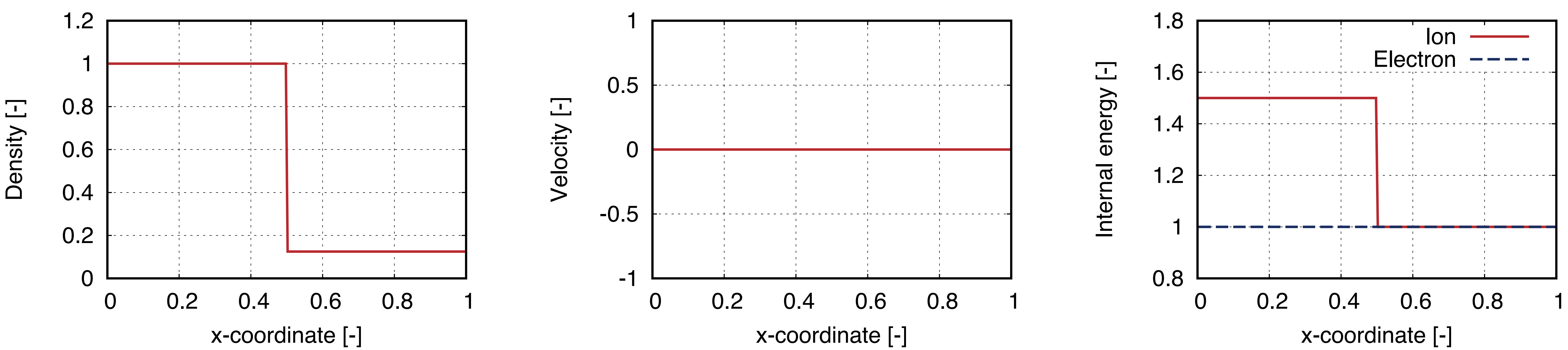
S. Kawai et al., J. Comput. Phys. (2008, 2010)

$$
B_\mathrm{s} = \frac{1}{2}u\beta^*\frac{\partial u}{\partial x} + \kappa_\mathrm{s}^*\frac{\partial e_\mathrm{s}}{\partial x} \equiv \frac{1}{2}u\beta^*\frac{\partial u}{\partial x} + C_\kappa \overline{\frac{\rho c_\mathrm{s}}{e_\mathrm{i}+e_\mathrm{e}}\left|\frac{\partial^r e_\mathrm{s}}{\partial x^r}\right|\Delta x^{r+1}}\frac{\partial e_\mathrm{s}}{\partial x}
$$

$\mathrm{s} \in \{\mathrm{i}, \mathrm{e}\}$

Numerical methods and conditions for the shock tube problem

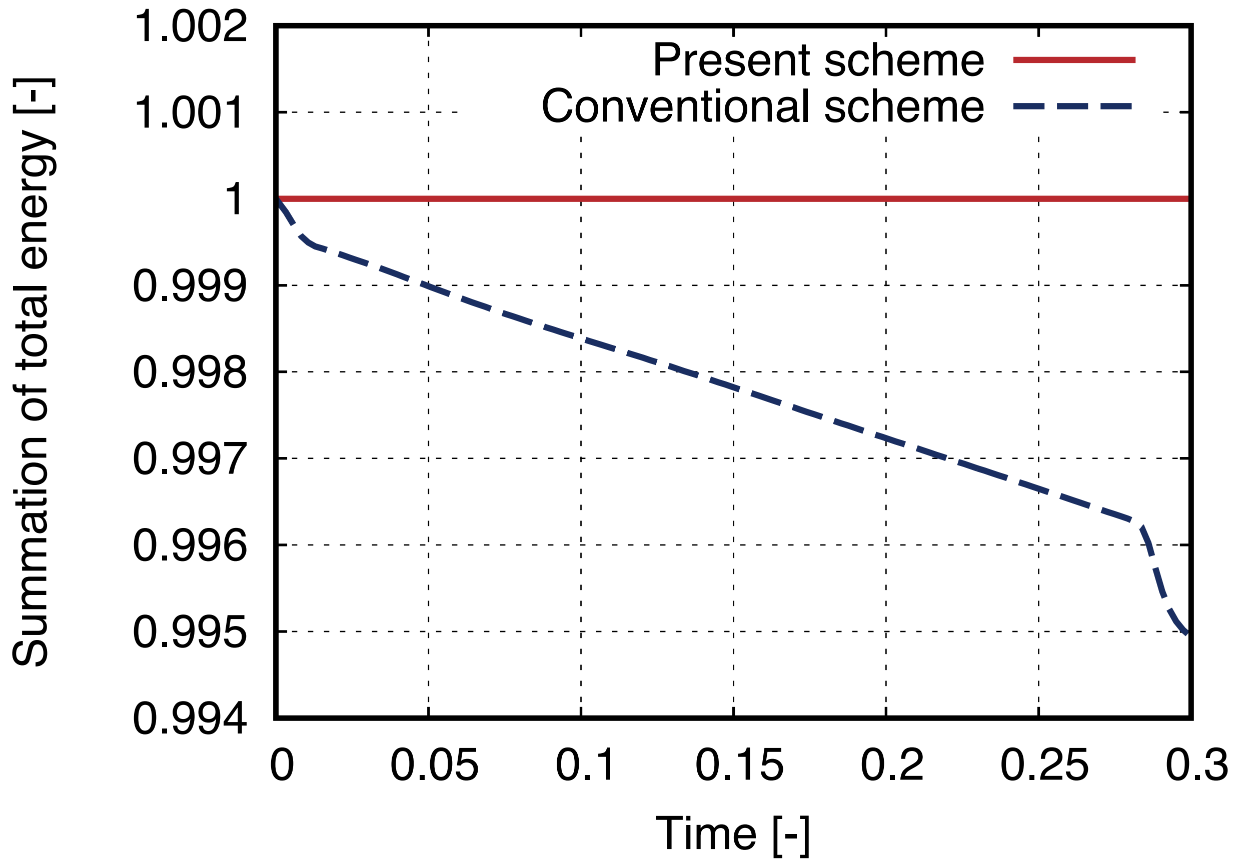
Governing equation	Compressible Euler equation
Midpoint interpolation	4th-order compact scheme
Time integration	3rd-order TVD Runge–Kutta method
Low-pass filter	Modified 8th-order compact filter (α=0.495)
Artificial dissipation	LAD scheme (C_β=2, C_κ=0.1)
Equation of state	Thermally and calorically ideal gas (γ=1.4)

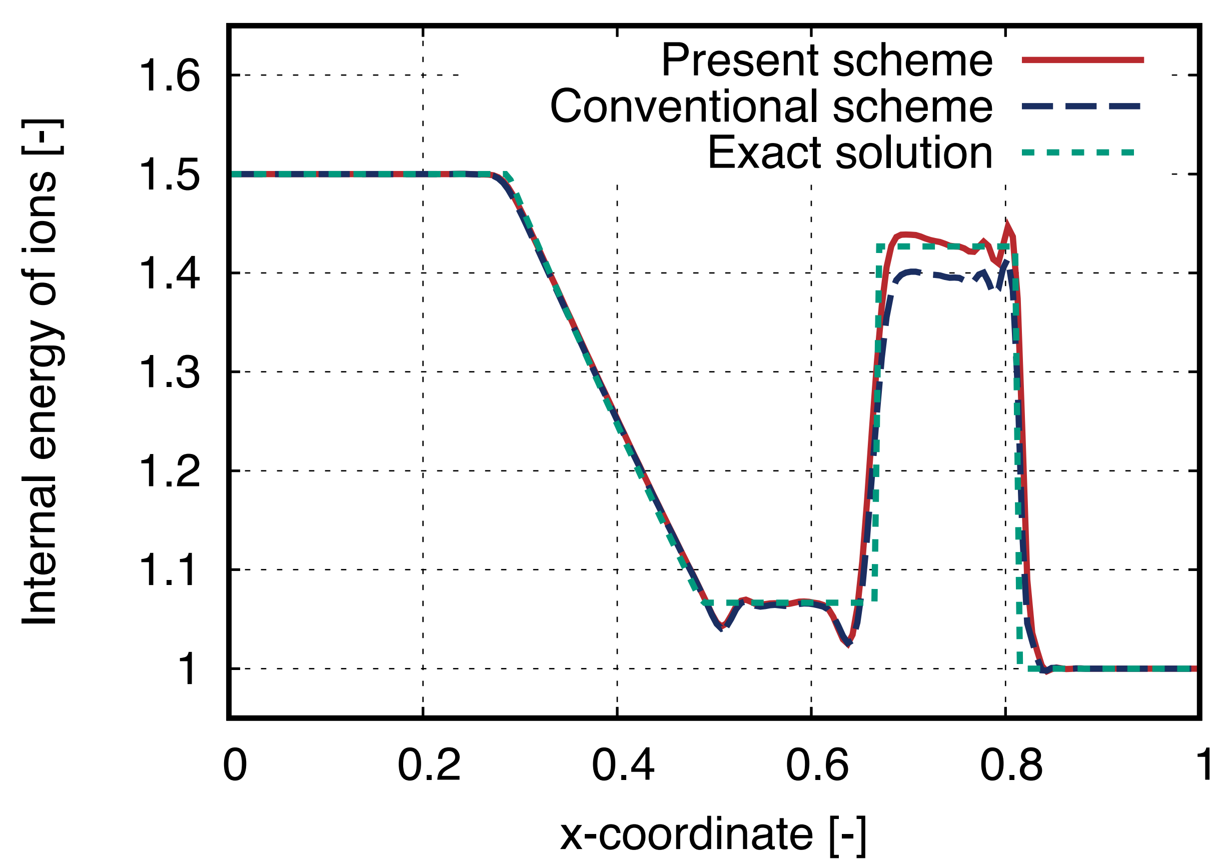


Proposed scheme could reproduce the Rankine–Hugoniot jump condition

✓New scheme achieved much better conservation

✓RH relationship cannot be satisfied without the concept of structure preservation





Flux-form compact filtering scheme

T. Shiroto, S. Kawai and N. Ohnishi, J. Comput. Phys.: Short note (in prep.)

- ✓ Modified filter is equivalent to conventional one but has different order of arithmetic operation
- ✓ Parallelization by domain decomposition manner

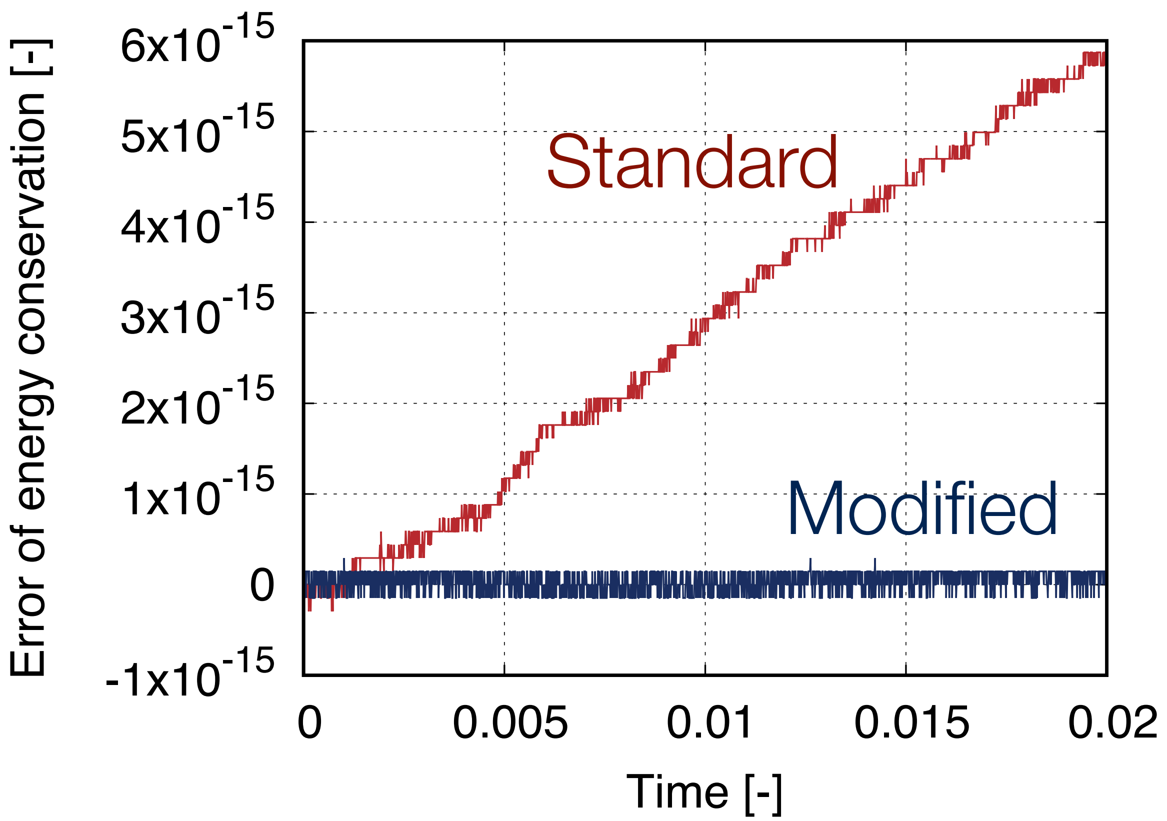
Standard compact filter

$$\alpha_\mathrm{f}\hat{q}_{i-1} + \hat{q}_i + \alpha_\mathrm{f}\hat{q}_{i+1} = \sum_{n=0}^{N} \frac{a_n}{2}(q_{i+n} + q_{i-n})$$

Modified compact filter

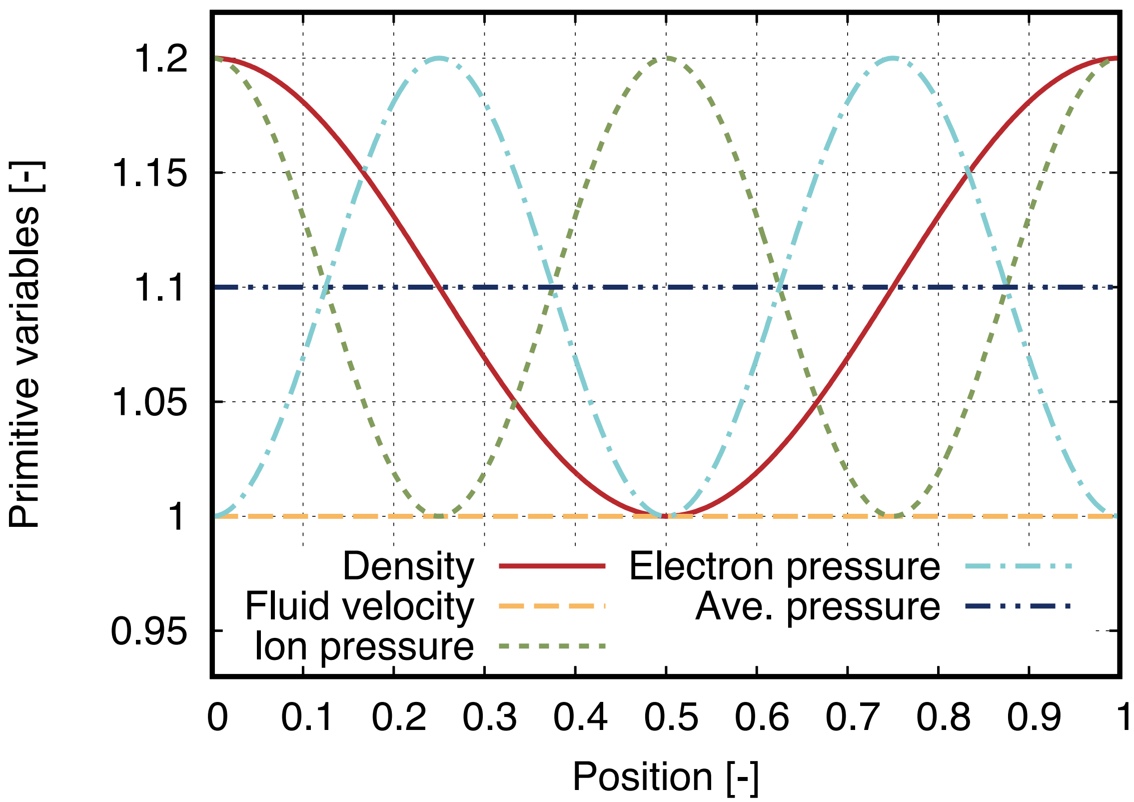
$$\alpha_\mathrm{f}\hat{f}_{i-\frac{1}{2}} + \hat{f}_{i+\frac{1}{2}} + \alpha_\mathrm{f}\hat{f}_{i+\frac{3}{2}} = \sum_{n=0}^{N} \frac{b_n}{2}(q_{i+1+n} - q_{i-n})$$

$$\hat{q}_i = q_i - (\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}})$$



Numerical conditions for linear advection: entropy waves of ions and electrons

- ✓ Total static pressure and velocity have no gradient
- ✓ Linear advection problem for entropy waves
- ✓ Initial conditions should be reproduced at t=1



Eigenvalues

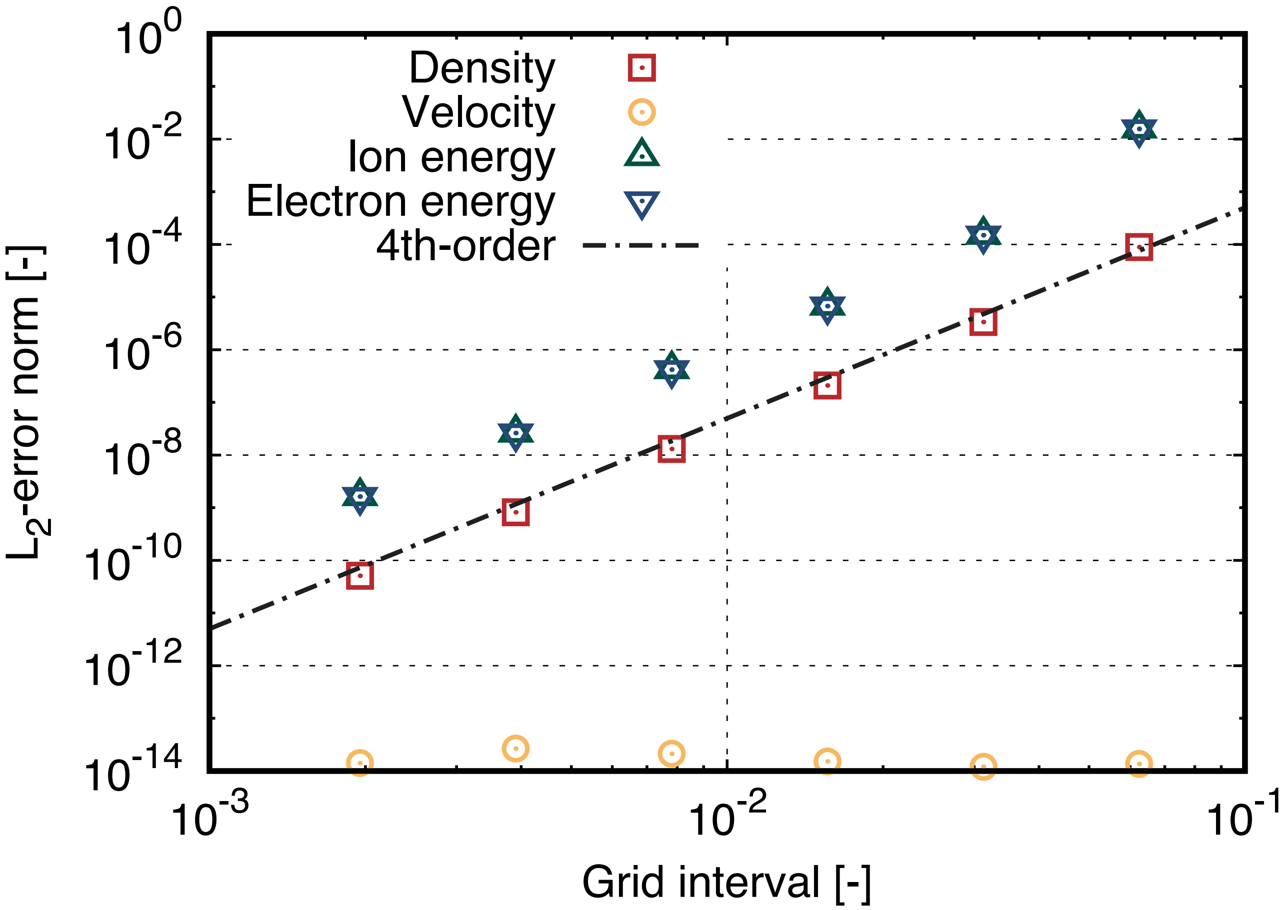
$$\lambda_1 = u - c_\mathrm{s}, \quad \lambda_2 = \lambda_3 = u, \quad \lambda_4 = u + c_\mathrm{s}$$

Eigenvectors

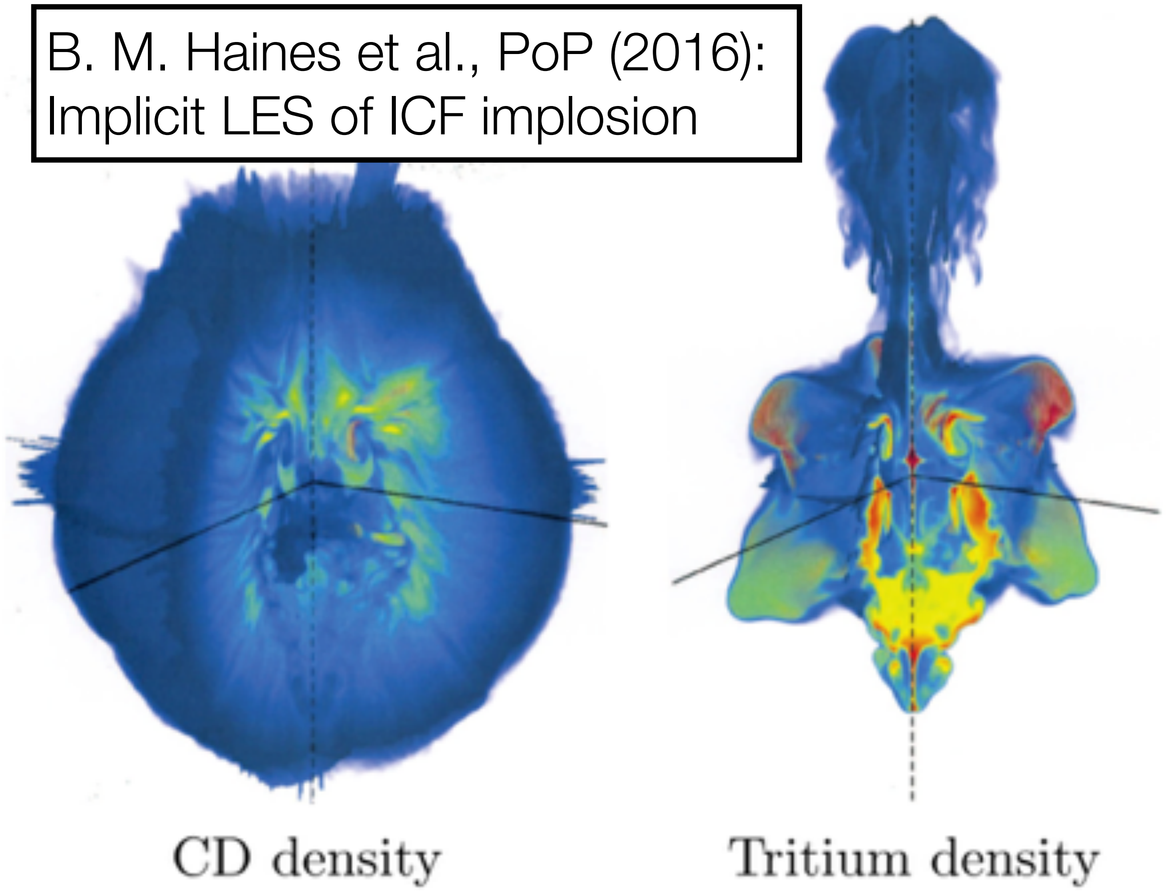
$$\mathbf{k} = \begin{bmatrix} \rho \\ -c_\mathrm{s} \\ (\gamma-1)e_\mathrm{i} \\ (\gamma-1)e_\mathrm{e} \end{bmatrix}, \begin{bmatrix} \rho \\ 0 \\ -e_\mathrm{i} \\ 0 \end{bmatrix}, \begin{bmatrix} \rho \\ 0 \\ 0 \\ -e_\mathrm{e} \end{bmatrix}, \begin{bmatrix} \rho \\ c_\mathrm{s} \\ (\gamma-1)e_\mathrm{i} \\ (\gamma-1)e_\mathrm{e} \end{bmatrix}$$

Proposed scheme can accept compact schemes which are used in LES

- ✓ Energy of ions and electrons have 4th-order spatial accuracy despite of indirect discretization
- ✓ 3D ICF code should be constructed for LES



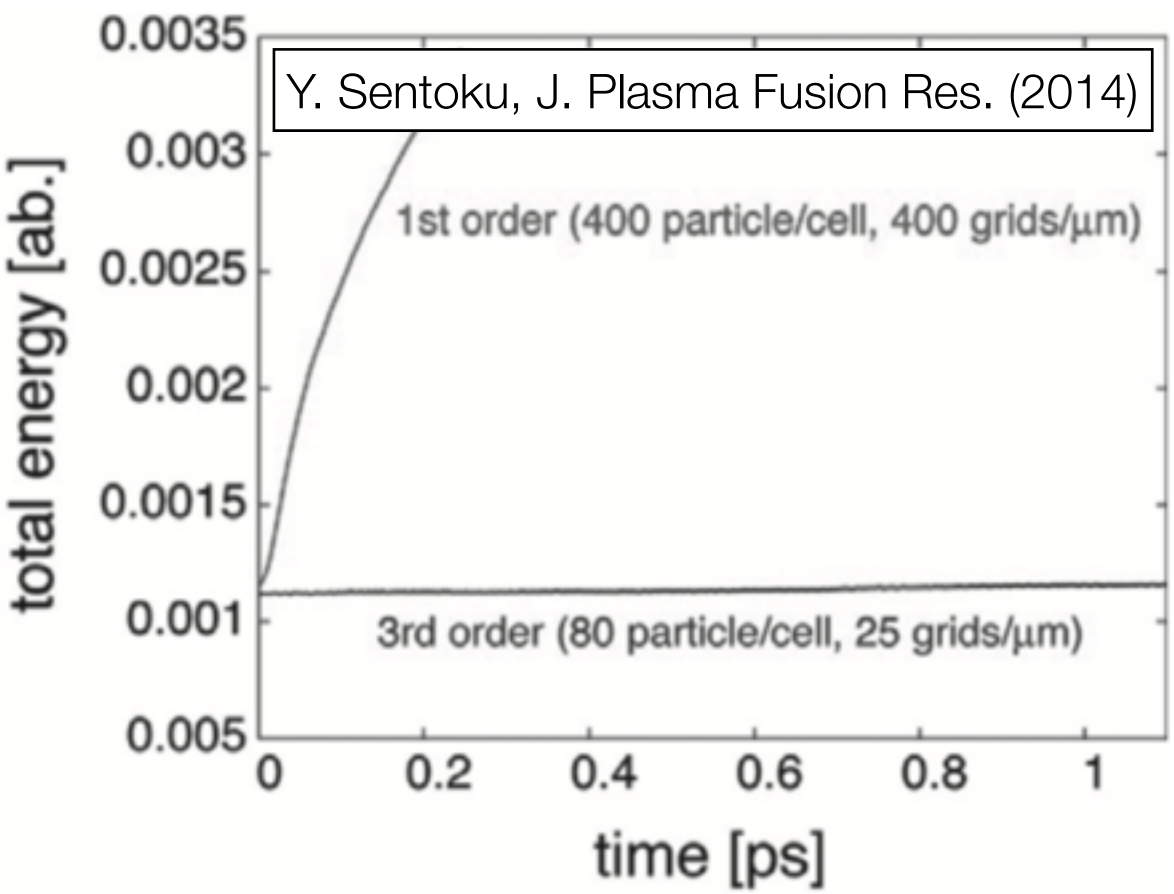
B. M. Haines et al., PoP (2016):
Implicit LES of ICF implosion



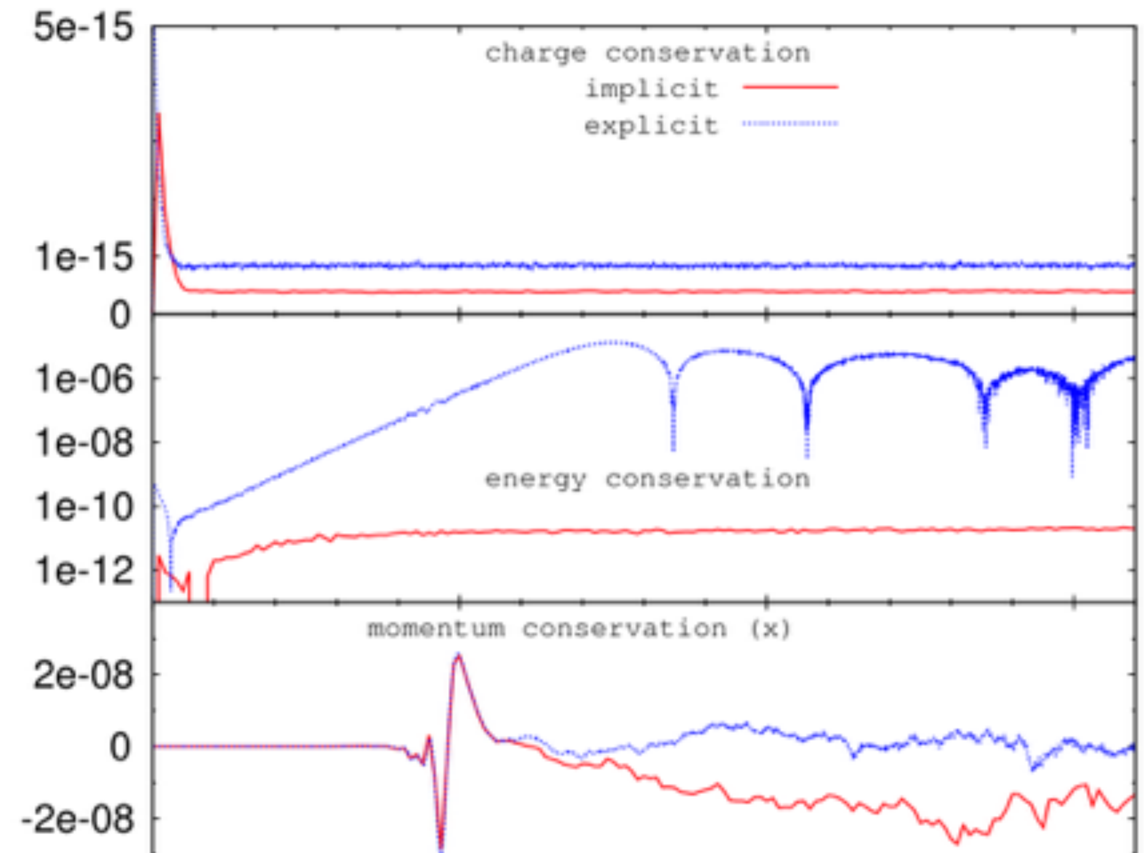
Preliminary report:
Structure-preserving PIC method

Modern Particle In Cell (PIC) still has a fatal problem of numerical heating

- ✓Today's PIC suffers from numerical instability
- ✓High-order particle has reduced statistical error
- ✓Implicit PIC marked good conservation property



Y. Sentoku, J. Plasma Fusion Res. (2014)



G. Chen and L. Chacon, Comput. Phys. Comm. (2015)

Why conservation law of electric charge is derived from Maxwell's equations?

Gauss's law: $\mathrm{div}\ \mathbf{E} = 4\pi\rho$

Partial differentiation with respect to time: $\dfrac{\partial}{\partial t}(\mathrm{div}\ \mathbf{E}) = 4\pi\dfrac{\partial \rho}{\partial t}$

Exchange the partial differential order: $\mathrm{div}\ \dfrac{\partial \mathbf{E}}{\partial t} = 4\pi\dfrac{\partial \rho}{\partial t}$

Substitute Ampere–Maxwell's law: $\mathrm{div}(c\ \mathrm{rot}\mathbf{B} - 4\pi\mathbf{J}) = 4\pi\dfrac{\partial \rho}{\partial t}$

Divergence of rotation is equal to zero: $\dfrac{\partial \rho}{\partial t} + \mathrm{div}\ \mathbf{J} = 0$

Why conservation law of energy is derived from Maxwell's equations?

Ampere–Maxwell and Faraday–Maxwell laws:

$$\frac{1}{c}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = \mathbf{E}\cdot\mathrm{rot}\ \mathbf{B} - \frac{4\pi}{c}\mathbf{J}\cdot\mathbf{E}$$

$$\frac{1}{c}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = -\mathbf{B}\cdot\mathrm{rot}\ \mathbf{E}$$

Summation of these two equations:

$$\frac{1}{c}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{c}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = \mathbf{E}\cdot\mathrm{rot}\ \mathbf{B} - \mathbf{B}\cdot\mathrm{rot}\ \mathbf{E} - \frac{4\pi}{c}\mathbf{J}\cdot\mathbf{E}$$

Product rule is utilized:

$$\frac{1}{8\pi}\frac{\partial(\mathbf{E}^2+\mathbf{B}^2)}{\partial t} + \frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{B}) = -\mathbf{J}\cdot\mathbf{E}$$

Conservative PIC method (1)

Solenoidal condition

$$\frac{(B_x)^n_{i+1} - (B_x)^n_i}{\Delta x} = 0$$

Gauss's law

$$\frac{(E_x)^n_{i+1} - (E_x)^n_i}{\Delta x} = 4\pi\rho^n_{i+\frac{1}{2}}$$

Ampere–Maxwell equation

$$\frac{1}{c}\frac{(E_x)^{n+1}_i - (E_x)^n_i}{\Delta t} = -\frac{4\pi}{c}(J_x)^{n+\frac{1}{2}}_i$$

$$\frac{1}{c}\frac{(E_y)^{n+1}_{i+} - (E_y)^n_{i+}}{\Delta t} = -\frac{(B_z)^{n^+}_{i+1} - (B_z)^{n^+}_i}{\Delta x} - \frac{4\pi}{c}(J_y)^{n+\frac{1}{2}}_{i+\frac{1}{2}}$$

$$\frac{1}{c}\frac{(E_z)^{n+1}_{i+} - (E_z)^n_{i+}}{\Delta t} = \frac{(B_y)^{n^+}_{i+1} - (B_y)^{n^+}_i}{\Delta x} - \frac{4\pi}{c}(J_z)^{n+\frac{1}{2}}_{i+\frac{1}{2}}$$

$$\boxed{F_{i+} \equiv \frac{F_{i+1} + F_i}{2} \neq F_{i+\frac{1}{2}}}$$

Implicit time integration is required!

Conservative PIC method (2)

Faraday–Maxwell equation

$$\frac{1}{c}\frac{(B_x)_i^{n+1}-(B_x)_i^n}{\Delta t}=0$$

$$\frac{1}{c}\frac{(B_y)_{i+}^{n+1}-(B_y)_{i+}^n}{\Delta t}=\frac{(E_z)_{i+1}^{n^+}-(E_z)_i^{n^+}}{\Delta x}$$

$$\frac{1}{c}\frac{(B_z)_{i+}^{n+1}-(B_z)_{i+}^n}{\Delta t}=-\frac{(E_y)_{i+1}^{n^+}-(E_y)_i^{n^+}}{\Delta x}$$

Product rule

$$\frac{F_{i+1}G_{i+1}-F_iG_i}{\Delta x}=\frac{F_{i+1}+F_i}{2}\frac{G_{i+1}-G_i}{\Delta x}+\frac{F_{i+1}-F_i}{\Delta x}\frac{G_{i+1}+G_i}{2}$$
$$=F_{i+}\frac{G_{i+1}-G_i}{\Delta x}+\frac{F_{i+1}-F_i}{\Delta x}G_{i+}$$

The law of charge conservation is composed without any approximation (1)

Proof.

Gauss's law:

$$\frac{(E_x)^n_{i+1} - (E_x)^n_i}{\Delta x} = 4\pi \rho^n_{i+\frac{1}{2}}$$

Partial finite-differentiation with respect to time:

$$\frac{(E_x)^{n+1}_{i+1} - (E_x)^{n+1}_i - (E_x)^n_{i+1} + (E_x)^n_i}{\Delta t \Delta x} = 4\pi \frac{\rho^{n+1}_{i+\frac{1}{2}} - \rho^n_{i+\frac{1}{2}}}{\Delta t}$$

$$\frac{1}{\Delta x}\left\{\frac{(E_x)^{n+1}_{i+1} - (E_x)^n_{i+1}}{\Delta t} - \frac{(E_x)^{n+1}_i - (E_x)^n_i}{\Delta t}\right\} = 4\pi \frac{\rho^{n+1}_{i+\frac{1}{2}} - \rho^n_{i+\frac{1}{2}}}{\Delta t}$$

The order of finite-differential is exchanged!

The law of charge conservation is composed without any approximation (2)

$$\frac{1}{\Delta x}\left\{\frac{(E_x)_{i+1}^{n+1}-(E_x)_{i+1}^{n}}{\Delta t}-\frac{(E_x)_{i}^{n+1}-(E_x)_{i}^{n}}{\Delta t}\right\}=4\pi\frac{\rho_{i+\frac{1}{2}}^{n+1}-\rho_{i+\frac{1}{2}}^{n}}{\Delta t}$$

Substitute x-component of Ampere–Maxwell's law:

$$\frac{1}{\Delta x}\left\{-4\pi(J_x)_{i+1}^{n+\frac{1}{2}}+4\pi(J_x)_{i}^{n+\frac{1}{2}}\right\}=4\pi\frac{\rho_{i+\frac{1}{2}}^{n+1}-\rho_{i+\frac{1}{2}}^{n}}{\Delta t}$$

Thus, the structure of charge conservation is preserved:

$$\boxed{\frac{\rho_{i+\frac{1}{2}}^{n+1}-\rho_{i+\frac{1}{2}}^{n}}{\Delta t}+\frac{(J_x)_{i+1}^{n+\frac{1}{2}}-(J_x)_{i}^{n+\frac{1}{2}}}{\Delta x}=0}$$

The law of energy conservation is composed without any approximation (1)

y-component of Ampere–Maxwell and z-component of Faraday–Maxwell laws:

$$\frac{1}{c}(E_y)_{i+}^{n+}\frac{(E_y)_{i+}^{n+1}-(E_y)_{i+}^{n}}{\Delta t}=-(E_y)_{i+}^{n+}\frac{(B_z)_{i+1}^{n^+}-(B_z)_{i}^{n^+}}{\Delta x}-\frac{4\pi}{c}(E_y)_{i+}^{n^+}(J_y)_{i+\frac{1}{2}}^{n+\frac{1}{2}}$$

$$\frac{1}{c}(B_z)_{i+}^{n+}\frac{(B_z)_{i+}^{n+1}-(B_z)_{i+}^{n}}{\Delta t}=-(B_z)_{i+}^{n+}\frac{(E_y)_{i+1}^{n^+}-(E_y)_{i}^{n^+}}{\Delta x}$$

Product rule is utilized:

$$\frac{(E_y^2+B_z^2)_{i+}^{n+1}-(E_y^2+B_z^2)_{i+}^{n}}{8\pi\Delta t}+\frac{c}{4\pi}\frac{(E_yB_z)_{i+1}^{n^+}-(E_yB_z)_{i}^{n^+}}{\Delta x}=-(E_y)_{i+}^{n^+}(J_y)_{i+\frac{1}{2}}^{n+\frac{1}{2}}$$

The law of energy conservation is composed without any approximation (2)

$$\frac{(E_y^2+B_z^2)_{i+}^{n+1}-(E_y^2+B_z^2)_{i+}^{n}}{8\pi\Delta t}+\frac{c}{4\pi}\frac{(E_yB_z)_{i+1}^{n^+}-(E_yB_z)_{i}^{n^+}}{\Delta x}=-(E_y)_{i+}^{n^+}(J_y)_{i+\frac{1}{2}}^{n+\frac{1}{2}}$$

These equations are derived in the same way:

$$\frac{(E_z^2+B_y^2)_{i+}^{n+1}-(E_z^2+B_y^2)_{i+}^{n}}{8\pi\Delta t}+\frac{c}{4\pi}\frac{(E_zB_y)_{i+1}^{n^+}-(E_zB_y)_{i}^{n^+}}{\Delta x}=-(E_z)_{i+}^{n^+}(J_z)_{i+\frac{1}{2}}^{n+\frac{1}{2}}$$

$$\frac{(E_x^2+B_x^2)_{i+}^{n+1}-(E_x^2+B_x^2)_{i+}^{n}}{8\pi\Delta t}=-(E_x)_{i+}^{n^+}(J_x)_{i+}^{n+\frac{1}{2}}$$

Thus, the structure of energy conservation is preserved:

$$\frac{1}{8\pi}\frac{(\mathbf{E}^2)_{i+}^{n+1}-(\mathbf{E}^2)_{i+}^{n}}{\Delta t}+\frac{1}{8\pi}\frac{(\mathbf{B}^2)_{i+}^{n+1}-(\mathbf{B}^2)_{i+}^{n}}{\Delta t}+\frac{c}{4\pi}\frac{(\mathbf{E}\times\mathbf{B})_{x,i+1}^{n^+}-(\mathbf{E}\times\mathbf{B})_{x,i}^{n^+}}{\Delta x}$$
$$=-(E_x)_{i+}^{n^+}(J_x)_{i+}^{n+\frac{1}{2}}-(E_y)_{i+}^{n^+}(J_y)_{i+\frac{1}{2}}^{n+\frac{1}{2}}-(E_z)_{i+}^{n^+}(J_z)_{i+\frac{1}{2}}^{n+\frac{1}{2}}$$

The law of momentum conservation is composed without any approximation (1)

y-component of Ampere–Maxwell and z-component of Faraday–Maxwell laws:

$$\frac{1}{c}(B_z)^{n^+}_{i+}\frac{(E_y)^{n+1}_{i+}-(E_y)^{n}_{i+}}{\Delta t}=-(B_z)^{n^+}_{i+}\frac{(B_z)^{n^+}_{i+1}-(B_z)^{n^+}_{i}}{\Delta x}-\frac{4\pi}{c}(J_y)^{n+\frac{1}{2}}_{i+\frac{1}{2}}(B_z)^{n^+}_{i+}$$

$$\frac{1}{c}(E_y)^{n^+}_{i+}\frac{(B_z)^{n+1}_{i+}-(B_z)^{n}_{i+}}{\Delta t}=-(E_y)^{n^+}_{i+}\frac{(E_y)^{n^+}_{i+1}-(E_y)^{n^+}_{i}}{\Delta x}$$

Product rule is utilized:

$$\frac{(E_yB_z)^{n+1}_{i+}-(E_yB_z)^{n}_{i+}}{4\pi c\Delta t}+\frac{(E_y^2+B_z^2)^{n^+}_{i+1}-(E_y^2+B_z^2)^{n^+}_{i}}{8\pi\Delta t}=-\frac{1}{c}(J_y)^{n+\frac{1}{2}}_{i+\frac{1}{2}}(B_z)^{n^+}_{i+}$$

The law of momentum conservation is composed without any approximation (2)

$$\frac{(E_yB_z)^{n+1}_{i+} - (E_yB_z)^{n}_{i+}}{4\pi c\Delta t} + \frac{(E_y^2+B_z^2)^{n^+}_{i+1} - (E_y^2+B_z^2)^{n^+}_{i}}{8\pi\Delta t} = -\frac{1}{c}(J_y)^{n+\frac{1}{2}}_{i+\frac{1}{2}}(B_z)^{n^+}_{i+}$$

These equations are derived in the same way:

$$\frac{(E_zB_y)^{n+1}_{i+} - (E_zB_y)^{n}_{i+}}{4\pi c\Delta t} - \frac{(E_z^2+B_y^2)^{n^+}_{i+1} - (E_z^2+B_y^2)^{n^+}_{i}}{8\pi\Delta t} = -\frac{1}{c}(J_z)^{n+\frac{1}{2}}_{i+\frac{1}{2}}(B_y)^{n^+}_{i+}$$

$$\frac{1}{8\pi}\frac{(E_x^2)^{n^+}_{i+1} - (E_x)^{n^+}_{i}}{\Delta x} = \rho^{n^+}_{i+\frac{1}{2}}(E_x)^{n^+}_{i+} \qquad \frac{1}{8\pi}\frac{(B_x^2)^{n^+}_{i+1} - (B_x)^{n^+}_{i}}{\Delta x} = 0$$

Thus, the structure of momentum conservation is preserved:

$$\frac{(\mathbf{E}\times\mathbf{B})^{n+1}_{x,i+} - (\mathbf{E}\times\mathbf{B})^{n}_{x,i+}}{4\pi c\Delta t} + \frac{(\mathbf{E}^2+\mathbf{B}^2-2E_x^2-2B_x^2)^{n^+}_{i+1} - (\mathbf{E}^2+\mathbf{B}^2-2E_x^2-2B_x^2)^{n^+}_{i}}{8\pi\Delta x}$$
$$= -\rho^{n^+}_{i+\frac{1}{2}}(E_x)^{n^+}_{i+} - \frac{1}{c}(J_y)^{n+\frac{1}{2}}_{i+\frac{1}{2}}(B_z)^{n^+}_{i+} + \frac{1}{c}(J_z)^{n+\frac{1}{2}}_{i+\frac{1}{2}}(B_y)^{n^+}_{i+}$$

The law of momentum conservation is composed without any approximation (3)

$$\frac{(\mathbf{E}\times\mathbf{B})^{n+1}_{x,i+} - (\mathbf{E}\times\mathbf{B})^{n}_{x,i+}}{4\pi c\Delta t} + \frac{(\mathbf{E}^2+\mathbf{B}^2-2E_x^2-2B_x^2)^{n^+}_{i+1} - (\mathbf{E}^2+\mathbf{B}^2-2E_x^2-2B_x^2)^{n^+}_{i}}{8\pi\Delta x}$$
$$= -\rho^{n^+}_{i+\frac{1}{2}}(E_x)^{n^+}_{i+} - \frac{1}{c}(J_y)^{n+\frac{1}{2}}_{i+\frac{1}{2}}(B_z)^{n^+}_{i+} + \frac{1}{c}(J_z)^{n+\frac{1}{2}}_{i+\frac{1}{2}}(B_y)^{n^+}_{i+}$$

These equations are derived in the same way:

$$\frac{(\mathbf{E}\times\mathbf{B})^{n+1}_{y,i+} - (\mathbf{E}\times\mathbf{B})^{n}_{y,i+}}{4\pi c\Delta t} - \frac{(E_xE_y+B_xB_y)^{n^+}_{i+1} - (E_xE_y+B_xB_y)^{n^+}_{i}}{4\pi\Delta x}$$
$$= -\rho^{n^+}_{i+\frac{1}{2}}(E_y)^{n^+}_{i+} - \frac{1}{c}(J_z)^{n+\frac{1}{2}}_{i+\frac{1}{2}}(B_x)^{n^+}_{i+} + \frac{1}{c}(J_x)^{n+\frac{1}{2}}_{i+}(B_z)^{n^+}_{i+}$$

$$\frac{(\mathbf{E}\times\mathbf{B})^{n+1}_{z,i+} - (\mathbf{E}\times\mathbf{B})^{n}_{z,i+}}{4\pi c\Delta t} - \frac{(E_xE_z+B_xB_z)^{n^+}_{i+1} - (E_xE_z+B_xB_z)^{n^+}_{i}}{4\pi\Delta x}$$
$$= -\rho^{n^+}_{i+\frac{1}{2}}(E_z)^{n^+}_{i+} - \frac{1}{c}(J_x)^{n+\frac{1}{2}}_{i+}(B_y)^{n^+}_{i+} + \frac{1}{c}(J_y)^{n+\frac{1}{2}}_{i+\frac{1}{2}}(B_x)^{n^+}_{i+}$$

Q. E. D.

Summary

- ✓ Violation of energy conservation on the two-temperature hydrodynamic model was under the control of machine epsilon
- ✓ Structure-preserving scheme also follows the jump condition of Rankine–Hugoniot relationship
- ✓ Conservative PIC scheme detests discrimination between the electric field and magnetic field